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GENERALIZED DERIVATIONS IN PRIME RINGS AND BANACH ALGEBRAS

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Abstract

Let R be a prime ring with extended centroid C, F a generalized derivation of R and $n \geq 1$, $m \geq 1$ fixed integers. In this paper we study the situations:

1. $(F(x \circ y))^m = (x \circ y)^n$ for all $x, y \in I$, where I is a nonzero ideal of R;
2. $(F(x \circ y))^n = (x \circ y)^n$ for all $x, y \in I$, where I is a nonzero right ideal of R.

Moreover, we also investigate the situation in semiprime rings and Banach algebras.

Keywords: prime ring, generalized derivation, extended centroid, Utumi quotient ring.

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1. Introduction

Throughout this paper, R always denotes an associative prime ring with center $Z(R)$ and with extended centroid C, U the Utumi quotient ring of R. For given $x, y \in R$, the symbol $[x, y]$ and $x \circ y$ stand for the Lie product $xy - yx$ and Jordan product $xy + yx$ respectively. Also $[x_1, x_2, \ldots, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n]$ for all $x_1, x_2, \ldots, x_n \in R$, for every positive integer $n \geq 2$. In particular, when $x_1 = x$ and $x_2 = x_3 = \cdots = x_n = y$, we use the notation to define the engel type polynomial $[x, y]_{n+1} = [[x, y]_n, y]$ instead of $[x, y, y, \ldots, y]$ for $n \geq 1$ and $[x, y]_1 = [x, y]$.

An additive mapping $d : R \to R$ is called a derivation, if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. In particular, d is an inner derivation induced by an element $a \in R$, if $d(x) = [a, x]$ for all $x \in R$. In [5], Brešar introduced the definition of generalized derivation. An additive mapping $F : R \to R$ is called a generalized derivation, if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Hence, the concept of generalized derivations covers both the concepts of a derivation and a left multplier (i.e. an additive mapping satisfying $f(xy) = f(x)y$ for all $x, y \in R$).

In [8], Daif and Bell proved that if R is a semiprime ring with a nonzero ideal I and d is a derivation of R such that $d([x, y]) = [x, y]$ for all $x, y \in I$, then $I \subseteq Z(R)$. In particular, if R is prime ring, then R must be commutative. In [25], Quadri *et al.* proved that if R is a prime ring, I a nonzero ideal of R and F a generalized derivation associated with a nonzero derivation d such that $F([x, y]) = [x, y]$ for all $x, y \in I$, then R is commutative. Further, this result of Quadri *et al.* is studied in semiprime ring by Dhara in [10]. Recently in [9], De Filippis and Huang studied the situation $(F([x, y]))^n = [x, y]$ for all $x, y \in I$, where I is a nonzero ideal in a prime ring R, F a generalized derivation of R and $n \geq 1$ fixed integer. In this case they conclude that either R is commutative or $n = 1$ and $F(x) = x$ for all $x \in R$. More recently in [14], Huang and Davvaz consider the situation $(F([x, y]))^m = [x, y]^n$ for all $x, y \in R$. More precisely, they proved the following:

Let R be a prime ring and m, n fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F([x, y]))^m = [x, y]^n$ for all $x, y \in R$, then R is commutative.

Note that in this result, the assumption $d \neq 0$ exists.

There is also ongoing interest to study the above identities replacing Lie product $[x, y]$ by Jordan product $x \circ y$, for $x, y \in R$. In this line of investigation, in [2], Ashraf and Rehman proved that if R is a prime ring, I a nonzero ideal of R and d is a derivation of R such that $d(x \circ y) = (x \circ y)$ for all $x, y \in I$, then R is commutative. Then Argac and Inceboz [1] generalized the above result by considering some power values. They proved that if R is a prime ring, I a nonzero ideal of R,

n a fixed positive integer and d a derivation of R such that $(d(x \circ y))^n = (x \circ y)$ for all $x, y \in I$, then R is commutative. Quadri *et al.* [25] proved that if R is a prime ring, I a nonzero ideal of R and F a generalized derivation associated with a nonzero derivation d such that $F(x \circ y) = x \circ y$ for all $x, y \in I$, then R is commutative. Recently, Huang [15] proved the following:

Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x \circ y))^n = x \circ y$ for all $x, y \in I$, then R is commutative.

Note that in this result again the assumption of $d \neq 0$ is existing. It is natural to ask what will happen in case $d = 0$.

