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CLIFFORD CONGRUENCES ON GENERALIZED QUASI-ORTHODOX GV-SEMIGROUPS

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Abstract

A semigroup S is said to be completely π-regular if for any $a \in S$ there exists a positive integer n such that a^n is completely regular. A completely π-regular semigroup S is said to be a GV-semigroup if all the regular elements of S are completely regular. The present paper is devoted to the study of generalized quasi-orthodox GV-semigroups and least Clifford congruences on them.

Keywords: Clifford semigroup, Clifford congruence, generalized quasi-orthodox semigroup.

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1. Introduction

The study of the structure of semigroups is essentially influenced by the study of the congruences defined on them. We know that the set of all congruences defined on a semigroup S is a partially ordered set with respect to inclusion and relative to this partial order it forms a lattice, the lattice of congruences on S. The study of lattice of congruences on different types of semigroups such as regular semigroups and eventually regular semigroups led to breakthrough innovations made by T.E. Hall [3], LaTorre [5], S.H. Rao and P. Lakshmi [10]. The congruences that they looked into were mostly group congruences. In paper [10], S.H. Rao and P. Lakshmi characterized group congruences on eventually regular semigroups in which they used self-conjugate subsemigroups. Further studies were continued by S. Sattayaporn [11] with weakly self-conjugate subsets.

Over the years, congruence structures have been an integral part of discussion in mathematics.

In this paper, we study various types of congruences on GV-semigroups. To be more precise, we characterize least Clifford congruences on generalized quasi-orthodox GV-semigroups.

2. Preliminaries

An element a in a semigroup $(S, \cdot)$ is said to be regular if there exists an element $x \in S$ such that $axa = a$. A semigroup $(S, \cdot)$ is said to be regular if every element of S is regular. In this case there also exists $y \in S$ such that $aya = a$ and $yay = y$. Such an element y is called an inverse of a. An element a in a semigroup $(S, \cdot)$ is said to be π-regular (or power regular) if there exists a positive integer n such that a^n is regular. Naturally, a semigroup $(S, \cdot)$ is said to be π-regular (or power regular) if every element of S is π-regular. An element a in a semigroup $(S, \cdot)$ is said to be completely regular if there exists an element $x \in S$ such that $a = axa$ and $ax = xa$. We know that an element a in a semigroup S is completely regular if and only if it belongs to a subgroup of G. We call a semigroup S, a completely regular semigroup if every element of S is completely regular.

An element a in a semigroup $(S, \cdot)$ is said to be completely π-regular if there exists a positive integer n such that a^n is completely regular. Naturally, a semigroup S is said to be completely π-regular if every element of S is completely π-regular.

Lemma 2.1 [7]**.** *Let S be a semigroup and let x be an element of S such that x^n belongs to a subgroup G of S for some positive integer n. Then, if e is the identity of G, we have*

(a) $ex = xe \in G$,

(b) $x^m \in G$ *for any integer* $m > n$.

Let a be a completely π-regular element in a semigroup S. Then a^n lies in a subgroup G of S for some positive integer n. The inverse of a^n in G is denoted by $(a^n)^{-1}$. From the above lemma, it follows that for a completely π-regular element a in a semigroup S, all its completely regular powers lie in the same subgroup of S. Let a^0 be the identity of this group and $\overline{a} = (aa^0)^{-1}$. Then clearly, $a^0 = a\overline{a} = \overline{a}a$ and $aa^0 = a^0a$.

Throughout this paper, we always let $E(S)$ be the set of all idempotents of the semigroup S. Also we denote the set of all inverses of a regular element a in a semigroup S by $V(a)$. For $a \in S$, by "a^n is a-regular" we mean that n is the smallest positive integer for which a^n is regular.

As usual, we denote the Green's relations [4] on the semigroup $(S,\cdot)$ by $\mathscr{L}$, $\mathscr{R}$, $\mathscr{D}$, $\mathscr{J}$ and $\mathscr{H}$. For any $a \in S$, we let H_a be the $\mathscr{H}$-class in S containing a. If $(S,\cdot)$ be a π-regular semigroup, we consider the relations $\mathscr{L}^*, \mathscr{R}^*$, $\mathscr{J}^*, \mathscr{H}^*$ and $\mathscr{D}^*$ defined by

$$a\,\mathscr{L}^*\,b \text{ if and only if } a^p\,\mathscr{L}\,b^q\,,$$
$$a\,\mathscr{R}^*\,b \text{ if and only if } a^p\,\mathscr{R}\,b^q\,,$$
$$a\,\mathscr{J}^*\,b \text{ if and only if } a^p\,\mathscr{J}\,b^q\,,$$
$$\mathscr{H}^* = \mathscr{L}^* \cap \mathscr{R}^* \qquad \text{and} \qquad \mathscr{D}^* = \mathscr{L}^*\, o\, \mathscr{R}^*$$

where a^p is a-regular and b^q is b-regular.

