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STRONG QUASI k-IDEALS AND THE LATTICE DECOMPOSITIONS OF SEMIRINGS WITH SEMILATTICE ADDITIVE REDUCT

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Abstract

Here we introduce the notion of strong quasi k-ideals of a semiring in SL^+ and characterize the semirings that are distributive lattices of t-k-simple(t-k-Archimedean) subsemirings by their strong quasi k-ideals. A quasi k-ideal Q is strong if it is an intersection of a left k-ideal and a right k-ideal. A semiring S in SL^+ is a distributive lattice of t-k-simple semirings if and only if every strong quasi k-ideal is a completely semiprime k-ideal of S. Again S is a distributive lattice of t-k-Archimedean semirings if and only if $\sqrt{Q}$ is a k-ideal, for every strong quasi k-ideal Q of S.

Keywords: quasi k-ideal, strong quasi k-ideal, strong quasi k-simple, t-k-simple, t-k-Archimedean.

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1. Introduction

The notion of semirings was introduced by Vandiver [21] in connection with the axiomatization of the arithmetic of natural numbers. Historically, the semirings appeared in mathematics implicitly as the semiring of all ideals of a ring, the

semiring of all endomorphisms of a commutative semigroup etc. long before than its axiomatic formulation. Though they appeared long before, but the semirings found their full place in mathematics recently e.g. in idempotent analysis [8, 15, 13] which are being used in theoretical physics, optimization [6, 7, 16] etc., various applications in theoretical computer science and algorithm theory [9, 14]. The underlying semirings both in idempotent analysis and theoretical computer science are such that the additive reduct is a semilattice, i.e. both idempotent and commutative.

This is a continuation of our study on the semirings whose additive reduct is a semilattice by their bi-ideals and quasi ideals [1, 11, 12]. Here we characterize such semirings by their strong quasi-k-ideals. The notion of quasi-ideals in rings and semigroups was introduced and developed by Otto Steinfeld [20]. Later, the idea has been generalized in semirings, ordered semigroups, Γ-semigroups [2, 11, 12] etc. In semigroups, quasi-ideals are precisely intersection of a left ideal and a right ideal. In rings though an intersection of a left ideal and a right ideal is a quasi-ideal but the converse is not true [22]. Thus a class of quasi-ideals which can be expressed as an intersection of a left ideal and a right ideal was distinguished and they are known as quasi-ideals with intersection property. Here we have shown that the case for the quasi k-ideals in semirings is the same as in the rings and we call the quasi k-ideals which are intersection of a left k-ideal and a right k-ideal as strong quasi k-ideal. Interestingly, though there are strong quasi k-ideals which are neither left k-ideal nor right k-ideal, still a semiring is strong quasi k-simple if and only if it is t-k-simple; which shows that the strong quasi k-ideals are potential to characterize the semirings which are distributive lattice of t-k-simple semirings and t-k-Archimedean semirings.

We show that S is a distributive lattice(chain) of t-k-simple subsemirings if and only if every strong quasi k-ideal is a completely semiprime (prime) k-ideal of S, where as S is a distributive lattice (chain) of t-k-Archimedean subsemirings if and only if radicals of every strong quasi k-ideal is a k-ideal (completely prime).

The preliminaries and prerequisites we need have been discussed in Section 2. In Section 4 we give different equivalent characterizations of the semirings which are distributive lattices (chains) of strong t-k-simple semirings. Section 5 is devoted to characterize semirings which are distributive lattices of t-k-Archimedean semirings.

2. Preliminaries

The terminology and basic notions in this section are according to [1, 19].

A semiring $(S, +, \cdot)$ is an algebra with two binary operations $+$ and $\cdot$ such that both the additive reduct $(S, +)$ and the multiplicative reduct $(S, \cdot)$ are semigroups

and the following distributive laws hold:

$$x(y+z) = xy + xz \text{ and } (x+y)z = xz + yz.$$

A semiring S is called an additively idempotent semiring if the additive reduct $(S,+)$ is an idempotent semigroup (band), i.e., if it satisfies the identity $x+x=x$. In addition, if the additive reduct $(S,+)$ is commutative, i.e., if it satisfies the identity $x+y=y+x$, then we say that S is a semiring with semilattice additive reduct. Throughout this paper, unless otherwise stated, S is always a semiring whose additive reduct is a semilattice and the variety of all such semirings is denoted by SL^+.

A non-empty subset A of S is called a k-subset of S if for $x \in S, a \in A, x+a \in A$ implies that $x \in A$. The k-closure $\overline{A}$ of a nonempty subset A of a semiring S is given by

$$\overline{A} = \{x \in S \mid \exists\ a, b \in A \text{ such that } x + a = b\}.$$

This is the smallest k-subset containing A. Thus A is a k-subset if $\overline{A} \subseteq A$. Since for every $a \in S$, $a + a = a$ we have $A \subseteq \overline{A}$.

A nonempty subset L of a semiring S is called a left k-ideal of S if $L + L \subseteq L$, $SL \subseteq L$ and $\overline{L} = L$. The right k-ideals are defined dually. A subset I of S is called a k-ideal of S if it is both a left and a right k-ideal of S. A semiring S is called (*resp.* left, right) k-simple if it has no non-trivial (*resp.* left, right) k-ideal. If S is both left k-simple and right k-simple then it is called t-k-simple.

The left (*resp.* right) k-ideal generated by a is denoted by $L_k(a)$ (resp. $R_k(a)$) and we have

$$L_k(a) = \{u \in S \mid u + a + sa = a + sa, \text{ for some } s \in S\},$$
$$R_k(a) = \{u \in S \mid u + a + as = a + as, \text{ for some } s \in S\}.$$

Sen and Bhuniya introduced analogues of Green's relations $\overline{\mathcal{L}}, \overline{\mathcal{R}}$ and $\overline{\mathcal{H}}$ on a semiring S in the following way: for $a, b \in S$

$$a\overline{\mathcal{L}}b \text{ if } L_k(a) = L_k(b), \quad a\overline{\mathcal{R}}b \text{ if } R_k(a) = R_k(b) \quad \text{and} \quad \overline{\mathcal{H}} = \overline{\mathcal{L}} \cap \overline{\mathcal{R}}.$$

These equivalences are additive congruences on S, whereas $\overline{\mathcal{L}}$ is multiplicative right and $\overline{\mathcal{R}}$ is multiplicative left congruence on S only. A semiring S is (left, right) t-k-simple if and only if $(\overline{\mathcal{L}}, \overline{\mathcal{R}})\ \overline{\mathcal{H}} = S \times S$.

A nonempty subset A of S is called completely prime (*resp.* semiprime) if for all $x, y \in S$ such that $xy \in A$ one has $x \in A$ or $y \in A$ (*resp.* $x^2 \in A$ implies that $x \in A$).