Our present paper is motivated by above results. In the present paper, we will investigate the situation when a prime ring R satisfies $(F(x \circ y))^m = (x \circ y)^n$ for all x, y in some suitable subsets of R, where F is a generalized derivation of R associated with a derivation d. Note that in our result the hypothesis $d \neq 0$ is deleted.

More precisely, we shall prove the following results:

Theorem 1. *Let R be a prime ring, F a generalized derivation of R and I a nonzero ideal of R. Suppose that $(F(x \circ y))^m = (x \circ y)^n$ for all $x, y \in I$, where $m \geq 1$ and $n \geq 1$ are fixed integers. Then one of the following holds:*

1. *R is commutative;*
2. *there exists $a \in C$ such that $F(x) = ax$ for all $x \in R$ with $a^m = 1$. Moreover, in this case if $m \neq n$, then either char $(R) = 2$ or char $(R) = 2^{|m-n|} - 1$.*

Theorem 2. *Let R be a prime ring, I a nonzero right ideal of R and F a generalized derivation of R. If $(F(x \circ y))^n = (x \circ y)^n$ for all $x, y \in I$, where $n \geq 1$ is fixed integer, then one of the following holds:*

1. *$[I, I]I = 0$;*
2. *there exists $a \in U$ and $\alpha \in C$ such that $F(x) = ax$ for all $x \in R$, with $(a - \alpha)I = 0$ and $\alpha^n = 1$.*

Theorem 3. *Let R be a semiprime ring and F a generalized derivation of R. If $(F(x \circ y))^m = (x \circ y)^n$ for all $x, y \in R$, where $m \geq 1$ and $n \geq 1$ are fixed integers, then R is commutative or $F(x) = ax + d(x)$ for all $x \in R$, where $a \in C$ and d is a derivation of R such that $d(R) \subseteq Z(R)$.*

In the last section we apply above results to Banach algebras. Here A will denote a complex non-commutative Banach algebras. By a Banach algebra we shall mean a complex normed algebra A whose underlying vector space is a Banach space. By $rad(A)$ we denote the Jacobson radical of A, which is the intersection of all primitive ideals of A. A is said to be semisimple, if $rad(A) = 0$.

In 1955, Singer and Wermer [27] gave an interesting result. They proved that every continuous derivation on a commutative Banach algebra maps the algebra into its radical. After thirty years, Thomas [28] proved the same result of Singer and Wermer without considering continuity of derivation. It is clear that the same result of Singer and Wermer does not hold in noncommutative Banach algebras because of inner derivations. It is still an open question whether the above result of Singer and Wermer is true or not in the noncommutative Banach algebra. Some partial solutions of this open question have been obtained by a number of authors.

Let A be a noncommutative Banach algebra and D be a continuous derivation on A. Brešar and Vukman [4] proved that if $[D(x), x] \in rad(A)$ for all $x \in A$, then D maps A into $rad(A)$. In [24], Mathieu proved the same conclusion if $[D(x), x]D(x) \in rad(A)$ for all $x \in A$. Vukman [29] proved also that the same conclusion holds if $[D(x), x]_3 \in rad(A)$ for all $x \in A$.

Continuing on this line, in [19] Kim proved that if d is a continuous linear Jordan derivation in a Banach algebra A, such that $[d(x), x]d(x)[d(x), x] \in rad(A)$, for all $x \in A$, then d maps A into $rad(A)$. In [14], Huang and Davvaz proved the following result:

Let A be a non-commutative Banach algebra with Jacobson radical $rad(A)$. Let $F = L_a + d$ be a continuous generalized derivation of R, where L_a denotes the left multiplication by some element $a \in A$ and d is a derivation of A. If $(F([x, y]))^m - ([x, y])^n \in rad(A)$ for all $x, y \in A$, then $d(A) \subseteq rad(A)$.

In the last section, finally we provide a result about continuous generalized derivations on Banach algebras which is as follows:

Theorem 4. *Let A be a noncommutative Banach algebra. Let $F = L_a + d$ be a continuous generalized derivation of A, where L_a denotes the left multiplication by some element $a \in A$ and d is a derivation on A. If $(F(x \circ y))^m - (x \circ y)^n \in rad(A)$ for all $x, y \in A$, then $d(A) \subseteq rad(A)$.*

2. Generalized Derivations on Ideals

We begin with the following:

Lemma 5. *Let R be a prime ring with extended centroid C, I a nonzero ideal of R and $a, b \in R$. Suppose that $(a(x \circ y) + (x \circ y)b)^m = (x \circ y)^n$ for all $x, y \in I$, where $m \geq 1, n \geq 1$ are fixed integers. Then one of the following holds:*