A semigroup $(S,\cdot)$ is said to be a band if each element of S is idempotent, i.e., $a^2 = a$ for all $a \in S$. A commutative band is called a semilattice. A band S is said to be a rectangular band if it satisfies the identity $axa = a$ for all $a, x \in S$. A congruence ρ on a semigroup S is called a semilattice congruence if S/ρ is a semilattice. A semigroup S is called a semilattice Y of semigroups $S_\alpha (\alpha \in Y)$ if S admits a semilattice congruence ρ on S such that $Y = S/\rho$ and each S_α is a ρ-class mapped onto α by the natural epimorphism $\rho^{\#} : S \longrightarrow Y$. We write $S = (Y; S_\alpha)$. For other notations and terminologies not given in this paper, the reader is referred to the texts of Bogdanovic [1] and Howie [4].

3. Completely Archimedean semigroups and GV-semigroups

In this section we recall some definitions and state some of their important properties.

Definition 3.1. A semigroup S is said to be an Archimedean semigroup if for any two elements $a, b \in S$ there exists a positive integer n such that $a^n \in SbS$.

Definition 3.2. An Archimedean semigroup S is said to be completely Archimedean if it is completely π-regular.

Definition 3.3. Let I be an ideal of a semigroup S. We define a relation ρ_I on S by $a\rho_I b$ if and only if either $a, b \in I$ or $a = b$ where $a, b \in S$. It is easy to verify that ρ_I is a congruence on S. This congruence is said to be Rees congruence on S and the quotient semigroup S/ρ_I contains a zero, namely I. This quotient semigroup S/ρ_I is said to be the Rees quotient semigroup and is denoted by S/I. In this case the semigroup S is said to be an ideal extension or simply an extension of I by the semigroup S/I. An ideal extension S of a semigroup I is said to be a nil-extension of I if S/I is nil semigroup, i.e., for any $a \in S$ there exists a positive integer n such that $a^n \in I$.

Theorem 3.4 [1]**.** *The following conditions on a semigroup are equivalent:*

(i) *S is a completely Archimedean semigroup;*

(ii) *S is a nil-extension of a completely simple semigroup.*

In a completely π-regular semigroup, regular elements may not be completely regular. A completely π-regular semigroup in which every regular element is completely regular is said to be a GV-semigroup. A completely π-regular semigroup containing a single idempotent is called a π-group. It is well known that a semigroup S is a π-group if and only if S is nil-extension of a group. The following theorem gives the complete characterization of GV-semigroups.

Theorem 3.5 [1]**.** *The following conditions on a semigroup S are equivalent:*

(i) *S is a GV-semigroup;*

(ii) *S is π-regular and every $\mathscr{H}^*$-class of S is a π-group;*

(iii) *S is semilattice of completely Archimedean semigroups.*

In this connection, it is interesting to mention that $\mathscr{J}^*$ is a semilattice congruence on a GV-semigroup S and each $\mathscr{J}^*$-class is a completely Archimedean subsemigroup of S.

4. Generalized quasi-orthodox GV-semigroups

In this section we define generalized quasi-orthodox semigroups and study some important properties of generalized quasi-orthodox GV-semigroups.

Definition 4.1. A semigroup $(S, \cdot)$ is said to be an orthodox semigroup if $E(S)$ forms a subsemigroup of S.

Definition 4.2. A semigroup $(S, \cdot)$ is said to be a generalized quasi-orthodox semigroup if for any two elements $e, f \in E(S)$, there exists a positive integer n such that $(ef)^n = (ef)^{n+1}$.

Clearly, every orthodox semigroup is generalized quasi-orthodox. But the converse is not true in general. We cite some examples to ensure that generalized quasi-orthodox semigroup may not be an orthodox semigroup.

Example 4.3 [6]**.** Let $S = \{e, f, a, 0\}$. On S we define a multiplication $'\cdot'$ with the following Cayley table:

$\cdot$	e	f	a	0
e	e	a	a	0
f	0	f	0	0
a	0	a	0	0
0	0	0	0	0

Then $(S, \cdot)$ is a generalized quasi-orthodox semigroup but not an orthodox.