A subsemiring F of S is called a filter of S if for any $a, b \in S$, $ab \in F$ implies that $a, b \in F$ and $a + b = b$, $a \in F$ implies that $b \in F$. For $a \in S$, $N(a)$ denotes the filter generated by a. Let $\mathcal{N}$ be the equivalence relation on S, defined by:

for $x, y \in S$,

$$x\mathcal{N}y \text{ if } N(x) = N(y).$$

As we can expect, we have the following result which plays a crucial role in this article.

Lemma 1 [4]**.** *Let S be a semiring in SL^+. Then $\mathcal{N}$ is the least distributive lattice congruence on S.*

A subsemiring B of S is called a k-bi-ideal of S if $BSB \subseteq B$ and $\overline{B} \subseteq B$. Every left k-ideal and right k-ideal is a k-bi-ideal of S. A semiring S is called k-b-simple if it has no non-trivial k-bi-ideal.

The k-bi-ideal generated by a is denoted by $B_k(a)$ and we have

$$B_k(a) = \{u \in S \mid u + a + a^2 + asa = a + a^2 + asa, \text{ for some } s \in S\}.$$

Let $\mathcal{C}$ be a class of semirings and we call the members of $\mathcal{C}$ as C-semirings. A semiring S is called a distributive lattice of $\mathcal{C}$-semirings if there exists a congruence ρ on S such that S/ρ is a distributive lattice and each ρ-class is a $\mathcal{C}$-semiring.

The following lemma summarizes some useful techniques for handling semirings with semilattice additive reduct which we will use frequently in this paper. We omit the proof as it can be done similarly to the Lemma 2.1 [3].

Lemma 2. *Let S be a semiring in SL^+. For $a, b, u, v, s, s_1, s_2, t, t_1, t_2 \in S$*

1. *$b + s_1as_2 = t_1at_2$ implies that there is $x = s_1 + s_2 + t_1 + t_2 \in S$ such that $b + xax = xax$.*
2. *If $a + b = b$ then*
   (i) *$u + sa = sa$ implies that $u + sb = sb$.*
   (ii) *$u + sa + a = sa + a$ implies that $u + sb + b = sb + b$.*
3. *$u + sa + a = sa + a$ and $v + bt + b = bt + b$ implies that there are $x = s + t \in S$ and $c = a + b \in S$ such that $u + xc + c = xc + c$ and $v + cx + c = cx + c$.*

We refer [5, 10] for the information we need concerning semigroup theory and [9] for notions concerning semiring theory.

3. Strong quasi k-ideals of a semiring

A subsemiring Q is called a quasi k-ideal of S if $QS \cap SQ \subseteq Q$ and $\overline{Q} = Q$. Intersection of a left k-ideal and a right k-ideal is a quasi k-ideal of S [11], but the converse is not true in general which we see in the following example. This example is motivated by that of given by Weinert in [22].

Example 3. Consider the two element lattice $\Gamma = \{0, 1\}$ with binary operations

$+$	0	1
0	0	1
1	1	1

$\cdot$	0	1
0	0	0
1	0	1

Consider the semialgebra S on Γ defined by the basis $\{e, a, b\}$ with the multiplication table

$\cdot$	e	a	b
e	e	a+b	0
a	b	0	0
b	b	0	0

Note that the semialgebra $S = \{r_1e + r_2a + r_3b \mid r_i \in \Gamma\} = \{0, e, a, b, e + a, a + b, e + b, e + a + b\}$ is a semiring with semilattice additive reduct. Consider the quasi k-ideal $Q = \{0, a\}$ of S. We show that this can not be expressed as an intersection of a left k-ideal and a right k-ideal. If possible, let L be a left k-ideal and R be a right k-ideal such that $Q = L \cap R$. Then $SQ = \{0, a + b\} \subseteq L$ shows that $a + b \in L$. Again $a \in Q \subseteq L$ shows that $b \in L$, since it is a left k-ideal. Also $QS = \{0, b\} \subseteq R$ shows that $b \in R$. Thus $b \in L \cap R$ but $b \notin Q$. This contradicts that $Q = L \cap R$.

We distinguish the quasi k-ideals which are intersection of a left k-ideal and a right k-ideal as follows:

Definition. A quasi k-ideal Q of S is said to be a strong quasi k-ideal if $Q = L \cap R$ for some left k-ideal L and right k-ideal R of S.

A semiring S is called strong quasi k-simple if it has no nontrivial strong quasi k-ideal.

In semigroups this is known as intersection property [2], but according to the perspective of the results of this article and for the sake of simplicity in expression we would like to call as strong quasi k-ideals.

The following equivalent conditions are direct consequences of the strongness.

Lemma 4. *Let Q be a quasi k-ideal of a semiring S. Then the following conditions are equivalent:*

1. *Q is a strong quasi k-ideal;*
2. *$L_k(a) \cap R_k(a) \subseteq Q$ for all $a \in Q$;*
3. *$L_k(a) \cap R_k(b) \subseteq Q$ for all $a, b \in Q$.*

Proof. (1) $\Rightarrow$ (3): Let $Q = L \cap R$ for some left k-ideal L and right k-ideal R of S and $a, b \in Q$. Then $a \in L$ implies that $L_k(a) \subseteq L$ and $b \in R$ implies that $R_k(b) \subseteq R$. Thus $L_k(a) \cap R_k(b) \subseteq L \cap R = Q$.

(3) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (1): Let $L = \{x \in S | x + sq + q = sq + q; s \in S, q \in Q\}$ and $R = \{x \in S | x + qs + q = qs + q; s \in S, q \in Q\}$. Then L is a left k-ideal and R is a right k-ideal of S. Also $Q \subseteq L \cap R$.

Now if $u \in L \cap R$, then there exist $s_1, s_2 \in S$ and $q_1, q_2 \in Q$ such that $u + s_1q_1 + q_1 = s_1q_1 + q_1$ and $u + q_2s_2 + q_2 = q_2s_2 + q_2$. This implies that $u + sq + q = sq + q$ and $u + qs + q = qs + q$ where $s = s_1 + s_2 \in S$ and $q = q_1 + q_2 \in Q$ by Lemma 2. Thus $u \in L_k(q) \cap R_k(q) \subseteq Q$ which shows that $L \cap R \subseteq Q$. Hence $L \cap R = Q$. ■

Intersection of any family of strong quasi k-ideals of a semiring S is a strong quasi k-ideal. Thus for any $a \in S$, there is the smallest strong quasi k-ideal of S containing a. We call this the strong quasi k-ideal generated by a and denote it by $Q_k(a)$.