1. *R is commutative;*
2. *$a, b \in C$ with $(a + b)^m = 1$. (In this case if $m \neq n$ and $m + n$ is odd, then char $(R) = 2$ and if $m \neq n$ and $m + n$ is even, then char $(R) = 2^{|m-n|} - 1$).*

Proof. If R is commutative, the conclusion (1) is obtained. So, we assume that R is noncommutative. Then by assumption, I satisfies the generalized polynomial identity

$$F(x, y) = (a(x \circ y) + (x \circ y)b)^m - (x \circ y)^n.$$

By Chuang [7, Theorem 2], this generalized polynomial identity (GPI) is also satisfied by U. We assume either $a \notin C$ or $b \notin C$ and prove that a number of contradictions follows. In this case $F(x, y) = 0$ is a nontrivial GPI for U. In case C is infinite, we have $F(x, y) = 0$ for all $x, y \in U \bigotimes_C \overline{C}$ where $\overline{C}$ is the algebraic closure of C. Since both U and $U \bigotimes_C \overline{C}$ are prime and centrally closed [11], we may replace R by U or $U \bigotimes_C \overline{C}$ according to C is finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed and $F(x, y) = 0$ for all $x, y \in R$. By Martindale's Theorem [23], R is then a primitive ring having nonzero $soc(R)$ with C as the associated division ring. Hence by Jacobson's Theorem [16], R is isomorphic to a dense ring of linear transformations of a vector space V over C. Assume first that $\dim_C V = k$, then the density of R on V implies that $R \cong M_k(C)$. Since R is noncommutative, $k \geq 2$.

In this case assuming $x = e_{ij}$ and $y = e_{jj}$, $i \neq j$, so that $x \circ y = e_{ij}$, we obtain that

$$(ae_{ij} + e_{ij}b)^m = (e_{ij})^n.$$

Left and right multiplying by e_{ij} respectively, above relation yields $0 = e_{ij}(ae_{ij})^m = a_{ji}^m e_{ij}$ implying $a_{ji} = 0$ and $0 = (e_{ij}b)^m e_{ij} = b_{ji}^m e_{ij}$ implying $b_{ji} = 0$. Thus both a and b are diagonal matrices in $M_k(C)$. Let $a = \sum_{i=1}^k a_{ii}e_{ii}$ and $b = \sum_{i=1}^k b_{ii}e_{ii}$. Now since for any automorphism φ of R, $(a^\varphi(x \circ y) + (x \circ y)b^\varphi)^m = (x \circ y)^n$ holds for all $x, y \in M_k(C)$, we can write by above arguments that a^φ and b^φ are diagonal. Hence for each $j \neq 1$, we have $(1 + e_{1j})a(1 - e_{1j}) = \sum_{i=1}^k a_{ii}e_{ii} + (a_{jj} - a_{11})e_{1j}$ and $(1 + e_{1j})b(1 - e_{1j}) = \sum_{i=1}^k b_{ii}e_{ii} + (b_{jj} - b_{11})e_{1j}$ both diagonal. Therefore, $a_{jj} = a_{11}$ and $b_{jj} = b_{11}$ that is, $a, b \in F.I_k$. Thus we have $a, b \in C$, a contradiction.

Next assume that $\dim_C V = \infty$. Since $a \notin C$ or $b \notin C$, they do not centralize the nonzero ideal $H = soc(R)$ and hence there exist $h, h' \in H$ such that $[a, h] \neq 0$ or $[b, h'] \neq 0$. Moreover, because of the infinite dimensionality, H does not satisfy the polynomial $[x, y]$, that is, there exist $h_1, h_2 \in H$ such that $[h_1, h_2] \neq 0$. By Litoff's theorem [12], there exists idempotent $e^2 = e \in H$ such that $ah, ha, bh', h'b, h, h', h_1, h_2 \in eRe$, moreover eRe is a central simple algebra finite dimensional over its center. Since R satisfies $(a(x \circ y) + (x \circ y)b)^m = (x \circ y)^n$, replacing x with e and y with $ex(1-e)$ we have that R satisfies $(aex(1-e) + ex(1-e)b)^m = (ex(1-e))^n$. Left multiplying by $(1-e)$, we get $(1-e)(aex(1-e))^m = 0$ for all $x \in R$, that is $((1-e)aex)^{m+1} = 0$ for all $x \in R$. Then by Levitzki's lemma [13, Lemma 1.1], we conclude that $(1-e)aex = 0$ for all $x \in R$ and so $(1-e)ae = 0$.

