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QUOTIENT HYPER PSEUDO BCK-ALGEBRAS

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Abstract

In this paper, we first investigate some properties of the hyper pseudo BCK-algebras. Then we define the concepts of strong and reflexive hyper pseudo BCK- ideals and establish some relationships among them and the other types of hyper pseudo BCK- ideals. Also, we introduce the notion of regular congruence relation on hyper pseudo BCK-algebras and investigate some related properties. By using this relation, we construct the quotient hyper pseudo BCK-algebra and give some related results.

Keywords: hyper pseudo BCK-algebra, normal hyper pseudo BCK-ideal, quotient hyper pseudo BCK-algebra.

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1. Introduction

The study of BCK-algebra initiated by Y. Imai and Iseki [10] in 1966 as a generalization of the concept of set theoretic difference and propositional calculi. In order to extend BCK-algebras in a non-commutative form, Georgescu and Iorgulesu [6] introduced the notion of pseudo BCK-algebras and studied their properties. The hyper structure theory (called also multi algebras) was introduced in 1934 by F. Marty [13] at the 8th congress of Scandinavian Mathematicians. Since then many researchers have worked on algebraic hyper structures and developed it. A recent book [5] contains a wealth of applications. Via this book, Corsini and Leoreanu presented some of the numerous applications of algebraic hyper structure, especially those from last fifteen years, to the following subjects: geometry, hyper graphs, binary relations, lattices, fuzzy sets and rough sets, automate, cryptography, cods, median algebras, relation algebras, artificial intelligence and probabilities. Hyper structures have many applications to several sectors of both pure and applied sciences. In [3, 12] R.A. Borzooei et al. applied the hyper structures to (pseudo) BCK- algebra, and introduced the notion of a (pseudo) hyper BCK-algebra which is a generalization of (pseudo) BCK-algebra and investigated some related properties. In this paper, we introduce some properties of pseudo hyper BCK-algebras. Also, we define the concepts of strong and reflexive hyper pseudo BCK-ideals of a hyper pseudo BCK-algebra and investigate some related properties. We follow [1] to define the notion of regular congruence relation, and then introduce the concept of quotient hyper pseudo BCK-algebra. Moreover, we show that $[0]_\rho$ is a reflexive hyper pseudo BCK-ideal of H if and only if $\frac{H}{\rho}$ is a pseudo BCK-algebra.

2. Preliminaries

Definition 2.1 [14]**.** Let X be a set with a binary operation "$*$" and a constant 0. Then $(X; *, 0)$ is called a BCK-algebra if it satisfies the following conditions:

(BCI-1) $((x * y) * (x * z)) * (z * y) = 0$,

(BCI-2) $(x * (x * y)) * y = 0$,

(BCI-3) $x * x = 0$,

(BCI-4) $x * y = 0$ and $y * x = 0$ imply $x = y$,

(BCK-5) $0 * x = 0$ for all $x, y, z \in X$.

Definition 2.2 [12]**.** By a hyper BCK-algebra we mean a nonempty set H endowed with hyperoperation "$\circ$" and a constant 0 satisfy the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$, (HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(HK3) $x \circ y \ll x$,

(HK4) $x \ll y$ and $y \ll x$ imply $x = y$

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A, \exists b \in B$ such that $a \ll b$.

Definition 2.3 [6]**.** A pseudo BCK-algebra is a structure $(X; *, \diamond, 0)$ where "$*$" and "$\diamond$" are binary operations on X and "0" is a constant element of X, that satisfies the following for all $x, y \in X$:

$(a-1)$ $(x * y) \diamond (x * z) \ll y$, $(x \diamond y) * (x \diamond z) \ll z \diamond y$,

$(a-2)$ $x * (x \diamond y) \ll y$, $x \diamond (x * y) \ll y$,

$(a-3)$ $x \ll x$,

$(a-4)$ $0 \ll x$,

$(a-5)$ $x \ll y, y \ll x$ implies $x = y$,

$(a-6)$ $x \ll y \Leftrightarrow x * y = 0 \Leftrightarrow x \diamond y = 0$.

