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CONGRUENCES AND BOOLEAN FILTERS OF QUASI-MODULAR p-ALGEBRAS

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Abstract

The concept of Boolean filters in p-algebras is introduced. Some properties of Boolean filters are studied. It is proved that the class of all Boolean filters $BF(L)$ of a quasi-modular p-algebra L is a bounded distributive lattice. The Glivenko congruence Φ on a p-algebra L is defined by $(x, y) \in \Phi$ iff $x^{**} = y^{**}$. Boolean filters $[F_a), a \in B(L)$, generated by the Glivenko congruence classes F_a (where F_a is the congruence class $[a]\Phi$) are described in a quasi-modular p-algebra L. We observe that the set $F_B(L) = \{[F_a) : a \in B(L)\}$ is a Boolean algebra on its own. A one-one correspondence between the Boolean filters of a quasi-modular p-algebra L and the congruences in $[\Phi, \nabla]$ is established. Also some properties of congruences induced by the Boolean filters $[F_a), a \in B(L)$ are derived. Finally, we consider some properties of congruences with respect to the direct products of Boolean filters.

Keywords: p-algebras, quasi-modular p-algebras, Boolean filters, direct products, congruences.

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1. Introduction

The notion of pseudo-complements was introduced in semi-lattices and distributive lattices by O. Frink [7] and G. Birkhof [3]. The pseudo-complements in Stone algebras were studied and discussed by R. Balbes [1], O. Frink [7] and G. Grätzer [4] etc. Recently, the concept of Boolean filter of pseudo-complemented distributive lattices was introduced by M. Sambasiva Rao and K.P. Shum in [9].

In this paper, we further study the Boolean filters in a p-algebra L and many properties of Boolean filters are also given. We observe that every maximal filter of L is a Boolean filter, however the converse of this statement is not true. It is observed that a filter F of a p-algebra L is a prime Boolean filter if and only if it is a maximal filter. We will give a characterization theorem of Boolean filters of a quasi-modular p-algebra L. We also notice that the set of all Boolean filters of a quasi-modular p-algebra forms a bounded distributive lattice. Then, we introduce a Boolean filter $[F_a)$ which is generated by the congruence class F_a (notice that $[x]\Phi$ is denoted by F_a) of the Glivenko congruence relation Φ on L, where a is a closed element of a quasi-modular p-algebra L. It is proved that the set $F_B(L) = \{[F_a) : a \in B(L)\}$ forms a Boolean algebra on its own. We also observe that $F_B(L)$ is isomorphic to $B(L)$. The relationship between the Boolean filters and the congruences in $[\Phi, \nabla]$ of a quasi-modular p-algebra L is introduced. It is proved that there is a one-one correspondence between the congruences in $[\Phi, \nabla]$ and the Boolean filters of L. We also prove that the Boolean algebras $F_B(L)$ and $Con_B(L) = \{\theta_{[F_a)} : a \in B(L)\}$ are isomorphic, where $\theta_{[F_a)}$ is the congruence on L induced by a Boolean filter F_a for a closed element a of L. Moreover, we show that the Boolean algebra $Con_B(L)$ can be embedded into the interval $[\Phi, \nabla]$ of $Con(L)$. It is proved that the lattice of all Boolean filters of a finite quasi-modular p-algebra L is isomorphic to the sublattice $[\Phi, \nabla]$ of $Con(L)$. Finally, some properties of congruences with respect to the direct products of Boolean filters are explored and investigated.

Several results of [9] are still possible for p-algebras or quasi-modular p-algebras. Namely, Lemma 3.2, Lemma 3.3, Theorem 3.4, Theorem 3.6 and Theorem 6.1 correspond respectively to Proposition 2.3, Corollary 2.4, Theorem 2.6, Theorem 2.8 and Theorem 2.10 from [9].

2. Preliminaries

In this section, we cite some known definitions and basic results which can be found in the papers [2, 5, 6, 7, 8] and [10].

A p-algebra is a universal algebra $(L, \vee, \wedge, ^*, 0, 1)$, where $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and the unary operation * is defined by $x \wedge a = 0 \Leftrightarrow x \leq a^*$.

It is well known that the class of all p-algebras is equational. We now call a p-algebra L distributive (modular) if the lattice $(L, \vee, \wedge, 0, 1)$ is distributive (modular). The variety of modular p-algebras contains the variety of distributive p-algebras. We call a p-algebra quasi-modular if $((x \wedge y) \vee z^{**}) \wedge x = (x \wedge y) \vee (z^{**} \wedge x)$. Clearly, the class of all modular p-algebras is a subclass of the class of quasi-modular p-algebras. If the Stone identity $x^* \vee x^{**} = 1$ holds in a p-algebra, then we simply call this p-algebra an S-algebra. We usually call a distributive S-algebra a Stone algebra.

An element a of a p-algebra L is called closed if $a^{**} = a$. Then $B(L) = \{a \in L : a = a^{**}\}$ is the set of all closed elements of L. It is known that $(B(L), \bigtriangledown, \wedge, 0, 1)$, where $a \bigtriangledown b = (a^* \wedge b^*)^*$, forms a Boolean algebra. The set $D(L) = \{x \in L : x^* = 0\} = \{x \vee x^* : x \in L\}$ of all dense elements of L is a filter of L. If L is an S-algebra, then $(x \wedge y)^* = x^* \vee y^*$ for all $x, y \in L$. It follows that $a \bigtriangledown b = a \vee b$ for all $a, b \in B(L)$.