Since R satisfies generalized identity $e\{(a(exe \circ eye) + (exe \circ eye)b)^m - (exe \circ eye)^n\}e = 0$, the subring eRe satisfies $(eae(x \circ y) + (x \circ y)ebe)^m - (x \circ y)^n = 0$. Since $[h_1, h_2] \neq 0$, eRe is not commutative and so $eRe \cong M_k(C)$ for $k \geq 2$. Then by the above finite dimensional case, eae and ebe are central element of eRe. Thus $ah = eaeh = heae = ha$ and $bh' = ebeh' = h'ebe = h'b$, which contradicts our assumption.

In light of previous argument, we have that both $a, b \in C$ and then our identity reduces to $(a+b)^m(x \circ y)^m = (x \circ y)^n$ for all $x, y \in R$. This is a polynomial identity. Then by [20, Lemma 1], there exists a field F such that $R \subseteq M_k(F)$, $k \geq 2$, such that $M_k(F)$ satisfies the identity $(a + b)^m(x \circ y)^m = (x \circ y)^n$. But by choosing $x = e_{12}, y = e_{21}$, above identity yields $(a + b)^m(e_{11} + e_{22}) = (e_{11} + e_{22})$. This implies that $(a+b)^m = 1$. Hence our identity reduces to $(x \circ y)^m = (x \circ y)^n$ for all $x, y \in M_k(F)$. If $m = n$, then the identity is trivial and then the proof is done.

Now we assume that $m \neq n$. Then we consider the following two cases:

(i) Let $m + n$ be odd. We have $(x \circ y)^m - (x \circ y)^n = 0$ for all $x, y \in M_k(F)$. In this case replacing y with $-y$, we have $(x \circ y)^m + (x \circ y)^n = 0$ for all $x, y \in M_k(F)$. By addition of above two identities, we get $2(x \circ y)^m = 0$ for all $x, y \in M_k(F)$. Replacing $x = y = I_k$, we have $0 = 2(I_k \circ I_k)^m = 2^{m+1}I_k$. This leads a contradiction, unless char $(R) = 2$.

(ii) Let $m + n$ be even. Replacing $x = y = e_{11}$, we obtain $0 = (x \circ y)^m - (x \circ y)^n = (2^m - 2^n)e_{11}$. This gives a contradiction, unless char $(R) = 2^{|m-n|} - 1$. ■

***Proof of Theorem* 1.** If $F = 0$, then $(x \circ y)^n = 0$. Note that this is a polynomial identity and hence there exists a field F such that $R \subseteq M_k(F)$, the ring of $k \times k$ matrices over a field F, where $k \geq 1$. If $k = 1$, then R must be commutative, and then we obtain our conclusion (1). So let $k \geq 2$. Moreover, R and $M_k(F)$ satisfy the same polynomial identity [20, Lemma 1] that is $(x \circ y)^m = 0$ for all $x, y \in M_k(F)$. But by choosing $x = e_{12}$, $y = e_{21}$ we get

$$0 = (x \circ y)^m = e_{11} + e_{22}$$

which is a contradiction. If $F \neq 0$, then by hypothesis we have, I satisfies the differential identity

$$(F(x \circ y))^m = (x \circ y)^n \quad \text{for all} \quad x, y \in I. \tag{1}$$

Since I and U satisfy the same differential identities [21], we may assume that $(F(x \circ y))^m = (x \circ y)^n$ for all $x, y \in U$. By Lee [22], we may assume that for all $x \in U$, $F(x) = bx + d(x)$ for some $b \in U$ and a derivation d of U. Hence U satisfies

$$(b(x \circ y) + d(x \circ y))^m = (x \circ y)^n. \tag{2}$$

Assume first that d is inner derivation of U, i.e., there exists $p \in U$ such that $d(x) = [p, x]$ for all $x \in U$. Then

$$(b(x \circ y) + [p, x \circ y])^m = (x \circ y)^n,$$

for all $x, y \in U$ that is

$$((b + p)(x \circ y) - (x \circ y)p)^m = (x \circ y)^n,$$

for all $x, y \in U$. By Lemma 5, one of the following holds:

(i) R is commutative and so conclusion (1) is obtained.

(ii) $b + p, p \in C$ and $b^m = 1$. Moreover, in this case if $m \neq n$, then either char $(R) = 2$ or char $(R) = 2^{|m-n|} - 1$. Thus $F(x) = bx$ for all $x \in R$ with $b^m = 1$, which is our conclusion (2).