Example 4.4. Let $S = \{0, 1, 2, 3, 4, 5\}$. On S we define a multiplication $'\cdot'$ with the following Cayley table:

$\cdot$	0	1	2	3	4	5
0	0	2	2	3	0	0
1	3	1	3	3	1	5
2	3	2	3	3	2	0
3	3	3	3	3	3	3
4	0	1	2	3	4	5
5	5	1	1	3	5	5

Then $(S, \cdot)$ is a semigroup with $E(S) = \{0, 1, 3, 4, 5\}$. Here, $0 \cdot 1 = 2 \notin E(S)$. Hence S is not an orthodox semigroup. But, one can easily verify that $(S, \cdot)$ is a generalized quasi-orthodox semigroup.

Remark 4.5. A completely π-regular semigroup S is generalized quasi-orthodox if and only if for any $e, f \in E(S)$, $(ef)(ef)^0 = (ef)^0$.

Lemma 4.6. *Let S be a GV-semigroup; $e, f \in E(S)$, $g = (ef)^0 e$ and $h = f(ef)^0$. Then $ef(ef)^0 = gh$, $g^2 = g$, $h^2 = h$ and $(ef) \mathscr{J}^* g \mathscr{J}^* h$.*

Proof. Firstly, $gh = (ef)^0(ef)(ef)^0 = (ef)(ef)^0$. Again, $g^2 = (ef)^0 e (ef)^0 e = (ef)^0 e = g$. Similarly, $h^2 = h$. Since S is a semilattice of its $\mathscr{J}^*$-classes [1], it follows that $(ef) \mathscr{J}^* g \mathscr{J}^* h$. ∎

Definition 4.7. A GV-semigroup S is said to be generalized quasi-orthodox GV-semigroup if S is a generalized quasi-orthodox semigroup.

Theorem 4.8. *Let $S = (Y; S_\alpha)$ be a GV-semigroup, where Y is a semilattice and $S_\alpha\,(\alpha \in Y)$ is a completely Archimedean semigroup. Then the following conditions are equivalent:*

(i) *S is generalized quasi-orthodox;*

(ii) *For all $\alpha \in Y$, S_α is orthodox;*

(iii) *For all $e \in E(S)$ and for all $x \in S$, there exist $m, n \in \mathbb{N}$ such that*

$$\left(x^{m-1}(x^m)^{-1}ex\right)^n = \left(x^{m-1}(x^m)^{-1}ex\right)^{n+1};$$

(iv) *For all $a, b \in S$ there exists a positive integer n such that $(a^0 b^0)^n = (a^0 b^0)^{n+1}$.*

Proof. (i)$\Longleftrightarrow$(ii) This follows from Theorem X:2.1 [1].

(i)$\Rightarrow$(iii) For $x \in S$ there exists a positive integer m such that x^m is x-regular. Let $e \in E(S)$. As S is quasi-orthodox and $x^0 \in E(S)$, then there exists a positive integer n such that $(ex^0)^{n-1} = (ex^0)^n$. Therefore,

$$\begin{aligned}\left(x^{m-1}(x^m)^{-1}ex\right)^n &= x^{m-1}(x^m)^{-1}(ex^0)^{n-1}ex \\ &= x^{m-1}(x^m)^{-1}(ex^0)^n ex \\ &= \left(x^{m-1}(x^m)^{-1}ex\right)^{n+1}.\end{aligned}$$

Hence, for all $e \in E(S)$ and for all $x \in S$, there exist positive integers $m, n \in \mathbb{N}$ such that $(x^{m-1}(x^m)^{-1}ex)^n = (x^{m-1}(x^m)^{-1}ex)^{n+1}$.

(iii)$\Rightarrow$(i) Let $e, f \in E(S)$. Then by the given condition we have, $(fef)^n = (fef)^{n+1}$, for some positive integer n.

Now, $(ef)^{n+1} = e(fef)^n = e(fef)^{n+1} = (ef)^{n+2}$. Hence, S is generalized quasi-orthodox.

(i)$\Rightarrow$(iv) Since S is generalized quasi-orthodox and for all $a, b \in S$, $a^0, b^0 \in E(S)$, hence there exists a positive integer n such that $(a^0b^0)^n = (a^0b^0)^{n+1}$.