For an arbitrary lattice L, the set $F(L)$ of all filters of L ordered by the set inclusion forms a lattice. It is known that $F(L)$ is modular (distributive) if and only if L is a modular (distributive) lattice. Let $a \in L$ and $[a)$ be the principal filter of L generated by a : $[a) = \{x \in L : x \geq a\}$. A proper filter P of L is called prime if $x \vee y \in P$ implies $x \in P$ or $y \in P$ for all $x, y \in L$. We call a proper filter M of L maximal if $M \subseteq G$ for no proper filter G.

The following results on quasi-modular p-algebras may be found in [8].

Let L be a quasi-modular p-algebra. Then every element $x \in L$ can be represented by $x = x^{**} \wedge (x \vee x^*)$, where $x^{**} \in B(L)$ and $x \vee x^* \in D(L)$. The relation Φ of a quasi-modular p-algebra L is defined by $(x, y) \in \Phi \Leftrightarrow x^{**} = y^{**}$ and is called the Glivenko congruence relation. It is known that the Glivenko congruence is indeed a congruence on L such that $L/\Phi \cong B(L)$ holds. Every congruence class of Φ contains exactly one element of $B(L)$ which is the greatest element in the congruence class, the greatest element of a congruence class $[x]\Phi$ is x^{**}. Hence Φ partitions L into $\{F_a : a \in B(L)\}$, where $F_a = \{x \in L : x^{**} = a\} = [a]\Phi$. It is clear that $F_0 = \{0\}$ and $F_1 = D(L)$.

We frequently use the following rules in the computations of p-algebras (see [5, 10]):

(1) $0^{**} = 0$ and $1^{**} = 1$;

(2) $a \wedge a^* = 0$;

(3) $a \leq b$ implies $b^* \leq a^*$;

(4) $a \leq a^{**}$;

(5) $a^{***} = a^*$;

(6) $(a \vee b)^* = a^* \wedge b^*$;

(7) $(a \wedge b)^* \geq a^* \vee b^*$;

(8) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$;

(9) $(a \vee b)^{**} = (a^* \wedge b^*)^* = (a^{**} \vee b^{**})^{**}$.

3. Boolean Filters of p-algebras

In this section, we introduce the concept of Boolean filter of a p-algebra. Some properties of Boolean filters in a p-algebra are derived. We show that the maximal filter and prime Boolean filter of a p-algebra are equivalent. A characterization theorem of Boolean filters of a quasi-modular p-algebra will be given. Also we will prove that the set of all Boolean filters of a quasi-modular p-algebra is a bounded distributive lattice.

Definition 3.1. Let L be a p-algebra. Then, we call a filter F of L a *Boolean filter* if $x \vee x^* \in F$ for each $x \in L$.

We now give some examples of Boolean filters of a p-algebra L.

Example 3.2. (1) Let L be a p-algebra. Then the filter $D(L)$ is a Boolean filter of L as $x \vee x^* \in D(L)$ for all $x \in L$. Moreover $D(L)$ is the smallest Boolean filter of L and L is the greatest Boolean filter of L;

(2) Let B be a Boolean algebra. Then any filter F of B is a principal Boolean filter as $x \vee x^* = 1 \in F$ for each $x \in B$;

(3) Let $C_4 = \{0, a, b, c : 0 < a, b < c\}$ be a four element Boolean lattice and a pentagon $N_5 = \{u, x, y, z, 1 : u < x < y < 1, u < z < 1, x \wedge z = y \wedge z = u, x \vee z = y \vee z = 1\}$.Clearly $L = C_4 \bigoplus N_5$ is a quasi-modular p-algebra where $\bigoplus$ stands for ordinal sum. Then the set of all Boolean filters of L is $\{\{c, u, x, y, z, 1\}, \{a, c, u, x, y, z, 1\}, \{b, c, x, y, z, 1\}, L\}$;

We observe that the filters $\{x, y, 1\}, \{y, 1\}, \{z, 1\}, \{u, x, y, z, 1\}$ and $\{1\}$ are not Boolean filters.

The results in Corollary 2.4, Theorem 2.6 and Theorem 2.8 from [9] are already stated for the class of all bounded distributive pseudocomplemented lattices. Now we have the following Lemma

Lemma 3.3. *Every maximal filter of a p-algebra L is a Boolean filter.*

Proof. Let M be a maximal filter of L. Suppose that $x \vee x^* \notin M$ for some $x \in L$. Then $M \vee [x \vee x^*) = L$. Hence $a \wedge b = 0$ for some $a \in M, b \in [x \vee x^*)$. Now we have the following implications:

$$\begin{aligned} a \wedge b = 0 \quad &\Rightarrow \quad 0 = a \wedge b \geq a \wedge (x \vee x^*) \geq (a \wedge x) \vee (a \wedge x^*) \\ &\Rightarrow \quad a \wedge x = 0 \text{ and } a \wedge x^* = 0 \\ &\Rightarrow \quad a \leq x^* \text{ and } a \leq x^{**} \\ &\Rightarrow \quad a \leq x^* \wedge x^{**} = 0 \end{aligned}$$

This result leads to $0 = a \in M$ which is a contradiction. Hence $x \vee x^* \in M$ for all $x \in L$. Therefore, M is a Boolean filter of L. ■

We note that it is not true that every Boolean filter of L is a maximal filter. For, in Example 3.1(3), the filter $\{c, u, x, y, z, 1\}$ of L is a Boolean filter but not a maximal filter of L.

The proof of Corollary 2.4 of [9] is still appropriate for the following Lemma.