Next assume that d is not U-inner. From (2), we have U satisfies

$$(b(x \circ y) + (d(x) \circ y) + (x \circ d(y)))^m = (x \circ y)^n. \tag{3}$$

Then by Kharchenko's theorem [18], we have

$$(b(x \circ y) + (z \circ y) + (x \circ t))^m = (x \circ y)^n \tag{4}$$

for all $x, y, z, t \in U$. In particular, for $y = 0$, we have $(x \circ t)^m = 0$ for all $x, t \in U$. Repeating by the same argument as above we get that R is commutative. Hence the theorem is proved.

Corollary 6. *Let R be a prime ring, F a generalized derivation of R associated to a nonzero derivation d of R and I a nonzero ideal of R. Suppose that $(F(x \circ y))^m = (x \circ y)^n$ for all $x \in I$, where $m \geq 1, n \geq 1$ are fixed integers. Then R is commutative.*

Corollary 7. *Let R be a prime ring, d a nonzero derivation of R and I a nonzero ideal of R. Suppose that $(d(x)y + xd(y) + d(y)x + yd(x))^m = (x \circ y)^n$ for all $x, y \in I$, where $m \geq 1, n \geq 1$ are fixed integers. Then R must be commutative.*

Example. Let S be any ring and $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in S \right\}$. Note that R is not prime ring, since $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = 0$. Let $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in S \right\}$ be a nonzero ideal of R. Define a map $F : R \longrightarrow R$ by $F(x) = 2e_{11}x - xe_{11}$. Then F is a generalized derivation with associated nonzero derivation $d(x) = [e_{11}, x]$ satisfies the property $(F(x \circ y))^m = (x \circ y)^n$ for all $x, y \in I$. Since R is not commutative, we conclude that the primeness hypothesis in Theorem 1 is not superfluous.

3. Generalized Derivations on Right Ideals

In this section we will prove our next Theorem for right sided ideals.

To prove this theorem, we need the following:

Lemma 8. *Let R be a prime ring with extended centroid C and I a nonzero right ideal of R. If for some $a, b \in R$, $(a(x \circ y) + (x \circ y)b)^n - (x \circ y)^n = 0$ for all $x, y \in I$, where $n \geq 1$ is fixed integer, then R satisfies a non-trivial generalized polynomial identity or there exists $\alpha \in C$ such that $(a - \alpha)I = 0$, $b \in C$ with $(b + \alpha)^n = 1$.*

Proof. By our hypothesis, for any $x_0 \in I$, R satisfies the following generalized identity

$$(a(x_0x \circ x_0y) + (x_0x \circ x_0y)b)^n - (x_0x \circ x_0y)^n = 0. \tag{5}$$

We assume that this is a trivial GPI for R, for otherwise we are done. If there exists $x_0 \in I$ such that $\{x_0, ax_0\}$ is linearly C-independent, then from above we have that R satisfies

$$a(x_0x \circ x_0y)(a(x_0x \circ x_0y) + (x_0x \circ x_0y)b)^{n-1} = 0. \tag{6}$$

Again, since $\{x_0, ax_0\}$ is linearly C-independent, we have from above relation that R satisfies

$$(a(x_0x \circ x_0y))^2(a(x_0x \circ x_0y) + (x_0x \circ x_0y)b)^{n-2} = 0 \tag{7}$$

and hence $(a(x_0x \circ x_0y))^n = 0$, which is nontrivial, a contradiction. Thus $\{x_0, ax_0\}$ is linearly dependent over C for all $x_0 \in I$, that is $(a - \alpha)I = 0$ for some $\alpha \in C$. Then (5) becomes

$$((x_0x \circ x_0y)(b + \alpha))^n - (x_0x \circ x_0y)^n = 0. \tag{8}$$

Since this is trivial GPI for R, we have that $b + \alpha \in C$, that is $b \in C$. Thus identity reduces to

$$((b + \alpha)^n - 1)(x_0x \circ x_0y)^n = 0. \tag{9}$$

Since this is trivial identity for R, we conclude $(b + \alpha)^n = 1$. ■

Lemma 9. *Let R be a prime ring with extended centroid C, I be a right ideal of R and F be an inner generalized derivation of R. If $(F(x \circ y))^n = (x \circ y)^n$ for all $x, y \in I$, where $n \geq 1$ is a fixed integer, then one of the following holds:*

1. $[I, I]I = 0$;
2. *there exists* $a \in U$ *and* $\alpha \in C$ *such that* $F(x) = ax$ *for all* $x \in R$, *with* $(a - \alpha)I = 0$ *and* $\alpha^n = 1$.