Theorem 5. *Let S be a semiring and $a \in S$. Then the strong quasi k-ideal of S generated by a is given by $Q_k(a) = L_k(a) \cap R_k(a)$.*

Proof. Let $a \in S$. Then $L_k(a) \cap R_k(a)$ is a strong quasi k-ideal of S. Now let Q be a strong quasi k-ideal of S such that $a \in Q$. Then $L_k(a) \cap R_k(a) \subseteq Q$, by Lemma 4, which shows that $L_k(a) \cap R_k(a)$ is the smallest quasi k-ideal of S containing a. Thus $Q_k(a) = L_k(a) \cap R_k(a)$. ■

Corollary 6. *Let S be a semiring. Then the following results are equivalent:*

1. *S is strong quasi k-simple;*
2. *$Q_k(a) = Q_k(b)$ for all $a, b \in S$;*
3. *$L_k(a) \cap R_k(a) = S$ for all $a \in S$;*
4. *$L_k(a) \cap R_k(b) = S$ for all $a, b \in S$.*

Proof. It is clear that (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2). So we have to prove (1) $\Rightarrow$ (4) and (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (4): Let S be a strong quasi k-simple semiring and $a, b \in S$. Then $L_k(a) \cap R_k(b)$ is a strong quasi k-ideal of S and hence $L_k(a) \cap R_k(b) = S$ for all $a, b \in S$.

(2) $\Rightarrow$ (1): Let Q be a strong quasi k-ideal of S and $a \in Q$. Then for every $b \in S$, $b \in Q_k(b) = Q_k(a) \subseteq Q$ implies that $Q = S$ and hence S is a strong quasi k-simple semiring. ■

Now for $a, b \in S$, it follows from the Theorem 5 that $a\overline{\mathcal{H}}b$ if and only if $Q_k(a) = Q_k(b)$ which in light of the Corollary 6 can be interpreted as: a semiring S is t-k-simple if and only if it is strong quasi-k-simple. Again recall that in the semigroups t-simplicity is equivalent to the bi-ideal simplicity. Let us see what actually happens here.

Every strong quasi k-ideal is a k-bi-ideal. That the converse is not true follows from the observation that there are k-bi-ideals which are not even quasi. But surprisingly we have the following results.

Lemma 7. *If S is a left k-simple* (resp. *right k-simple*) *semiring then each k-bi-ideal of S is a right k-ideal* (resp. *left k-ideal*).

Proof. Let S be a left k-simple semiring and B be a k-bi-ideal of S. Then $\overline{SB}$ is a left k-ideal, and so $\overline{SB} = S$. Now $BS = \overline{B}\,\overline{SB} \subseteq \overline{BSB} \subseteq \overline{B} = B$ shows that B is a right k-ideal of S. The result for right k-simple semirings follows dually. ■

Proposition 8. *In a semiring S the following conditions are equivalent:*

1. *S is strong quasi k-simple;*
2. *S is t-k-simple;*
3. *S is k-b-simple.*

Proof. (1) $\Rightarrow$ (2): Let S be a strong quasi k-simple semiring and consider $a, b \in S$. Then by Corollary 6, $L_k(a) \cap R_k(b) = S = L_k(b) \cap R_k(a)$. Thus $a \in L_k(b)$, $b \in L_k(a)$, $a \in R_k(b)$ and $b \in R_k(a)$ imply $L_k(a) = L_k(b)$ and $R_k(a) = R_k(b)$. Therefore S is left k-simple as well as right k-simple and hence S is a t-k-simple semiring.

(2) $\Rightarrow$ (3): Let B be a k-bi-ideal of S. Since S is left k-simple, by Lemma 7, it follows that B is a right k-ideal of S. Also S is right k-simple and hence $B = S$. Thus S is a k-b-simple semiring.

(3) $\Rightarrow$ (1): Since each quasi k-ideal is a k-bi-ideal, it follows directly. ■

4. Distributive lattices of t-k-simple semirings

Thus, in the semirings with semilattice additive reduct, the notions of t-k-simplicity, k-b-simplicity and strong quasi k-simplicity are equivalent. So the semirings which are distributive lattices of t-k-simple semirings, are also distributive lattices of k-b-simple as well as of strong quasi k-simple semirings. In [18], Mondal and Bhuniya characterized such semirings by their k-bi-ideals. Above proposition motivates us to characterize the same by their strong quasi k-ideals.

Theorem 9. *The following conditions are equivalent on a semiring S in SL^+ :*

1. *S is a distributive lattice of t-k-simple subsemirings;*
2. *for all $a, b \in S$, $ab, ba \in Q_k(a)$ and $a \in Q_k(a^2)$;*
3. *for all $a \in S$, $Q_k(a)$ is a completely semiprime k-ideal of S;*
4. *every strong quasi k-ideal of S is a completely semiprime k-ideal of S;*
5. *for all $a, b \in S$, $Q_k(ab) = Q_k(a) \cap Q_k(b)$;*
6. *for all $a \in S$, $N(a) = \{x \in S | a \in Q_k(x)\}$;*
7. *$\overline{\mathcal{H}} = \mathcal{N}$ is the least distributive lattice congruence of S such that each of its congruence classes is t-k-simple subsemiring.*

Proof. (1) $\Rightarrow$ (2): Let S be a distributive lattice D of t-k-simple subsemirings S_α, $\alpha \in D$. Consider $a, b \in S$. Then there are $\alpha, \beta \in D$ such that $a \in S_\alpha$, $b \in S_\beta$, and so $aba, ab, ba \in S_\alpha S_\beta \subseteq S_{\alpha\beta}$. Again $S_{\alpha\beta}$ being a t-k-simple semiring, by Corollary 6 and Proposition 8, it follows that $ab \in Q_k(ab) = Q_k(aba) = L_k(aba) \cap R_k(aba) \subseteq L_k(a) \cap R_k(a) = Q_k(a)$. Similarly, $ba \in Q_k(a)$. Also $a, a^2 \in S_\alpha$ implies that $a \in Q_k(a^2)$.

(2) $\Rightarrow$ (3): Let $a \in S$. Consider an element $q \in Q_k(a)$ and $s \in S$. Then sq, $qs \in Q_k(q) \subseteq Q_k(a)$ implies that $Q_k(a)$ is a k-ideal of S. Now let $u^2 \in Q_k(a)$. Then $u \in Q_k(u^2) \subseteq Q_k(a)$ shows that $Q_k(a)$ is a completely semiprime k-ideal of S.

(3) $\Rightarrow$ (4): Follows similarly.