Lemma 3.4. *A proper filter of a p-algebra L which contains either x or x^* for all $x \in L$ is a Boolean filter.*

Now, we characterize the maximal filters of a p-algebra.

Theorem 3.5. *Let F be a proper filter of a p-algebra L. Then the following conditions are equivalent*

(1) *F is a maximal of L,*

(2) *$x \notin F$ implies $x^* \in F$ for all $x \in L$,*

(3) *F is prime Boolean.*

Proof. The most proof of Theorem 2.6 in [9] is still appropriate for this Theorem, we need only to prove that F is a prime filter of L without using distributivity in (2) $\Rightarrow$ (3). Suppose that F is not prime. Let $x \vee y \in F$ such that $x \notin F$ and $y \notin F$. By condition (2), we immediately see that $x^* \in F$ and $y^* \in F$. Hence $(x \vee y)^* = x^* \wedge y^* \in F$. Therefore $0 = (x \vee y) \wedge (x \vee y)^* \in F$, a contradiction (as F is a proper filter of L). This shows that F is prime. ∎

By Definition of Boolean filter, the following lemma is obvious.

Lemma 3.6. *Let L be a p-algebra. Then the following statements hold.*

(1) *Any filter of L containing a Boolean filter is a Boolean filter,*

(2) *The class $BF(L)$ of all Boolean filters of L is a $\{1\}$-sublattice of the lattice $F(L)$.*

We now characterize the Boolean filters of a quasi-modular p-algebra L.

Theorem 3.7. *Let F be a proper filter of a quasi-modular p-algebra L. Then the following conditions are equivalent.*

(1) *F is a Boolean filter;*

(2) *$x^{**} \in F$ implies $x \in F$;*

(3) *For $x, y \in L, x^* = y^*$ and $x \in F$ imply $y \in F$.*

Proof. We prove only that (1) $\Rightarrow$ (2) without using distributivity. Assume that F is a Boolean filter of L. Suppose that $x^{**} \in F$. Since F is a Boolean filter, we have $x \vee x^* \in F$ and so $x^{**} \wedge (x \vee x^*) \in F$. Since L is a quasi-modular p-algebra, it follows that $x = x^{**} \wedge (x \vee x^*) \in F$ and condition (2) hold.

The proofs (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are given in Theorem 2.8 of [9]. ∎

Theorem 3.8. *The class of Boolean filters $BF(L)$of a quasi-modular p-algebra L forms a bounded distributive lattice on its own.*

Proof. Clearly $(BF(L), \vee, \wedge, D(L), L)$ is a bounded lattice. For any $F, H, G \in BF(L)$, we have $(F \cap G) \vee (H \cap G) \subseteq (F \vee H) \cap G$. Then we have to prove that $(F \vee H) \cap G \subseteq (F \cap G) \vee (H \cap G)$. Let $x \in (F \vee H) \cap G$. Then by the distributivity of $B(L)$ and the fact that $(F \cap G) \vee (H \cap G)$ is a Boolean filter we deduce the following implications.

$$\begin{aligned}
x \in (F \vee H) \cap G \quad &\Rightarrow \quad x \in F \vee H \text{ and } x \in G \\
&\Rightarrow \quad x \geq f \wedge h \text{ for some } f \in F, h \in H \\
&\Rightarrow \quad x^{**} \geq (f \wedge h)^{**} = f^{**} \wedge h^{**} \\
&\Rightarrow \quad x^{**} = x^{**} \bigtriangledown (f^{**} \wedge h^{**}) \\
&\Rightarrow \quad x^{**} = (x^{**} \bigtriangledown f^{**}) \wedge (x^{**} \bigtriangledown h^{**}) \\
&\Rightarrow \quad x^{**} = (x \vee f)^{**} \wedge (x \vee h)^{**} \in (F \cap G) \vee (H \cap G) \\
&\Rightarrow \quad x \in (F \cap G) \vee (H \cap G) \text{ by Theorem 3.6 (2).}
\end{aligned}$$

Notice that $(x \vee f)^{**} \geq (x \vee f) \in F \cap G$ and $(x \vee h)^{**} \geq (x \vee h) \in H \cap G$. It follows that $(BF(L), \vee, \wedge, D(L), L)$ is a bounded distributive lattice. ■

4. Boolean filters via Glivenko congruence classes

In this section, we will show that, for every closed element a of a quasi-modular p-algebra L, the congruence class $F_a = [a]\Phi$ of the Glivenko congruence relation Φ on L generates a Boolean filter $[F_a)$. Many properties of the Boolean filters $[F_a)$ for all $a \in B(L)$ are discovered. Also, we derive that the set $F_B(L) = \{[F_a) : a \in B(L)\}$ forms a Boolean algebra. It is proved that $F_B(L)$ is isomorphic to $B(L)$. Also we express a Boolean filter as a union of certain elements of $F_B(L)$.