Proof. Since F is inner, there exist $a, b \in U$ such that $F(x) = ax + xb$ for all $x \in R$. If R does not satisfy any non-trivial (GPI), then by Lemma 8, we conclude that there exists $\alpha \in C$ such that $(a - \alpha)I = 0$, $b \in C$, $(b + \alpha)^n = 1$. In this case $F(x) = ax + xb = (a + b)x$ for all $x \in R$, where $(a - \alpha)I = 0$, $b \in C$ with $(b+\alpha)^n = 1$. Assuming $a' = a+b$ and $\beta = b+\alpha$, we can write that $F(x) = a'x$ for all $x \in R$, with $(a' - \beta)I = 0$ for some $\beta \in C$ and $\beta^n = 1$. This is our conclusion (2).

So we assume that R satisfies a non-trivial GPI. If $I = R$, then by Lemma 5, either R is commutative or $a, b \in C$ with $(a + b)^m = 1$. In the last case we have $F(x) = \lambda x$ for all $x \in R$, with $\lambda^m = 1$. Thus conclusions (1) and (2) are obtained.

Now let $I \neq R$. In this case we want to prove that either $[I, I]I = 0$ or there exist $\alpha, \beta \in C$ such that $(a - \alpha)I = 0$ and $(b - \beta)I = 0$. To prove this, by contradiction, we suppose that there exist $c_1, c_2, \ldots, c_5 \in I$ such that

- $[c_1, c_2]c_3 \neq 0$;
- $(a - \alpha)c_4 \neq 0$ for all $\alpha \in C$ or $(b - \beta)c_5 \neq 0$ for all $\beta \in C$.

Now we show that this assumption leads a number of contradictions. Since R satisfies nontrivial GPI, by [23], RC is a primitive ring having a nonzero socle H with a nonzero right ideal $J = IH$. Notice that H is simple, $J = JH$ and J satisfies the same basic conditions as I. Thus we replace R by H and I by J.

Then since R is a regular ring, for $c_1, c_2, \ldots, c_5 \in I$ there exists $e^2 = e \in R$ such that

$$eR = c_1R + c_2R + c_3R + c_4R + c_5R.$$

Then $e \in I$ and $ec_i = c_i$ for $i = 1, \ldots, 5$. Let $x \in R$. Then by our hypothesis we have

$$(a(e \circ ex(1 - e)) + (e \circ ex(1 - e))b)^n = (e \circ ex(1 - e))^n \tag{10}$$

Left multiplying by $(1 - e)$ we have $((1 - e)aex)^n(1 - e) = 0$, that is $((1 - e)aex)^{n+1} = 0$ for all $x \in R$. By Levitzki's lemma [13, Lemma 1.1], we have $(1 - e)aeR = 0$ implying $(1 - e)ae = 0$. Analogously, right multiplying by e, we get $(1 - e)be = 0$. Therefore $ae = eae$ and $be = ebe$. Moreover, since R satisfies

$$e\{(a(x \circ y) + (x \circ y)b)^n - (x \circ y)^n\}e = 0,$$

eRe satisfies

$$(eae(x \circ y) + (x \circ y)ebe)^n - (x \circ y)^n = 0.$$

Then by Lemma 5, one of the following holds: (1) $[eRe, eRe] = 0$, (2) $eae, ebe \in Ce$. Now $[eRe, eRe] = 0$ implies $[eR, eR]eR = 0$ which contradicts with the choices of c_1, c_2, c_3. Thus $eae = ae \in Ce$ and $ebe = be \in Ce$. Therefore, there exist $\alpha, \beta \in C$ such that $(a - \alpha)e = 0$ and $(b - \beta)e = 0$. This gives $(a - \alpha)eR = 0$ and $(b - \beta)eR = 0$. In any case this contradicts with the choices of c_4 and c_5.

In case $[I, I]I = 0$, conclusion (1) is obtained. Let $(a-\alpha)I = 0$ and $(b-\beta)I = 0$ for some $\alpha, \beta \in C$. Then our hypothesis