(iv)$\Rightarrow$(i) This part is obvious. ■

Lemma 4.9. *Let $S = (Y; S_\alpha)$ be a generalized quasi-orthodox GV-semigroup, where Y is a semilattice and S_α $(\alpha \in Y)$ is a completely Archimedean semigroup. Then for all $\alpha \in Y$, $E(S_\alpha)$ is a rectangular band and for any two elements $a, b \in S_\alpha$, $e \in E(S_\beta)$, $a^0b^0 = a^0eb^0$, where $\alpha, \beta \in Y$ with $\alpha \leq \beta$.*

Proof. Since each S_α is a completely Archimedean semigroup, then S_α has a completely simple kernel K_α. As every element $e \in E(S_\alpha)$ is completely regular, it follows that $e \in E(K_\alpha)$ and thus $E(S_\alpha) = E(K_\alpha)$. Since K_α is completely simple and orthodox, so $E(K_\alpha)$ is a rectangular band (Corollary III.5.3 [9]). Hence $E(S_\alpha)$ is a rectangular band.

Now a^0, a^0e, $b^0 \in K_\alpha$. Since S is generalized quasi-orthodox, so there exists a positive integer n such that $(a^0e)^n = (a^0e)^{n+1}$. Here $(a^0e)^n \in E(K_\alpha)$.

Therefore,

$$\begin{aligned}a^0b^0 &= a^0(a^0e)^n b^0 \\ &= a^0(a^0e)^{n+1} b^0 \\ &= a^0(a^0e)^n a^0eb^0 \\ &= a^0eb^0.\end{aligned}$$

■

Next we prove a very important result on generalized quasi-orthodox GV-semigroup.

Theorem 4.10. *Let $S = (Y; S_\alpha)$ be a generalized quasi-orthodox GV-semigroup, where Y is a semilattice and $S_\alpha\,(\alpha \in Y)$ is a completely Archimedean semigroup. Let $a \in S_\alpha$ be a completely regular element and $b \in S_\beta$ where $\alpha \leq \beta$. Then ab is completely regular.*

Proof. For all $\alpha \in Y$, S_α is the nil-extension of its kernel K_α, that is completely simple.

Clearly, $(ab) \in S_\alpha$. We show that $(ab) \in K_\alpha$. As S is a GV-semigroup, then there exists a positive integer n such that $(ab)^n$ is completely regular and $(ab)^n$ is in a subgroup $G \subseteq K_\alpha$. Let g be the identity of G. Then $abg = gab \in G$.

Now,

$$\begin{aligned} ab = &\ a^0ab, \\ = &\ (a^0ga^0)ab, \end{aligned}$$

since $E(S_\alpha)$ is a rectangular band and $a^0, g \in E(S_\alpha)$.

Therefore,

$$\begin{aligned} ab = &\ (a^0ga^0)ab \\ = &\ a^0(ga^0ab) \\ = &\ a^0(gab) \in K_\alpha, \end{aligned}$$

since $a^0 \in K_\alpha$, $gab \in G \subseteq K_\alpha$. Consequently, ab is completely regular. ■

5. LEAST CLIFFORD CONGRUENCES

In this section, we characterize least Clifford congruences on generalized quasi-orthodox GV-semigroups. For this purpose we define a relation ν on a GV-semigroup and finally we establish a necessary and sufficient condition for ν to be a least Clifford congruence on a GV-semigroup.

Recall that a regular semigroup in which idempotents are central is said to be a Clifford semigroup. It is interesting to mention that a semigroup S is a Clifford semigroup if and only if S is semilattice of groups.

In order to characterize further the least Clifford congruence on a GV-semigroup, we define the following relation ν.

Definition 5.1. Let $S = (Y; S_\alpha)$ be GV-semigroup, where Y is a semilattice and $S_\alpha\,(\alpha \in Y)$ is a completely Archimedean semigroup.

On S we define a relation ν as follows. For $a, b \in S$,

$$a\,\nu\,b \text{ if and only if } aa^0 = a^0bb^0a^0 \text{ and } bb^0 = b^0aa^0b^0.$$

Theorem 5.2. *Let $S = (Y; S_\alpha)$ be GV-semigroup, where Y is a semilattice and $S_\alpha\,(\alpha \in Y)$ is a completely Archimedean semigroup. Then the relation ν, as* *defined in Definition* 5.1 *is the least Clifford congruence on* S *if and only if* S *is generalized quasi-orthodox.*

Proof. Let $S = (Y; S_\alpha)$ be a generalized quasi-orthodox GV-semigroup. We prove that ν is the least Clifford congruence on S.