(4) $\Rightarrow$ (5): Let $a, b \in S$. Since $a \in Q_k(a)$ is a k-ideal so $ab \in Q_k(a)$ and similarly $ab \in Q_k(b)$ and so $ab \in Q_k(a) \cap Q_k(b)$ and so $Q_k(ab) \subseteq Q_k(a) \cap Q_k(b)$. Now let $x \in Q_k(a) \cap Q_k(b)$. Then $x \in R_k(a)$ implies that $x + a + as_1 = a + as_1$ for some $s_1 \in S$. Then $x^2 + ax + as_1x = ax + as_1x$ implies that $x^2 + a(x + s_1x) = a(x + s_1x)$. Again $Q_k(a) \cap Q_k(b)$ is a k-ideal of S implies that $s_1x \in Q_k(a) \cap Q_k(b)$ and so $s_1x \in R_k(b)$. Thus $x + xs_1 \in R_k(b)$ which shows that $x + s_1x + bs_2 + b = bs_2 + b$ for some $s_2 \in S$. Then it follows by Lemma 2 that $x^2 + abs_2 + ab = abs_2 + ab$ and hence $x^2 \in R_k(ab)$.

Similarly, $x^2 \in L_k(ab)$ and so $x^2 \in Q_k(ab)$ which yields that $x \in Q_k(ab)$. Thus $Q_k(a) \cap Q_k(b) \subseteq Q_k(ab)$ and hence $Q_k(a) \cap Q_k(b) = Q_k(ab)$.

(5) $\Rightarrow$ (6): Let $F = \{x \in S | a \in Q_k(x)\}$. Consider $x, y \in F$. Then $a \in Q_k(x) \cap Q_k(y)$, implies by Theorem 5 and Lemma 2, that there is $s \in S$ such that $a + sx + x = sx + x, a + xs + x = xs + x$ and $a + sy + y = sy + y, a + ys + y = ys + y$, from which we have $a + (x + y)s + (x + y) = (x + y)s + (x + y)$ and $a + s(x + y) + (x + y) = s(x + y) + (x + y)$, and hence $x + y \in F$. Again $a \in Q_k(x) \cap Q_k(y) = Q_k(xy)$ implies that $xy \in F$. Thus F is a subsemiring of S.

Let $x, y \in S$ be such that $xy \in F$. Then $a \in Q_k(xy) = Q_k(x) \cap Q_k(y)$ implies that $x, y \in F$. Now let $x \in S$ and $y \in F$ be such that $y + x = x$. Then $a \in Q_k(y)$ and so by Theorem 5, there is $s \in S$ such that $a+sy+y = sy+y$ and $a+ys+y = ys+y$ which imply that $a+sx+x = sx+x$ and $a+xs+x = xs+x$ by Lemma 2. Thus $a \in Q_k(x)$ i.e., $x \in F$. Hence F is a filter of S.

Let T be a filter of S containing a and $u \in F$. Then there exists $s \in S$ such that $a+su+u = su+u$ and $a+us+u = us+u$. Then $su+u, us+u \in T$, which implies that $a(su+u) \in T$ i.e., $(as+a)u \in T$. This again shows that $u \in T$. Thus $F \subseteq T$ and so $F = N(a)$.

(6) $\Rightarrow$ (7): For $x, y \in S$, $Q_k(x) = Q_k(y) \Leftrightarrow x \in N(y)$ and $y \in N(x) \Leftrightarrow N(x) = N(y)$ and hence $\overline{\mathcal{H}} = \mathcal{N}$ is the least distributive lattice congruence on S, by Lemma 1.

Now consider an $\overline{\mathcal{H}}$-class H. Since $\overline{\mathcal{H}}$ is a distributive lattice congruence, H is a subsemiring of S. Let $a, b \in S$ be such that $a, b \in H$. Then $a^2 \in H$ implies that $b\overline{\mathcal{H}}a^2$. Thus there is $s \in S$ such that $b + sa^2 + a^2 = sa^2 + a^2$ and $b + a^2s + a^2 = a^2s + a^2$. Since $\overline{\mathcal{H}}$ is a distributive lattice congruence on S, $u_1 = (as+a)\overline{\mathcal{H}}a$ and $u_2 = (sa+a)\overline{\mathcal{H}}a$ which again implies that $u = (u_1+u_2)\overline{\mathcal{H}}a$ i.e., $u \in H$. Now $b+u_2a = u_2a$ and $b+au_1 = au_1$ implies that $b+ua+a = ua+a$ and $b+au+a = au+a$. Similarly, there is $v \in H$ such that $a+vb+b = vb+b$ and $a+bv+b = bv+b$. Thus $Q_k(a) = Q_k(b)$ in H and hence H is a t- k-simple semiring.

(7) $\Rightarrow$ (1): Obvious. ■

In view of this theorem and Proposition 8 the following characterizations of the semirings S which are chain of t-k-simple semirings can be done by the radical of strong quasi k-ideals of S. We omit the proof as it is similar to the above theorem and Theorem 3.3 of [18].

Theorem 10. *The following conditions are equivalent on a semiring S in SL^+:*

1. *S is a chain of t-k-simple subsemirings;*
2. *for all $a, b \in S$, $ab, ba \in Q_k(a)$ and $a \in Q_k(ab)$ or $b \in Q_k(ab)$;*
3. *for all $a \in S$, $Q_k(a)$ is a completely prime k-ideal of S;*
4. *every strong quasi k-ideal of S is a completely prime k-ideal of S;*
5. *for all $a, b \in S$, $Q_k(ab) = Q_k(a) \cap Q_k(b)$ and $Q_k(a) \subseteq Q_k(b)$ or $Q_k(b) \subseteq Q_k(a)$;*
6. *for all $a, b \in S$, $N(a) = \{x \in S | a \in Q_k(x)\}$ and $N(ab) = N(a) \cup N(b)$;*
7. *$\overline{\mathcal{H}} = \mathcal{N}$ is the least chain congruence of S such that each of its congruence classes is t-k-simple subsemiring.*

5. Distributive lattice of t-k-Archimedean semirings

In this section, we characterize the semirings which are distributive lattices of t-k-Archimedean semirings by their strong quasi k-ideals, in fact by the k-radical of their strong quasi k-ideals.

For a non-empty subset A of S, the k-radical $\sqrt{A}$ of A in S is defined by $\sqrt{A} = \{x \in S | (\exists n \in \mathbb{N}) x^n \in \overline{A}\}$. Equivalently, $\sqrt{A} = \{x \in S | (\exists n \in \mathbb{N}) x^n + a = a \text{ for some } a \in A\}$.