Theorem 4.1. *Let L be a quasi-modular p-algebra. Then for any two closed elements a, b of L, the following statements hold.*

(1) $[F_a) = \{x \in L : x^{**} \geq a\} = [a) \vee D(L)$,
(2) $[F_a)$ *is a Boolean filter of* L,
(3) $a \leq b$ *in* $B(L)$ *if and only if* $[F_b) \subseteq [F_a)$ *in* $F_B(L)$,
(4) *The set* $F_B(L)$ *forms a Boolean algebra on its own.*
Moreover, $B(L) \cong F_B(L)$,
(5) $[F_{a \wedge b}) = [F_a) \vee [F_b)$,
(6) $[F_{a \bigtriangledown b}) = [F_a) \cap [F_b)$,
(7) $[F_{a \vee b}) = [F_a) \cap [F_b)$ *whenever* L *is a quasi-modular* S*-algebra.*

Proof. (1) $x \in [F_a)$ if and only if there exists a positive integer n and $f_1, f_2, \ldots,$ $f_n \in F_a$ such that $x \geq f_1 \wedge f_2 \wedge \ldots \wedge f_n$. Then $f_i^{**} = a, i = 1, \ldots, n$. Let $H = \{x \in L : x^{**} \geq a\}$. Clearly H is a filter of L. Firstly we verify that $[F_a) = H$. Let $x \in [F_a)$. Now we have the following implications.

$$\begin{aligned} x \in [F_a) &\Rightarrow x \geq f_1 \wedge f_2 \wedge \ldots \wedge f_n \text{ for some } f_1, \ldots, f_n \in F_a \\ &\Rightarrow x^{**} \geq (f_1 \wedge f_2 \wedge \ldots \wedge f_n)^{**} \\ &\Rightarrow x^{**} \geq f_1^{**} \wedge f_2^{**} \wedge \ldots \wedge f_n^{**} \\ &\Rightarrow x^{**} \geq a \text{ as } f_i^{**} = a \\ &\Rightarrow x \in H. \end{aligned}$$

Then $[F_a) \subseteq H$. Conversely, suppose that $H \nsubseteq [F_a)$. Then there exists $y \in [F_a)$ with $y \notin H$. Hence $y \geq f_1 \wedge f_2 \wedge \ldots f_n$ for some $f_1, f_2, \ldots, f_n \in F_a$. It follows that $y^{**} \geq (f_1 \wedge f_2 \wedge \ldots \wedge f_n)^{**} = f_1^{**} \wedge f_2^{**} \wedge \ldots \wedge f_n^{**} = a$ as $f_i^{**} = a$. Then $y^{**} \geq a$, which is a contradiction. Consequently $H \subseteq [F_a)$. Therefore $[F_a) = \{x \in L : x^{**} \geq a\}$.

Now we prove that $[F_a) = [a) \vee D(L)$. Let $x \in [F_a)$. Then we have the following implications.

$$\begin{aligned} x \in [F_a) &\Rightarrow x^{**} \geq a \\ &\Rightarrow x = x^{**} \wedge (x \vee x^*) \geq a \wedge (x \vee x^*) \\ &\Rightarrow x \in [a) \vee D(L) \text{ as } x \vee x^* \in D(L). \end{aligned}$$

Then $[F_a) \subseteq [a) \vee D(L)$. Conversely, let $y \in [a) \vee D(L)$. Then $y \geq a \wedge z$ for some $z \in D(L)$. It follows that $y^{**} \geq (a \wedge z)^{**} = a^{**} \wedge z^{**} = a \wedge 1 = a$ as $a^{**} = a$ and z is a dense element of L. Therefore $y \in [F_a)$ and $[a) \vee D(L) \subseteq [F_a)$.

(2) By (1) above, $D(L) \subseteq [F_a)$ for all $a \in B(L)$. Now, by Lemma 3.6(1), $[F_a)$ is a Boolean filter of L.

(3) Let $a \leq b$ in $B(L)$. Let $x \in [F_b)$. Then $x^{**} \geq b \geq a$. Hence $x \in [F_a)$ and $[F_b) \subseteq [F_a)$ Conversely, suppose that $[F_b) \subseteq [F_a)$. Since $b \in F_b \subseteq [F_b) \subseteq [F_a)$, then $b = b^{**} \geq a$.

(4) Define the mapping $g : B(L) \to F_B(L)$ by $g(a) = [F_a)$. It follows easily from (3) above that g is an order anti-isomorphism between $B(L)$ and $F_B(L)$. Then $F_B(L)$ is a Boolean algebra and g is a Boolean anti-isomorphism. It follows that the mapping $f : B(L) \to F_B(L)$ defined by $f(a) = [F_{a^*})$ is a Boolean isomorphism. Therefore $B(L) \cong F_B(L)$.

(5), (6) Since $g : B(L) \to F_B(L)$ defined by $g(a) = [F_a)$ is an anti-isomorphism by(4) above between Boolean algebras $B = (B, \triangledown, \wedge, ^*, 0, 1)$ and $F_B(L) = (F_B(L), \vee, \cap, ^-, D(L), L)$, where $\overline{[F_a)} = [F_{a^*})$, we get

$$\begin{aligned}[F_{a\wedge b}) &= g(a\wedge b)\\ &= g(a)\vee g(b)\\ &= [F_a)\vee[F_b)\end{aligned}$$

and

$$\begin{aligned}[F_{a\bigtriangledown b}) &= g(a\bigtriangledown b)\\ &= g(a)\cap g(b)\\ &= [F_a)\cap[F_b)\end{aligned}$$

(7) If L is a quasi-modular S-algebra, then $(x\wedge y)^* = x^*\vee y^*$ for all $x, y\in L$. Hence for all $a, b\in B(L)$ we have $a\bigtriangledown b = (a^*\wedge b^*)^* = a^{**}\vee b^{**} = a\vee b$. Therefore $[F_{a\vee b}) = [F_a)\cap[F_b)$ immediately follows from (6). ∎

Corollary 4.2. *Let L be a finite quasi-modular p-algebra. Then we have.*

(1) *Every Boolean filter can be expressed as* $[F_a)$ *for some* $a\in B(L)$*;*

(2) $BF(L)\cong F_B(L)$.