We first show that ν is an equivalence relation on S. Clearly, ν is reflexive and symmetric.

Let $a \,\nu\, b$ and $b \,\nu\, c$ holds for $a, b, c \in S$. Then $a, b, c \in S_\alpha$ for some $\alpha \in Y$.

Now, $a \,\nu\, b$ implies $aa^0 = a^0bb^0a^0$ and $bb^0 = b^0aa^0b^0$. Also, $b \,\nu\, c$ implies $bb^0 = b^0cc^0b^0$ and $cc^0 = c^0bb^0c^0$.

Therefore,

$$aa^0 = a^0b^0cc^0b^0a^0,$$

i.e.,

$$\begin{aligned} c^0aa^0c^0 &= c^0a^0b^0cc^0b^0a^0c^0 \\ &= c^0a^0b^0cc^0, \quad \text{[since } E(S_\alpha) \text{ is a rectangular band]} \\ &= c^0a^0b^0c^0c, \\ &= c^0c, \\ &= cc^0. \end{aligned}$$

Similarly, $aa^0 = a^0cc^0a^0$. Thus, $a \,\nu\, c$. Hence ν is an equivalence relation.

Let $a \,\nu\, b$ and $c \in S$. Let $a, b \in S_\alpha$ and $c \in S_\beta$. Therefore, $aa^0 = a^0bb^0a^0$ and $bb^0 = b^0aa^0b^0$. By using Lemma 4.9., we have

$$\begin{aligned} (ca)^0(cb)(cb)^0(ca)^0 &= (ca)^0(cb)b^0(cb)^0(ca)^0 \\ &= (ca)^0cb^0aa^0b^0(cb)^0(ca)^0 \\ &= (ca)^0caa^0b^0(ca)^0 \\ &= (ca)^0caa^0(ca)^0 \\ &= (ca)^0ca(ca)^0 \\ &= (ca)(ca)^0. \end{aligned}$$

Similarly, $(cb)^0(ca)(ca)^0(cb)^0 = (cb)(cb)^0$. Therefore, $(ca) \,\nu\, (cb)$. Dually, we can obtain, $(ac) \,\nu\, (bc)$. Consequently, ν is a congruence on S.

Now, for any $a \in S$, $e \in E(S)$ we have $e(ae)^0 \in E(S)$.

Also,

$$\begin{aligned} (ea)^0(ae)(ae)^0(ea)^0 &= (ea)^0(ae)(ea)^0 \\ &= (ea)^0e(ae)(ea)^0 \\ &= (ea)^0(ea)(ea)^0 \\ &= (ea)(ea)^0. \end{aligned}$$

Similarly, $(ae)^0(ea)(ea)^0(ae)^0 = (ae)(ae)^0$. Therefore, $(ae) \,\nu\, (ea)$. Thus, we conclude that ν is a Clifford congruence on S.

Now, we verify that ν is the least Clifford congruence on S.

Let ρ be any other Clifford congruence on S and $a\,\nu\,b$, for $a, b \in S$. Then, $(aa^0)\,\nu\,(bb^0)$. Now, $aa^0 = a^0bb^0a^0 = (a^0b^0aa^0b^0a^0)\,\rho\,(b^0a^0aa^0a^0b^0) = b^0aa^0b^0 = bb^0$. Hence, $(aa^0)\,\rho\,(bb^0)$. Now, $a\,\rho\,(aa^0)\,\rho\,(bb^0)\,\rho\,b$ implies, $\nu \subseteq \rho$. Hence, ν is the least Clifford congruence on S.

To prove the converse, let ν be the least Clifford congruence on a GV-semigroup S. We show that S is generalized quasi-orthodox. Now, let $e, f \in E(S)$. Then, by definition, $(ef)\,\nu\,(fe)$. This implies, $(ef)\nu = (fe)\nu$. Let $(ef)^n$ be (ef)-regular. Then, $(ef)^{n+1}\nu = (fe)\nu(ef)^n\nu$, i.e., $(ef)^{n+1}\nu = (ef)^n\nu$. This implies $((ef)(ef)^0)\nu = (ef)^0\nu$, i.e., $(ef)(ef)^0 = (ef)^0$, i.e., $(ef)^{n+1} = (ef)^n$. This proves that S is generalized quasi-orthodox. ■

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