In [3], Bhuniya and Mondal defined a semiring S to be k-Archimedean (left, right) if $S = \sqrt{SaS}(\sqrt{Sa}, \sqrt{aS})$ for all $a \in S$. If S is both left and right k-Archimedean then it is called t-k-Archimedean. Equivalently, S is t-k-Archimedean if $S = \sqrt{Sa} \cap \sqrt{aS} = \sqrt{Sa \cap aS}$. Thus S is t-k-Archimedean if and only if for all $a, b \in S$ there are $n \in \mathbb{N}$ and $x \in S$ such that

$$b^n + xa = xa \text{ and } b^n + ax = ax.$$

Equivalently, a semiring S is left (right) k-Archimedean if $S = \sqrt{L_k(a)}(\sqrt{R_k(a)})$ for all $a \in S$. Thus S may be called strong quasi k-Archimedean (k-b-Archimedean) if $S = \sqrt{Q_k(a)}$ $(\sqrt{B_k(a)})$ for all $a \in S$. Now we show that the strong quasi k-Archimedean semirings as well as the k-b-Archimedean semirings are nothing but the t-k-Archimedean semirings.

Recall that for every $a \in S$, $Q_k(a) = L_k(a) \cap R_k(a)$. Though in general neither $L_k(a) = \overline{Sa}$ nor $R_k(a) = \overline{aS}$, still we have:

Lemma 11. *Let S be a semiring. Then $\sqrt{Q_k(a)} = \sqrt{Sa \cap aS}$ for all $a \in S$.*

Proof. Let $b \in \sqrt{Q_k(a)}$. Since $Q_k(a) = L_k(a) \cap R_k(a)$, there are $n \in \mathbb{N}, s \in S$ such that

$$b^n + a + sa = a + sa \text{ and } b^n + a + as = a + as$$

which implies that

$$b^{n+1} + (b + bs)a = (b + bs)a \text{ and } b^{n+1} + a(b + sb) = a(b + sb).$$

Thus $b \in \sqrt{Sa \cap aS}$ and so $\sqrt{Q_k(a)} \subseteq \sqrt{Sa \cap aS}$. Again $\overline{Sa} \subseteq L_k(a)$ and its left-right dual implies that $\sqrt{Sa \cap aS} \subseteq \sqrt{L_k(a) \cap R_k(a)} = \sqrt{Q_k(a)}$. Thus $\sqrt{Q_k(a)} = \sqrt{Sa \cap aS}$ for all $a \in S$. ■

Proposition 12. *Let $S \in SL^+$. Then the following conditions are equivalent:*

1. *S is t-k-Archimedean;*
2. *$b \in \sqrt{B_k(a)}$ for all $a, b \in S$ i.e., $S = \sqrt{B_k(a)}$ for all $a \in S$;*
3. *$b \in \sqrt{Q_k(a)}$ for all $a, b \in S$ i.e., $S = \sqrt{Q_k(a)}$ for all $a \in S$.*

Proof. (1) $\Rightarrow$ (2): Let S be a t-k-Archimedean semiring. Then for all $a, b \in S$, $\sqrt{Sa \cap aS} = S = \sqrt{Sb \cap bS}$. Then $b \in \sqrt{Sa \cap aS}$ implies that there are $n \in \mathbb{N}$ and $x \in S$ such that $b^n + xa = xa$ and $b^n + ax = ax$. Using these two relations we get $b^{2n} + b^n xa = b^n xa \Rightarrow b^{2n} + ax^2a = ax^2a$. Which gives $b^{2n} + a + a^2 + ax^2a = a + a^2 + ax^2a$ and so $b \in \sqrt{B_k(a)}$.

(2) $\Rightarrow$ (3): Follows trivially since $B_k(a) \subseteq Q_k(a)$ for every $a \in S$.

(3) $\Rightarrow$ (1): Let $a, b \in S$. Then $b \in \sqrt{Q_k(a)}$ shows that there are $m \in \mathbb{N}$ and $y \in S$ such that $b^m + a + ya = a + ya$ and $b^m + a + ay = a + ay$. Now $b^{2m} + ab^m + ayb^m = ab^m + ayb^m$ implies that $b^{2m} + a(a + ya) + ay(a + ya) = a(a + ya) + ay(a + ya)$. Using additive idempotent property and rearranging the terms we get $b^{2m} + a(a + ya + y^2a) = a(a + ya + y^2a)$ and $b^{2m} + (a + ay + ay^2)a = (a + ay + ay^2)a$. Thus $b \in \sqrt{Sa \cap aS}$. ■

Thus the notions of t-k-Archimedean semirings, k-b-Archimedean semirings and strong quasi k-Archimedean semirings all are equivelent. In [17] Mondal characterized the semirings which are distributive lattices of t-k-Archimedean semirings by the radicals of their k-bi-ideals. The Proposition 12 shows that such semiring can also be characterized by their strong quasi k-ideals.

Lemma 13. *Let S be a semiring such that for all $a, b \in S$, $ab \in \sqrt{Sa} \cap \sqrt{bS}$. Then*

1. *for all $a, b \in S, a \in \overline{Sb \cap bS}$ implies that $a \in \sqrt{Q_k(b^{2^r})}$ for all $r \in \mathbb{N}$;*
2. *for all $a, b \in S, a \in \sqrt{Q_k(b)}$ implies that $\sqrt{Q_k(a)} \subseteq \sqrt{Q_k(b)}$.*

Proof. (1) Let $a, b \in S$ be such that $a \in \overline{Sb \cap bS}$. Then $a + sb = sb$ and $a + bs = bs$ for some $s \in S$. Now, by hypothesis, there are $n \in \mathbb{N}$ and $t \in S$ such that $(bs)^n + tb = tb$ and $(sb)^n + bt = bt$. Again $a + sb = sb$ and $a + bs = bs$ implies that $a^{n+1} + (sb)^{n+1} = (sb)^{n+1}$ and $a^{n+1} + (bs)^{n+1} = (bs)^{n+1}$. Then we have $a^{n+1} + stb^2 = stb^2$ and $a^{n+1} + b^2ts = b^2ts$ and hence $a \in \sqrt{Q_k(b^2)}$, by Lemma 11. Thus the result is true for $r = 1$. Assume that $k \in \mathbb{N}$ is such that $a \in \sqrt{Q_k(b^{2^k})}$. Then $a^m + ub^{2^k} = ub^{2^k}$ and $a^m + b^{2^k}u = b^{2^k}u$ for some $m \in N$, $u \in S$. Then proceeding as above we get $a \in \sqrt{Q_k(b^{2^{k+1}})}$. Therefore, by the principle of induction, we have $a \in \sqrt{Q_k(b^{2^r})}$, for all $r \in \mathbb{N}$.