Definition 2.4 [3]**.** A hyper pseudo BCK-algebra is a structure $(H; \circ, *, 0)$ where "$\circ$" and"$*$" are hyperoperations on H and "0" is a constant element that satisfy the following axioms:

(PHK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$, $\quad (x * z) * (y * z) \ll x * y$,

(PHK2) $(x \circ y) * z = (x * z) \circ y$,

(PHK3) $x \circ y \ll x$, $\quad x * y \ll x$,

(PHK4) $x \ll y$ and $y \ll x$ imply $x = y$

for all $x, y, z \in H$, where $x \ll y \Leftrightarrow 0 \in x \circ y \Leftrightarrow 0 \in x * y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A, \exists b \in B$ such that $a \ll b$.

Theorem 2.5 [3]**.** *Any BCK-algebra and hyper BCK-algebra is a hyper pseudo BCK-algebra.*

Proposition 2.6 [3]**.** *In any hyper pseudo BCK-algebra H, the following holds for all $x, y, z, c \in H$ and $A, B \subseteq H$:*

(i) $0 \circ 0 = 0$, $\quad 0 * 0 = 0 \quad x \circ 0 = x$, $\quad x * 0 = x$,

(ii) $0 \ll x$, $\quad x \ll x$, $\quad A \ll A$,

(iii) $0 \circ x = 0$, $\quad 0 * x = 0$, $\quad 0 \circ A = 0$, $\quad 0 * A = 0$,

(iv) $A \subseteq B$ *implies* $A \ll B$,

(v) $A \ll 0$ *implies* $A = \{0\}$,

(vi) $y \ll z$ *implies* $x \circ z \ll x \circ y$ *and* $x * z \ll x * y$,

(vii) $x \circ y = \{0\}$ *implies* $(x \circ z) \circ (y \circ z) = \{0\}$, *that is,* $x \circ z \ll y \circ z$; $x * z = \{0\}$ *implies* $(x * z) * (y * z) = \{0\}$, *that is,* $x * z \ll y * z$,

(viii) $A \circ \{0\} = \{0\}$ *implies* $A = \{0\}$, *and* $A * \{0\} = \{0\}$ *implies* $A = \{0\}$,

(ix) $(A \circ c) \circ (B \circ c) \ll A \circ B$, $(A * c) * (B * c) \ll A * B$.

Definition 2.7 [3]**.** Let H be a hyper pseudo BCK-algebra. For any subset I of H and any element $y \in H$, we denote,

(1) $*(y, I)^{\ll} = \{x \in H | x * y \ll I\}$,

(2) $*(y, I)^{\subseteq} = \{x \in H | x * y \subseteq I\}$,

(3) $\circ(y, I)^{\ll} = \{x \in H | x \circ y \ll I\}$,

(4) $\circ(y, I)^{\subseteq} = \{x \in H | x \circ y \subseteq I\}$.

Definition 2.8 [3]**.** Let H be a hyper pseudo BCK-algebra, $\emptyset \neq I \subseteq H$ and $0 \in I$. Then I is said to be a hyper pseudo BCK-ideal of

(i1) type (1), if for any $y \in I, *(y, I)^{\ll} \subseteq I$ and $\circ(y, I)^{\ll} \subseteq I$;

(i2) type (2), if for any $y \in I, *(y, I)^{\subseteq} I$ and $\circ(y, I)^{\ll} \subseteq I$;

(i3) type (3), if for any $y \in I, *(y, I)^{\ll} \subseteq I$ and $\circ(y, I)^{\subseteq} \subseteq I$;

(i4) type (4), if for any $y \in I, *(y, I)^{\subseteq} \subseteq I$ and $\circ(y, I)^{\subseteq} \subseteq I$;

(i5) type (5), if for any $y \in I, *(y, I)^{\ll} \subseteq I$ or $\circ(y, I)^{\ll} \subseteq I$;

(i6) type (6), if for any $y \in I, *(y, I)^{\subseteq} \subseteq I$ or $\circ(y, I)^{\ll} \subseteq I$;

(i7) type (7), if for any $y \in I, *(y, I)^{\ll} \subseteq I$ or $\circ(y, I)^{\subseteq} \subseteq I$;