Now, we are going to represent a Boolean filter of a quasi-modular p-algebra L as a union of certain elements of $F_B(L)$. We have the following theorem.

Theorem 4.3. *Let F be a Boolean filter of L. Then* $F=\bigcup_{x\in F}[F_{x^{**}})$.

Proof. Let $x\in F$. Then $x^{**}\in F$ and $x\vee x^*\in D(L)\subseteq F$. Thus $x = x^{**}\wedge(x\vee x^*)\in[x^{**})\vee D(L) = [F_{x^{**}})$. Then $F\subseteq\bigcup_{x\in F}[F_{x^{**}})$. Conversely, let $y\in\bigcup_{x\in F}[F_{x^{**}})$. Then $y\in[F_{z^{**}})$ for some $z\in F$. Hence $y^{**}\geq z^{**}\in F$. Then $y^{**}\in F$, which implies $y\in F$ as F is Boolean. Therefore $\bigcup_{x\in F}[F_{x^{**}})\subseteq F$. ∎

5. Boolean filters and congruences

In this section we investigate the relationships between the set of Boolean filters of a quasi-modular p-algebra L and the set of congruences on the interval $[\Phi,\nabla]$, where ∇ is the universal congruence on L.

We first state the following lemma.

Lemma 5.1. *Let θ be a congruence relation on a quasi-modular p-algebra L such that $\theta\in[\Phi,\nabla]$. Then $Coker\theta$ is a Boolean filter of L.*

Proof. Obviously $Coker\theta = \{x\in L : (x,1)\in\theta\}$ is a filter of L. For every $x\in L$, $(x\vee x^*)^{**} = 1 = 1^{**}$. Then $(x\vee x^*,1)\in\Phi\subseteq\theta$. Hence $x\vee x^*\in Coker\theta$. Therefore, $Coker\theta$ is a Boolean filter of L. ∎

For a Boolean filter F of a quasi-modular p-algebra L, define a relation θ_F on L as follows :

$$(x, y) \in \theta_F \Leftrightarrow x^{**} \wedge a = y^{**} \wedge a \text{ for some } a \in F \cap B(L)$$

We now establish the following theorem for a Boolean filter of L.

Theorem 5.2. *Let F be a Boolean filter of a quasi-modular p-algebra L. Then the following statements hold.*

(1) *θ_F is a congruence on L such that $\Phi \subseteq \theta_F$;*
(2) *$[x^{**}]\theta_F = [x]\theta_F$, for all $x \in L$;*
(3) *$Coker\theta_F = F$;*
(4) *$\theta_{D(L)} = \Phi$ and $\theta_L = \nabla$ whenever F is identical with $D(L)$, respectively, L;*
(5) *L/θ_F is a Boolean algebra.*

Proof. (1) Clearly, θ_F is an equivalence relation on L. Now we prove that θ_F is a lattice congruence on L. Let $(x, y), (c, d) \in \theta_F$. Then $x^{**} \wedge a = y^{**} \wedge a$ and $c^{**} \wedge b = d^{**} \wedge b$ for some $a, b \in F \cap B(L)$. Now we have the following equalities.

$$\begin{aligned} (x \wedge c)^{**} \wedge (a \wedge b) &= x^{**} \wedge c^{**} \wedge a \wedge b \\ &= y^{**} \wedge d^{**} \wedge a \wedge b \\ &= (y \wedge d)^{**} \wedge (a \wedge b). \end{aligned}$$

Then $(x \wedge c, y \wedge d) \in \theta_F$. Now by distributivity of $B(L)$ we have

$$\begin{aligned} (x \vee c)^{**} \wedge (a \wedge b) &= (x^* \wedge c^*)^* \wedge (a \wedge b) \\ &= (x^{***} \wedge c^{***})^* \wedge (a \wedge b) \\ &= (x^{**} \bigtriangledown c^{**}) \wedge (a \wedge b) \\ &= (x^{**} \wedge a \wedge b) \bigtriangledown (c^{**} \wedge a \wedge b) \\ &= (y^{**} \wedge a \wedge b) \bigtriangledown (d^{**} \wedge a \wedge b) \\ &= (y^{**} \bigtriangledown d^{**}) \wedge (a \wedge b) \\ &= (y \vee d)^{**} \wedge (a \wedge b). \end{aligned}$$

Then $(x \vee c, y \vee d) \in \theta_{F_a}$ as $a \wedge b \in F \cap B(L)$. Now we show that θ_F preserves the operation *. Let $(x, y) \in \theta_F$. Then $x^{**} \wedge a = y^{**} \wedge a$ for some $a \in F \cap B(L)$. Now by the distributivity of $B(L)$ we have the following set of implications.

$$\begin{aligned} x^{**} \wedge a = y^{**} \wedge a &\Rightarrow (x^{**} \wedge a) \bigtriangledown a^* = (y^{**} \wedge a) \bigtriangledown a^* \\ &\Rightarrow (x^{**} \bigtriangledown a^*) \wedge (a \bigtriangledown a^*) = (y^{**} \bigtriangledown a^*) \wedge (a \bigtriangledown a^*) \\ &\Rightarrow x^{**} \bigtriangledown a^* = y^{**} \bigtriangledown a^* \end{aligned}$$

$$
\begin{aligned}
&\Rightarrow (x^{***} \wedge a^{**})^* = (y^{***} \wedge a^{**})^* \\
&\Rightarrow (x^{***} \wedge a)^{**} = (y^{***} \wedge a)^{**} \\
&\Rightarrow x^{***} \wedge a = y^{***} \wedge a \\
&\Rightarrow (x^*, y^*) \in \theta_F.
\end{aligned}
$$

It is immediate that θ_F is a congruence on L. Let $(x, y) \in \Phi$. Then $x^{**} = y^{**}$. Hence, $x^{**} \wedge a = y^{**} \wedge a$, for some $a \in F \cap B(L)$. Thus $(x, y) \in \theta_F$ and $\Phi \subseteq \theta_F$.