(2) Let $a \in \sqrt{Q_k(b)}$. Then there are $n \in \mathbb{N}$ and $s \in S$ such that $a^n + sb = sb$ and $a^n + bs = bs$. Consider $x \in \sqrt{Q_k(a)}$. Then there is $m \in \mathbb{N}$ such that $x^m \in \overline{Sa \cap aS}$. Let $r \in \mathbb{N}$ be such that $2^r > n$. Then by (1) we find $p \in \mathbb{N}$ and $t \in S$ such that $x^p + ta^{2^r} = ta^{2^r}$ and $x^p + a^{2^r}t = a^{2^r}t$ which implies that $x^p + ta^{2^r-n}sb = ta^{2^r-n}sb$ and $x^p + bsa^{2^r-n}t = bsa^{2^r-n}t$. Then $x \in \sqrt{Q_k(b)}$ by Lemma 11. ■

Recall that for $a, b \in S$,

$$a\overline{\mathcal{H}}b \text{ if and only if } Q_k(a) = Q_k(b).$$

Let us define $\sqrt{\overline{\mathcal{H}}}$, the radical of $\overline{\mathcal{H}}$ on S by: for $a, b \in S$,

$$a\sqrt{\overline{\mathcal{H}}}b \text{ if } \sqrt{Q_k(a)} = \sqrt{Q_k(b)}.$$

Now we have the main theorem of this section:

Theorem 14. *The following conditions are equivalent on a semiring S:*

1. *S is a distributive lattice of t-k-Archimedean semirings;*
2. *for all $a, b \in S, b \in \overline{SaS}$ implies $b \in \sqrt{Q_k(a)}$;*
3. *for all $a, b \in S, ab \in \sqrt{Sa} \cap \sqrt{bS}$;*
4. *$\sqrt{Q}$ is a k-ideal of S for every strong quasi k-ideal Q of S;*
5. *$\sqrt{Q_k(a)}$ is a k-ideal of S, for all $a \in S$;*
6. *$N(a) = \{x \in S | a \in \sqrt{Q_k(x)}\}$ for all $a \in S$;*
7. *$\mathcal{N} = \sqrt{\overline{\mathcal{H}}}$ is the least distributive lattice congruence and each of its congruence classes is a t-Archimedean semiring.*

Proof. We prove this theorem in the following scheme: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (1) and (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (3)

(1) $\Rightarrow$ (2): Let γ be a distributive lattice congruence on S such that the γ-classes $T_\alpha : \alpha \in S/\gamma$ are t-k-Archimedean semirings. Let $a, b \in S$ be such that $b \in \overline{SaS}$. Then $b + sas = sas$, for some $s \in S$ by Lemma 2 and hence $b^3 + bsasb = bsasb$ which implies $b^3 + ubu = ubu$, where $u = bs + sb$. Now $uau\gamma au^2\gamma au\gamma ua$ implies that $uau, au, ua \in T_\alpha$ for some $\alpha \in D$. Since T_α is a t-k-Archimedean semiring, there exist $m \in \mathbb{N}$ and $v \in T_\alpha$ such that $(uau)^m + auv = auv$ and $(uau)^m + vua = vua$. Therefore $b^{3m} + (uau)^m = (uau)^m$ implies $b^{3m} + auv = auv$ and $b^{3n} + vua = vua$. Thus $b \in \sqrt{Sa \cap aS} = \sqrt{Q_k(a)}$.

(2) $\Rightarrow$ (3): Let $a, b \in S$. Now $(ab)^2 + abab = abab$ implies that $(ab)^2 \in \overline{SaS} \cap \overline{SbS}$. Then there exist $m, n \in \mathbb{N}$ such that $(ab)^{2m} \in \overline{Sa}$ and $(ab)^{2n} \in \overline{bS}$. Thus $ab \in \sqrt{Sa} \cap \sqrt{bS}$.

(3) $\Rightarrow$ (6): Let $a \in S$ and $F = \{x \in S | a \in \sqrt{Q_k(x)}\}$. Let $y, z \in F$. Then there are $n \in \mathbb{N}$ and $s \in S$ such that $a^n + sy = sy, a^n + sz = sz$ and $a^n + ys = ys, a^n + zs = zs$. Then $a^n + s(y + z) = s(y + z)$ and $a^n + (y + z)s = (y + z)s$ implies that $y + z \in F$. Again, by hypothesis, there are $m \in \mathbb{N}$ and $t \in S$ such that $(ys)^m + ty = ty$ and $(sz)^m + zt = zt$. Then we have $a^{n(m+1)} + (sy)^m sz = (sy)^m sz$ and $a^{n(m+1)} + ys(zs)^m = ys(zs)^m$. Then $a^{n(m+1)} + s(ys)^m z = s(ys)^m z$ and

$a^{n(m+1)} + y(sz)^m s = y(sz)^m s$ implies that $a^{n(m+1)} + styz = styz$ and $a^{n(m+1)} + yzts = yzts$. Thus $yz \in F$ and hence F is a subsemiring of S.

Consider $u \in F$ and $v \in S$ such that $u + v = v$. Now $a \in \sqrt{Q_k(u)}$ implies that there are $n \in \mathbb{N}$ and $s \in S$ such that $a^n + su = su$ and $a^n + us = us$. From these we have $a^n + sv = sv$ and $a^n + vs = vs$ and hence $v \in F$. Now let $x, y \in S$ be such that $xy \in F$. Then there are $m \in \mathbb{N}$ and $s_1 \in S$ such that

$$a^m + s_1xy = s_1xy \tag{1}$$

$$\text{and } a^m + xys_1 = xys_1. \tag{2}$$

Again we have $p \in \mathbb{N}$ and $s_2 \in S$ such that $(xy)^p + s_2x = s_2x$ and $(xy)^p + ys_2 = ys_2$, by (3). Since F is a subsemiring, $(xy)^p \in F$; and so there are $q \in \mathbb{N}$ and $s_3 \in S$ such that $a^q + s_3(xy)^p = s_3(xy)^p$ and $a^q + (xy)^p s_3 = (xy)^p s_3$. Then we have

$$a^q + s_3s_2u = s_3s_2u \tag{3}$$

$$\text{and } a^q + us_2s_3 = us_2s_3. \tag{4}$$

Now (1) and (4) together implies that $y \in F$ and (2) and (3) together implies that $x \in F$. Thus F is a filter and $a \in F$.

Suppose T is a filter $a \in T$. Consider $z \in F$. Then there exist $n \in \mathbb{N}$ and $s \in S$ such that $a^n + sz = sz$ and $a^n + zs = zs$. Since T is a filter, $a^n \in T$ and so $a^n + sz = sz$ and $a^n + zs = zs$ which implies that $z \in T$. Thus F is the smallest filter that contains a i.e., $F = N(a)$.