$$(a(x \circ y) + (x \circ y)b)^n - (x \circ y)^n = 0 \tag{11}$$

for all $x, y \in I$ gives

$$((x \circ y)(b + \alpha))^n - (x \circ y)^n = 0 \tag{12}$$

for all $x, y \in I$ and so

$$(x \circ y)^n(\beta + \alpha)^{n-1}(b + \alpha) - (x \circ y)^n = 0 \tag{13}$$

for all $x, y \in I$. Right multiplying by $x \circ y$, (13) reduces to

$$(x \circ y)^{n+1}(\beta + \alpha)^n - (x \circ y)^{n+1} = 0 \tag{14}$$

and hence $\{(\beta + \alpha)^n - 1\}(x \circ y)^{n+1} = 0$ for all $x, y \in I$. This implies either $(\beta+\alpha)^n = 1$ or $(x \circ y)^n = 0$ for all $x, y \in I$. The last relation implies $(x \circ y)z = 0$ for all $x, y, z \in I$ (see [6, Lemma 2 (II)]). Now we assume first that $(\beta + \alpha)^n = 1$. Then multiplying $\beta + \alpha$ in (13), we have

$$(x \circ y)^n(b + \alpha) - (\beta + \alpha)(x \circ y)^n = 0 \tag{15}$$

for all $x, y \in I$, that is $(x \circ y)^n(b - \beta) = 0$ for all $x, y \in I$. Then again by [6], this relation yields either $b - \beta = 0$ that is $b = \beta \in C$ or $(x \circ y)z = 0$ for all $x, y, z \in I$. In case $(a - \alpha)I = 0$, $b = \beta \in C$ and $(\beta + \alpha)^n = 1$, we can write $F(x) = ax + xb = (a + \beta)x$ for all $x \in R$, with $(a - \alpha)I = 0$ and $(\beta + \alpha)^n = 1$. This gives our conclusion (2).

On the other hand, if $(x \circ y)z = 0$ i.e., $xyz = -yxz$ for all $x, y, z \in I$, then we have for all $x, y, z, u \in I$ that $uxyz = -xuyz = xyuz$. This implies $0 = [u, xy]z$. Replacing $y = yt$, it gives $0 = [u, xyt]z = xy[u, t]z + [u, xy]tz = xy[u, t]z$ for all $x, y, z, u, t \in I$. Since R is prime and I is a nonzero right ideal of R, this relation gives $[u, t]z = 0$ for all $z, u, t \in I$, i.e., $[I, I]I = 0$, which is our conclusion (1). ■

Now we are in a position to prove our main theorem for right ideals.

***Proof of Theorem* 2.** If F is inner generalized derivation of R, then by Lemma 9, we are done. Now let F be not inner. By Lee [22], we have $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U. Let $x, y \in I$. Then by [21], U satisfies

$$(a(xX \circ yY) + d(xX \circ yY))^n - (xX \circ yY)^n = 0$$

that is

$$(a(xX \circ yY) + (d(x)X + xd(X)) \circ yY) + (xX \circ (d(y)Y + yd(Y))))^n - (xX \circ yY)^n = 0.$$

Since F is not inner, d is also not inner derivation. Then by Kharchenko's Theorem [18], U satisfies

$$(a(xX \circ yY) + (d(x)X + xZ_1) \circ yY) + (xX \circ (d(y)Y + yZ_2)))^n - (xX \circ yY)^n = 0.$$

In particular for $X = 0$, we have $(xZ_1 \circ yY)^n = 0$ for all $Z_1, Y \in U$. In particular, $(x \circ y)^n = 0$ for all $x, y \in I$. Then by [6, Lemma 2 (II)], $(x \circ y)z = 0$ for all $x, y, z \in I$. Then by same argument as in Lemma 9, we conclude $[I, I]I = 0$, which is our conclusion (1).

Example. Let $R = \begin{pmatrix} GF(2) & GF(2) \\ 0 & GF(2) \end{pmatrix}$. We define maps $F, d : R \to R$, by $F\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. Then F is a generalized derivation associated with the derivation d of R. Note that R is not prime for $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$. We see that for $n = 2$ and $I = R$, $(F(x \circ y))^n = (x \circ y)^n$ for all $x, y \in I$. Since $[I, I]I \neq 0$ and $F(x) \neq \pm x$ for all $x \in R$, we conclude that the primeness hypothesis in Theorem 2 is not superfluous.

4. GENERALIZED DERIVATIONS ON SEMIPRIME RINGS AND BANACH ALGEBRAS

Now we prove our rest theorems in semiprime ring and Banach algebras.

Let R be a semiprime ring and U be its right Utumi quotient ring. It is well known that any derivation of a semiprime ring R can be uniquely extended to a derivation of U and so any derivation of R can be defined on the whole of U [21, Lemma 2].

By the standard theory of orthogonal completions for semiprime rings, we have the following lemma.