(2) Since $x^{****} \wedge a = x^{**} \wedge a$, $(x^{**}, x) \in \theta_{F_a}$, and thereby $[x^{**}]\theta_F = [x]\theta_F$, $\forall x \in L$.

(3) It is known that $Coker\theta_F = [1]\theta_{F_a}$. Let $x \in Coker\theta_F$. Then we get the following implications.

$$
\begin{aligned}
x \in Coker\theta_F &\Rightarrow (x, 1) \in \theta_F \\
&\Rightarrow x^{**} \wedge a = 1^{**} \wedge a \text{ for some } a \in F \cap B(L) \\
&\Rightarrow x^{**} \wedge a = a \text{ as } 1^{**} = 1 \\
&\Rightarrow x^{**} \geq a \in F \\
&\Rightarrow x^{**} \in F \\
&\Rightarrow x \in F \text{ as } F \text{ is a Boolean filter of } L.
\end{aligned}
$$

Then $Coker\theta_F \subseteq F$. Conversely, let $y \in F$. Then

$$
\begin{aligned}
y \in F &\Rightarrow y^{**} \wedge y^{**} = y^{**} = 1^{**} \wedge y^{**} \\
&\Rightarrow (y, 1) \in \theta_F \text{ as } y^{**} \in F \cap B(L) \\
&\Rightarrow y \in Coker\theta_F.
\end{aligned}
$$

Then $F \subseteq Coker\theta_F$.

(4) Since $D(L) \cap B(L) = \{1\}$ and $L \cap B(L) = B(L)$, we deduce the following equalities:

$$
\begin{aligned}
\theta_{D(L)} &= \{(x, y) \in L \times L : x^{**} \wedge 1 = y^{**} \wedge 1\} = \{(x, y) \in L \times L : x^{**} = y^{**}\} = \Phi, \\
\theta_L &= \{(x, y) \in L \times L : x^{**} \wedge 0 = y^{**} \wedge 0\} = \{(x, y) \in L \times L : x, y \in L\} \\
&= L \times L = \nabla.
\end{aligned}
$$

(5) From (2) we have, $L/\theta_F = \{[x]\theta_F : x \in L\} = \{[x^{**}]\theta_F : x \in L\}$. Let $[x]\theta_F, [y]\theta_F, [z]\theta_F \in L/\theta_F$. Then

$$
\begin{aligned}
[x]\theta_F \wedge ([y]\theta_F \vee [z]\theta_F) &= [x \wedge (y \vee z)]\theta_F \\
&= [(x \wedge (y \vee z))^{**}]\theta_F \\
&= [x^{**} \wedge (y \vee z)^{**}]\theta_F
\end{aligned}
$$

$$\begin{aligned}
&= [x^{**} \wedge (y^{**} \bigtriangledown z^{**})]\theta_F \\
&= [(x^{**} \wedge y^{**}) \bigtriangledown (x^{**} \wedge z^{**})]\theta_F \\
&= [(x \wedge y)^{**} \bigtriangledown (x \wedge z)^{**}]\theta_F \\
&= [((x \wedge y) \vee (x \wedge z))^{**}]\theta_F \\
&= [(x \wedge y) \vee (x \wedge z)]\theta_F \\
&= [x \wedge y]\theta_F \vee [x \wedge z]\theta_F \\
&= ([x]\theta_F \wedge [y]\theta_F) \vee ([x]\theta_F \wedge [z]\theta_F).
\end{aligned}$$

This shows that L/θ_F is a distributive lattice. Clearly, $[0]\theta_F$ and $[1]\theta_F = F$ are the zero and the unit elements of L/θ_F. This shows that L/θ_F is a bounded distributive lattice. Now we proceed to show that every $[x]\theta_F$ of L/θ_F has a complement. Since $x \wedge x^* = 0$, $[x]\theta_F \wedge [x^*]\theta_F = [x \wedge x^*]\theta_F = [0]\theta_F$. Since F is a Boolean filter, $x \vee x^* \in F$. Hence, we have $[x]\theta_F \vee [x^*]\theta_F = [x \vee x^*]\theta_F = F$. Thus we have proved that L/θ_F is a Boolean algebra. ■

Now, let $F = [F_a)$ be a Boolean filter of L for some $a \in B(L)$. Then $a \in F \cap B(L)$. For brevity, we write θ_{F_a} instead of $\theta_{[F_a)}$.

In the following Corollary, we state some congruence properties of a quasi-modular p-algebra.

Corollary 5.3. *Let L be a quasi-modular p-algebra. Then the following statements hold.*

(1) $(x, y) \in \theta_{F_a} \Leftrightarrow x^{**} \wedge a = y^{**} \wedge a$,

(2) $Coker\theta_{F_a} = [F_a)$ *and* $Ker\theta_{F_a} = (a^*]$,

(3) $\theta_{F_1} = \Phi$ *and* $\theta_{F_0} = \nabla$.