(6) $\Rightarrow$ (7): Consider $a, b \in S$. Since $N(ab)$ is a filter and contains ab, we have $a, b \in N(ab)$. Then $ab \in \sqrt{Q_k(a)} \cap \sqrt{Q_k(b)} \subseteq \sqrt{Sa} \cap \sqrt{bS}$. Now for $a, b \in S$, the following holds.

$$\begin{aligned}
a\mathcal{N}b &\Leftrightarrow N(a) = N(b) \\
&\Leftrightarrow a \in N(b) \text{ and } b \in N(a) \\
&\Leftrightarrow b \in \sqrt{Q_k(a)} \text{ and } a \in \sqrt{Q_k(b)}, \text{ by (6)} \\
&\Leftrightarrow \sqrt{Q_k(b)} \subseteq \sqrt{Q_k(a)} \text{ and } \sqrt{Q_k(a)} \subseteq \sqrt{Q_k(b)}, \text{ by (1) of Lemma 13} \\
&\Leftrightarrow a\sqrt{\overline{\mathcal{H}}}b
\end{aligned}$$

which shows that $\mathcal{N} = \sqrt{\overline{\mathcal{H}}}$ is the least distributive lattice congruence.

Let T be an $\mathcal{N}$-class in S. Since $\mathcal{N}$ is a distributive lattice congruence, T is a subsemiring. Consider $a, b \in T$. Then $a\mathcal{N}b$ implies that $N(a) = N(b)$, and by (6) we have $b \in \sqrt{Q_k(a)}$. Thus there are $n \in \mathbb{N}$ and $s \in S$ such that $b^n + a + sa = a + sa$ and $b^n + a + as = a + as$ which implies that $b^{n+1} + (b + bs)a = (b + bs)a$ and $b^{n+1} + a(b + sb) = a(b + sb)$. Now since $\mathcal{N}$ is a distributive lattice congruence

$(b+bs)\mathcal{N}b\mathcal{N}(b+sb)$ which implies that $t_1 = b+bs \in T$ and $t_2 = b+sb \in T$. Thus $b \in \sqrt{Ta} \cap \sqrt{aT}$ and hence T is a t-k-Archimedean semiring.

(7) $\Rightarrow$ (1): Follows directly.

(3) $\Rightarrow$ (4): Let Q be a strong quasi k-ideal of S. Consider $u \in \sqrt{Q}$ and $c \in S$. Then there are $n \in \mathbb{N}, q \in Q$ such that $u^n = q$. Now by (3) there exist $x, y \in S, n_1, m_1 \in \mathbb{N}$ such that $(uc)^{n_1} = xu$ and $(ux)^{m_1} = yu$. Then $(uc)^{n_1(m_1+1)} = x(ux)_1^m u = x(yu)u = xyu^2$ implies $(uc)_2^n = x_1u^2$ where $n_2 = n_1(m_1+1)$ and $x_1 = xy$. Also, there exist $s \in S, m_2 \in \mathbb{N}$ such that $(u^2x_1)m_2 = su^2$. Proceeding as above we find by iteration that every $r \in \mathbb{N}$, there exists $p \in \mathbb{N}$ such that $(uc)^p = x_r u^{2^r}$. Let $r \in \mathbb{N}$ be such that $2^r > n$. Then there exists $q \in \mathbb{N}$ such that $(uc)^q = x_r u^{2^r}$. By (3), there exist $m \in \mathbb{N}$ and $z \in S$ such that $(x_r u^{2^r})^{(m+1)} = (x_r u^{2^r})^{(m+1)} = u^{2^r} z x_r u^{2^r} = q u^{2^r - n} z x_r u^{2^r - n} q$. Hence $uc \in \sqrt{Q}$. Similarly, $cu \in \sqrt{Q}$. Hence $\sqrt{Q}$ is a k-ideal of S.

(4) $\Rightarrow$ (5): Trivial.

(5) $\Rightarrow$ (3): Let $a, b \in S$. Then $\sqrt{Q_k(a)}$ and $\sqrt{Q_k(b)}$ are k-ideals of S. Then $ab \in \sqrt{Q_k(a)} \cap \sqrt{Q_k(b)}$ and hence $ab \in \sqrt{Sa} \cap \sqrt{bS}$. ■

Theorem 15. *The following conditions on a semiring S are equivalent:*

1. *S is a chain of t-k-Archimedean semirings;*
2. *S is a distributive lattice of t-k-Archimedean semirings and for all $a, b \in S$,*
$$b \in \sqrt{Q_k(a)} \quad \text{or} \quad a \in \sqrt{Q_k(b)};$$
3. *For all $a, b \in S$, $N(a) = \{x \in S | a \in \sqrt{Q_k(x)}\}$ and $N(ab) = N(a) \cup N(b)$;*
4. *$\mathcal{N} = \sqrt{\overline{\mathcal{H}}}$ is the least chain congruence of S such that each of its congruence classes is a t-k-Archimedean semiring.*

Proof. (1) $\Rightarrow$ (2): Let S be a chain $\mathcal{C}$ of t-k-Archimedean semirings $S_\alpha (\alpha \in \mathcal{C})$. Then S is a distributive lattice of t-k-Archimedean semirings. Let $a, b \in S$. Then $a \in S_\alpha$ and $a \in S_\beta$ for some $\alpha, \beta \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, either $\alpha\beta = \alpha$ or $\alpha\beta = \beta$. If $\alpha\beta = \alpha$ then $a, ab \in S_\alpha$; and since S_α is a t-k-Archimedean semiring, there exist $n \in \mathbb{N}$ and $x \in S_\alpha$ such that $a^n + xab = xab$ and $a^n + abx = abx$. Now, by Theorem 14, as S is a distributive lattice of t-k-Archimedean semiring, there are $m \in \mathbb{N}$ and $s \in S$ such that $(abx)^m + bxs = bxs$. Then we have $a^{nm} + s_1 b = s_1 b$ and $a^{nm} + bxs = bxs$ for some $s_1 \in S$ and hence $a \in \sqrt{Q_k(b)}$. Again if $\alpha\beta = \beta$, then $b, ab \in S_\beta$ and similarly as above we have $b \in \sqrt{Q_k(a)}$.

(2) $\Rightarrow$ (3): Since S is a distributive lattice of t-k-Archimedean semirings, we have $N(a) = \{x \in S | a \in \sqrt{Q_k(x)}\}$, by Theorem 14. Let $a, b \in S$. Then $ab \in N(ab)$ implies that $a \in N(ab)$ and $b \in N(ab)$, and $N(a) \cup N(b) \subseteq N(ab)$.