Lemma 10 ([3, Lemma 1 and Theorem 1] or [21, p. 31–32])**.** *Let R be a 2-torsion free semiprime ring and P a maximal ideal of C. Then PU is a prime ideal of U invariant under all derivations of U. Moreover, $\bigcap\{PU \mid P$ is a maximal ideal of C with U/PU 2-torsion free $\} = 0$.*

***Proof of Theorem* 3.** We known the fact that any derivation of a semiprime ring R can be uniquely extended to a derivation of its right Utumi quotient ring U and so any derivation of R can be defined on the whole of U [21, Lemma 2]. Moreover R and U satisfy the same GPIs (see [7]) as well as same differential identities (see [21]). Thus by [22], we have $F(x) = ax + d(x)$ for some $a \in U$, a derivation d on U and hence $(a(x \circ y) + d(x \circ y))^m = (x \circ y)^n$ for all $x, y \in U$.

Let $M(C)$ be the set of all maximal ideals of C and $P \in M(C)$. Now by the standard theory of orthogonal completions for semiprime rings (see [21, p.31-32]), we have PU is a prime ideal of U invariant under all derivations of U. Moreover, $\bigcap\{PU \mid P \in M(C)\ \} = 0$. Set $\overline{U} = U/PU$. Then derivation d canonically induces a derivation $\overline{d}$ on $\overline{U}$ defined by $\overline{d}(\overline{x}) = \overline{d(x)}$ for all $x \in U$. Therefore,

$$(\overline{a}(\overline{x} \circ \overline{y}) + \overline{d}(\overline{x} \circ \overline{y}))^m = (\overline{x} \circ \overline{y})^n$$

for all $\overline{x}, \overline{y} \in \overline{U}$. By Theorem 1 for prime ring case, we have for each $P \in M(C)$, either $[U, U] \subseteq PU$ or $[a, U] \subseteq PU$ and $d(U) \subseteq PU$. This gives that $[a, U][U, U] \subseteq PU$ for all $P \in M(C)$ and $d(U)[U, U] \subseteq PU$ for all $P \in M(C)$. Since $\bigcap\{PU \mid P \in M(C)\ \} = 0$, $[a, U][U, U] = 0$ and $d(U)[U, U] = 0$. In particular, $[a, R][R, R] = 0$ and $d(R)[R, R] = 0$. First case implies $a \in C$ and second case implies $d(R) \subseteq Z(R)$. Hence $F(x) = ax + d(x)$ for all $x \in R$, where $a \in C$ and d is a derivation of R such that $d(R) \subseteq Z(R)$.

By a Banach algebra, we shall mean a complex normed algebra A whose underlying vector space is Banach space. The Jacobson radical of A is the intersection of all primitive ideals of A and is denoted by $rad(A)$. Now we prove our theorem for Banach algebras.

***Proof of Theorem* 4.** By hypothesis F is continuous generalized derivation. Since we know that left multiplication map is continuous, we get that d is continuous. In [26], Sinclair proved that any continuous derivation of a Banach algebra leaves the primitive ideals invariant. Hence, for any primitive ideal P of A, it is obvious that $F(P) \subseteq aP + d(P) \subseteq P$. It means that continuous generalized derivation F leaves the primitive ideals invariant. Denote $A/P = \bar{A}$ for any primitive ideal P. Thus we can define the generalized derivation $F_p : \bar{A} \to \bar{A}$ by $F_p(\bar{x}) = F_p(x+P) = F(x)+P = ax+d(x)+P$ for all $\bar{x} \in \bar{A}$, where $A/P = \bar{A}$ is a factor Banach algebra. Since P is primitive ideal, the factor algebra $\bar{A}$ is primitive and so it is prime and semisimple. The hypothesis $(F(x \circ y))^m - (x \circ y)^n \in rad(A)$ yields that $(F_p(\bar{x} \circ \bar{y}))^m - (\bar{x} \circ \bar{y})^n = \bar{0}$ for all $\bar{x}, \bar{y} \in \bar{A}$. By Theorem 1, we have either $\bar{A}$ is commutative or $\bar{d} = \bar{0}$.

Assume first that $\bar{A}$ is commutative. By a result of Johnson and Sinclair [17] every linear derivation on a semisimple Banach algebra is continuous. Thus $\bar{d}$ is continuous in $\bar{A}$. In [27], Singer and Werner proved that any continuous linear derivation on a commutative Banach algebra maps the algebra into the radical. Hence $\bar{d} = \bar{0}$ in $\bar{A}$.

Therefore, in any case we have that $\bar{d} = \bar{0}$ in $\bar{A}$, that is $d(A) \subseteq P$ for any primitive ideal P of A and hence we get $d(A) \subseteq rad(A)$.

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