Proof. (1) Let $(x, y) \in \theta_{F_a}$. Then

$$\begin{aligned}
(x, y) \in \theta_{F_a} &\Rightarrow x^{**} \wedge b = y^{**} \wedge b \text{ for some } b \in [F_a) \cap B(L) \\
&\Rightarrow x^{**} \wedge b \wedge a = y^{**} \wedge b \wedge a \\
&\Rightarrow x^{**} \wedge a = y^{**} \wedge a \text{ as } b = b^{**} \geq a
\end{aligned}$$

Conversely, let $x^{**} \wedge a = y^{**} \wedge a$. Then $(x, y) \in \theta_{F_a}$ as $a \in [F_a) \cap B(L)$.

(2) By Theorem 5.2(3), we have $Coker\theta_{F_a} = [F_a)$. Now we prove the second equality in (2) as follows:

$$\begin{aligned}
Ker\theta_{F_a} &= \{x \in L : (x, 0) \in \theta_{F_a}\} \\
&= \{x \in L : x^{**} \wedge a = 0^{**} \wedge a\} \\
&= \{x \in L : x^{**} \wedge a = 0\} \text{ as } 0^{**} = 0 \\
&= \{x \in L : x \leq x^{**} \leq a^*\} \\
&= (a^*].
\end{aligned}$$

(3) Using Theorem 5.2(4), we get $\theta_{F_1} = \theta_{D(L)} = \Phi$ and $\theta_{F_0} = \theta_{[F_0)} = \theta_L = \nabla$ ■

By combining Lemma 5.1 and Theorem 5.2(1), (3) we establish the following characterization theorem of a Boolean filter of L.

Theorem 5.4. *A filter F of a quasi-modular p-algebra L is a cokernel of a congruence $\theta \in [\Phi, \nabla]$ if and only if F is a Boolean filter.*

Consider $Con_B(L) = \{\theta_{F_a} : a \in B(L)\}$, we observe that $Con_B(L)$ is a partially ordered set under set inclusion. We now study properties of the elements in the set $Con_B(L)$.

Theorem 5.5. *Let L be a quasi-modular p-algebra. Then for every $a, b \in B(L)$, the following statement hold in $Con_B(L)$.*

(1) $a \le b$ *if and only if* $\theta_{F_b} \subseteq \theta_{F_a}$,

(2) *The set* $Con_B(L)$ *is a Boolean algebra on its own. Moreover,* $F_B(L) \cong Con_B(L)$,

(3) $\theta_{F_a} \sqcup \theta_{F_b} = \theta_{F_{a\wedge b}}$ *and* $\theta_{F_a} \sqcap \theta_{F_b} = \theta_{F_{a\triangledown b}}$,

(4) $\theta_{F_a} \sqcap \theta_{F_{a^*}} = \Phi$ *and* $\theta_{F_a} \sqcup \theta_{F_{a^*}} = \nabla$.

Proof. (1) Let $a \le b$ and $(x, y) \in \theta_{F_b}$. Then $x^{**} \wedge b = y^{**} \wedge b$. Hence $x^{**} \wedge b \wedge a = y^{**} \wedge b \wedge a$. This leads to $x^{**} \wedge a = y^{**} \wedge a$. Thus $(x, y) \in \theta_{F_a}$ and $\theta_{F_b} \subseteq \theta_{F_a}$. Conversely, let $\theta_{F_b} \subseteq \theta_{F_a}$. Then we have $(b, 1) \in \theta_{F_b} \subseteq \theta_{F_a}$. This implies that $b \wedge a = 1 \wedge a = a$. Thus $a \le b$.

(2) Define the mapping $\Psi : B(L) \to Con_B(L)$ as follows :

$$\Psi(a) = \theta_{F_a} \text{ for all } a \in B(L).$$

By (1) above, Ψ is an order anti-isomorphism between $B(L)$ and $Con_B(L)$. This immediately implies that $Con_B(L)$ is a Boolean algebra. Now if we define the mapping $f : B(L) \to Con_B(L)$ by $f(a) = \theta_{F_{a^*}}$, then f is an isomorphism between Boolean algebras $B(L)$ and $Con_B(L)$. Then $B(L) \cong Con_B(L)$ and $B(L) \cong F_B(L)$ imply $F_B(L) \cong Con_B(L)$.

(3) Since by (2) above Ψ is a anti-isomorphism, we have $\Psi(a \wedge b) = \Psi(a) \sqcup \Psi(b)$ and $\Psi(a \triangledown b) = \Psi(a) \sqcap \Psi(b)$, where $\sqcup$ and $\sqcap$ are the join and meet operations on $Con_B(L)$. Now

$$\theta_{F_a} \sqcup \theta_{F_b} = \Psi(a) \sqcup \Psi(b) = \Psi(a \wedge b) = \theta_{F_{a\wedge b}}$$

and

$$\theta_{F_a} \sqcap \theta_{F_b} = \Psi(a) \sqcap \Psi(b) = \Psi(a \triangledown b) = \theta_{F_{a\triangledown b}}.$$

(4) From (3) above we have

$$\theta_{F_a} \sqcap \theta_{F_{a^*}} = \theta_{F_{a \triangledown a^*}} = \theta_{F_1} = \Phi$$

and

$$\theta_{F_a} \sqcup \theta_{F_{a^*}} = \theta_{F_{a \wedge a^*}} = \theta_{F_0} = \nabla.$$

Therefore $Con_B(L) = (Con_B(L), \sqcup, \sqcap, ^-, \Phi, \nabla)$, where $\overline{\theta}_{F_a} = \theta_{F_{a^*}}$ is the complement of θ_{F_a} on $Con_B(L)$ and Φ, ∇ are the smallest and greatest elements of $Con_B(L)$ respectively. ■

In the following Corollary an isomorphism between the sublattice $[\Phi, \nabla]$ of $Con(L)$ and the lattice $BF(L)$ of all Boolean filters of L is obtained.