Again, either $a \in \sqrt{Q_k(b)}$ or $b \in \sqrt{Q_k(a)}$. If $a \in \sqrt{Q_k(b)}$, then there are $n \in \mathbb{N}$ and $s \in S$ such that $a^n + bs = bs$ and so $a^{n+1} + abs = abs$. Since S is a distributive lattice of t-k-Archimedean semirings, there exist $m \in \mathbb{N}$ and $t \in S$ such that $(abs)^m + tab = tab$, by Theorem 14. Then we have $a^{(n+1)m} + tab = tab$ and $a^{(n+1)m} + abt_1 = abt_1$ for some $t_1 \in S$. Then $a \in \sqrt{Q_k(ab)}$ which implies that $ab \in N(a)$. Thus $N(ab) \subseteq N(a)$. If $b \in \sqrt{Q_k(a)}$, then similarly we have $N(ab) \subseteq N(b)$, which shows that $N(ab) \subseteq N(a) \cup N(b)$. Hence $N(ab) = N(a) \cup N(b)$.

(3) $\Rightarrow$ (4): It follows by Theorem 14 that $\mathcal{N} = \sqrt{\overline{\mathcal{H}}}$ is the least distributive lattice congruence on S and each $\mathcal{N}$-class is a t-k-Archimedean semiring.

Now consider $a, b \in S$. Then $ab \in N(a) \cup N(b)$ shows that $ab \in N(a)$ or $ab \in N(b)$. Again $N(a) \subseteq N(ab)$ and $N(b) \subseteq N(ab)$. Thus either $N(ab) \subseteq N(a) \subseteq N(ab)$ or $N(ab) \subseteq N(b) \subseteq N(ab)$, i.e., either $a\mathcal{N}ab$ or $b\mathcal{N}ab$. Hence $\mathcal{N}$ is a chain congruence on S. Since every chain is a distributive lattice and $\mathcal{N}$ is the least distributive lattice congruence, it is the least chain congruence on S.

(4) $\Rightarrow$ (1): Trivial. ■

Theorem 16. *Let S be a semiring. Then the following conditions on S are equivalent:*

1. *S is a chain of t-k-Archimedean semirings;*
2. *$\sqrt{Q}$ is a completely prime k-ideal of S, for every strong quasi k-ideal Q of S;*
3. *$\sqrt{Q_k(a)}$ is a completely prime k-ideal of S, for every $a \in S$;*
4. *$\sqrt{Q_k(ab)} = \sqrt{Q_k(a)} \cap \sqrt{Q_k(b)}$ for all $a, b \in S$ and every strong quasi k-ideal of S is semiprimary.*

Proof. (1) $\Rightarrow$ (2): Let S be a chain $\mathcal{C}$ of t-k-Archimedean semirings $\{S_\alpha | \alpha \in \mathcal{C}\}$. Consider a strong quasi k-ideal Q of S. Then $\sqrt{Q}$ is a k-ideal of S, by Theorem 14. Let $x, y \in S$ be such that $xy \in \sqrt{Q}$. Then there is $n \in \mathbb{N}$ such that $(xy)^n = u =\in \overline{Q} = Q$. Suppose $x \in S_\alpha$ and $y \in S_\beta$, $\alpha, \beta \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, either $\alpha\beta = \alpha$ or $\alpha\beta = \beta$. If $\alpha\beta = \alpha$, then $x, u \in S_\alpha$ shows that $x \in \sqrt{Q_k(u)} \subseteq \sqrt{Q}$. Similarly, if $\alpha\beta = \beta$, then $y \in \sqrt{Q}$. Hence $\sqrt{Q}$ is a completely prime k-ideal of S.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (4): Let $a, b \in S$. Then, $\sqrt{Q_k(a)}, \sqrt{Q_k(b)}$ and $\sqrt{Q_k(ab)}$ are completely prime k-ideals of S. Consider $x \in \sqrt{Q_k(ab)}$. Then there exist $n \in \mathbb{N}$ and $s \in S$ such that $x^n + abs = abs$ and $x^n + sab = sab$. Again, by Theorem 14, there are $m \in \mathbb{N}$ and $t \in S$ such that $(abs)^m + ta = ta$. Then we have $x^{nm} + ta = ta$. Also $x^{nm} + as_1 = as_1$ for some $s_1 \in S$. Then $x \in \sqrt{Q_k(a)}$. Thus $\sqrt{Q_k(ab)} \subseteq \sqrt{Q_k(a)}$. Similarly, considering $x^n + sab = sab$ we have $\sqrt{Q_k(ab)} \subseteq \sqrt{Q_k(b)}$. Therefore

$\sqrt{Q_k(ab)} \subseteq \sqrt{Q_k(a)} \cap \sqrt{Q_k(b)}$. Now consider $y \in \sqrt{Q_k(a)} \cap \sqrt{Q_k(b)}$. Then there exist $p \in \mathbb{N}$, $u \in S$ such that $y^p + ua = ua, y^p + au = au$ and $y^p + ub = ub, y^p + bu = bu$. Then we have $y^{2p} + uabu = uabu$. Also there exist $q_1, q_2 \in \mathbb{N}$ and $v_1, v_2 \in S$ such that $(uabu)^{q_1} + abuv_1 = abuv_1$ and $(uabu)^{q_2} + v_2uab = v_2uab$. Then we get $y^{2pq_1} + abuv_1 = abuv_1$ and $y^{2pq_2} + v_2uab = v_2uab$ and so $y \in \sqrt{Q_k(ab)}$. Hence $\sqrt{Q_k(ab)} = \sqrt{Q_k(a)} \cap \sqrt{Q_k(b)}$.

Now let Q be a strong quasi k-ideal of S. Let $a, b \in S$ be such that $ab \in Q$. Then $ab \in \sqrt{Q_k(ab)}$ implies that $a \in \sqrt{Q_k(ab)}$ or $b \in \sqrt{Q_k(ab)}$, since $\sqrt{Q_k(ab)}$ is completely prime. Then $a^n \in \overline{Q_k(ab)} = Q_k(ab) \subseteq Q$ or $b^n \in \overline{Q_k(ab)} = Q_k(ab) \subseteq Q$ for some $n \in \mathbb{N}$. Hence Q is semiprimary.

(4) $\Rightarrow$ (1): Consider $a, b \in S$. Then $ab \in \sqrt{Q_k(ab)} = \sqrt{Q_k(a)} \cap \sqrt{Q_k(b)} \subseteq \sqrt{Sa} \cap \sqrt{bS}$ shows that S is a distributive lattice t-k-Archimedean semirings. Again since $\sqrt{Q_k(ab)}$ is completely prime, $a \in \sqrt{Q_k(ab)}$ or $b \in \sqrt{Q_k(ab)}$ which implies that $a \in \sqrt{Q_k(b)}$ or $b \in \sqrt{Q_k(a)}$. Thus S is a chain of t-k-Archimedean semirings by Theorem 15 (2). ■

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