Corollary 5.6. *Let L be a finite quasi-modular p-algebra. Then $[\Phi, \nabla] \cong BF(L)$.*

Proof. Since L is finite, $BF(L) = F_B(L)$ and hence $Con_B(L) = [\Phi, \nabla]$. By the above Theorem 5.5 (2), we deduce that $BF(L) \cong [\Phi, \nabla]$. ■

6. CONGRUENCES AND DIRECT PRODUCT OF BOOLEAN FILTERS

Let L_1 and L_2 be two p-algebras. Then the direct product $L_1 \times L_2$ is also a p-algebra, where * is defined on $L_1 \times L_2$ by $(a, b)^* = (a^*, b^*)$. Now we study the direct product of Boolean filters of p-algebras. Some properties of congruences with respect to direct product are given.

We first consider the Boolean filters of the p-algebras in the following theorem.

Theorem 6.1. *If F_1 and F_2 are Boolean filters of p-algebras L_1 and L_2 respectively, then $F_1 \times F_2$ is a Boolean filter of $L_1 \times L_2$. Conversely, every Boolean filter F of $L_1 \times L_2$ can be expressed as $F = F_1 \times F_2$ where F_1 and F_2 are Boolean filters of L_1 and L_2 respectively.*

Proof. Let F_1 and F_2 be Boolean filters of L_1 and L_2 respectively. Then, it is clear that $F_1 \times F_2$ is a filter of $L_1 \times L_2$. Since F_1 and F_2 are Boolean filters of L_1 and L_2 respectively, we get $a \vee a^* \in F_1$ for each $a \in L_1$ and $b \vee b^* \in F_2$ for each $b \in L_2$. Hence we have $(a, b) \vee (a, b)^* = (a, b) \vee (a^*, b^*) = (a \vee a^*, b \vee b^*) \in F_1 \times F_2$ This shows that $F_1 \times F_2$ is a Boolean filter of $L_1 \times L_2$. Conversely, if F is a Boolean filter of $L_1 \times L_2$, then we consider F_1 and F_2 as follows:

$$F_1 = \{x \in L_1 : (x, 1) \in F\} \text{ and } F_2 = \{y \in L_2 : (1, y) \in F\}$$

Clearly F_1 and F_2 are filters of L_1 and L_2 respectively. We now prove that F_1 and F_2 are Boolean filters of L_1 and L_2 respectively. For all $x \in F_1$, we have $(x,1) \in F$. Since F is Boolean, $(x \vee x^*, 1) = (x,1) \vee (x,1)^* \in F$. Hence, we have $x \vee x^* \in F_1$. Therefore, F_1 is a Boolean filter of L_1. Similarly, F_2 is a Boolean filter of L_2. Now we prove that $F = F_1 \times F_2$. For this purpose, we let $(x,y) \in F$. Then we have the following implications.

$$\begin{aligned} (x,y) \in F \quad &\Rightarrow \quad (x,1) \in F \text{ and } (1,y) \in F \\ &\Rightarrow \quad x \in F_1 \text{ and } y \in F_2 \\ &\Rightarrow \quad (x,y) \in F_1 \times F_2. \end{aligned}$$

Hence, $F \subseteq F_1 \times F_2$. Conversely, if $(x,y) \in F_1 \times F_2$, then the following implications hold.

$$\begin{aligned} (x,y) \in F_1 \times F_2 \quad &\Rightarrow \quad x \in F_1 \text{ and } y \in F_2 \\ &\Rightarrow \quad (x,1) \in F \text{ and } (1,y) \in F \\ &\Rightarrow \quad (x,y) = (x,1) \wedge (1,y) \in F. \end{aligned}$$

Consequently, we have $F_1 \times F_2 \subseteq F$. This shows that $F_1 \times F_2 = F$. ∎

In closing this paper, we state two equalities concerning Boolean filters of quasi-modular p-algebras.

Theorem 6.2. *Let $[F_a)$ and $[F_b)$ be two Boolean filters of the quasi-modular p-algebras L_1 and L_2, respectively. Then*

(1) $[F_a) \times [F_b) = [F_{(a,b)})$

(2) $\theta_{F_a \times F_b} = \theta_{F_{(a,b)}}$.

Proof. (1) From the above Theorem 6.1, we see immediately that $[F_a) \times [F_b)$ is a Boolean filter of $L_1 \times L_2$. Now, we have

$$\begin{aligned} (x,y) \in [F_a) \times [F_b) \quad &\Leftrightarrow \quad x \in [F_a) \text{ and } y \in [F_b) \\ &\Leftrightarrow \quad x^{**} \geq a \text{ and } y^{**} \geq b \\ &\Leftrightarrow \quad (x,y)^{**} = (x^{**}, y^{**}) \geq (a,b) \\ &\Leftrightarrow \quad (x,y) \in [F_{(a,b)}). \end{aligned}$$

Therefore, $[F_a) \times [F_b) = [F_{(a,b)})$.

(2) By (1), we obtain $\theta_{F_a \times F_b} = \theta_{[F_a) \times [F_b)} = \theta_{[F_{(a,b)})} = \theta_{F_{(a,b)}}$. ∎

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