(i8) type (8), if for any $y \in I, *(y, I)^{\subseteq} \subseteq I$ or $\circ(y, I)^{\subseteq} \subseteq I$;

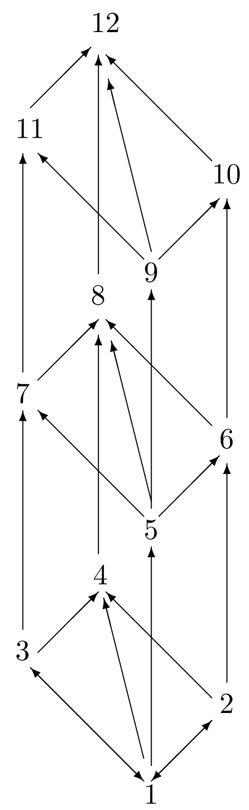
(i9) type (9), if for any $y \in I, *(y, I)^{\ll} \cap \circ(y, I)^{\ll} \subseteq I$;

(i10) type (10), if for any $y \in I, *(y, I)^{\subseteq} \cap \circ(y, I)^{\ll} \subseteq I$;

(i11) type (11), if for any $y \in I, *(y, I)^{\ll} \cap \circ(y, I)^{\subseteq} \subseteq I$;

(i12) type (12), if for any $y \in I, *(y, I)^{\subseteq} \cap \circ(y, I)^{\subseteq} I$.

Note 1. The relationship among all of types of hyper pseudo BCK-ideals is given by the following diagram (see [3]).



Note. From now on, in this paper we let H be a hyper pseudo BCK-algebra.

3. Some properties of hyper pseudo BCK-algebra

Proposition 3.1. *Let A be a subset of H, and $x, y, z \in H$. If $(x \circ y) * z \ll A$, then $a * z \ll A$ for all $a \in x \circ y$. Similarly, if $(x * y) \circ z \ll A$, then $a \circ z \ll A$ for all $a \in x * y$.*

Proof. For every $a \in x \circ y$, we have $a * z \subseteq (x \circ y) * z \ll A$. This implies $a * z \ll A$. The proof of the other case is similar. ■

Definition 3.2. Let S be a nonempty subset of H. If S is a hyper pseudo BCK-algebra with respect to both the hyperoperations "$\circ$" and "$*$" on H, then we say that S is a hyper subalgebra of H.

Proposition 3.3. *Let S be a nonempty subset of H. Then S is a hyper subalgebra of H if and only if both $x \circ y \subseteq S$ and $x * y \subseteq S$ for all $x, y \in S$.*

Proof. If S is a hyper subalgebra of H, then S is closed with respect to the hyperoperations "$\circ$" and "$*$" and so $x \circ y \subseteq S$ and $x * y \subseteq S$ for all $x, y \in S$. Conversely, let $x \circ y \subseteq S$ and $x * y \subseteq S$ for all $x, y \in S$. It is clear that S satisfies the axioms of hyper pseudo BCK-algebra H. Hence we need only to show that $0 \in S$. Since S is a nonempty subset of H, there exists $a \in S$. By Proposition 2.6 (ii), we have $a \ll a$, that is, $0 \in a \circ a$. Since $a \circ a \subseteq S$, we get $0 \in S$. ■

Definition 3.4. We define the following subsets of H:

$$S_*(H) = \{x \in H |\ x * x = \{0\}\}, S_\circ(H) = \{x \in H |\ x \circ x = \{0\}\}$$

$$S(H) = \{x \in H |\ x \circ x = x * x = \{0\}\} = S_\circ(H)\ \cap S_*(H).$$

Note that the equality $S_*(H) = S_\circ(H)$ is not true in general, as shown in the following example.

Example 3.5. Let $H = \{0, a, b\}$. Hyperoperations "$*$" and "$\circ$" given by the following tables:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a, b\}$	$\{0, b\}$

$*$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a\}$	$\{0\}$

Then, $(H; \circ, *, 0)$ is a hyper pseudo BCK-algebra. It is easy to check that $a \in S_*(H)$ but $a \notin S_\circ(H)$.

Theorem 3.6.

(i) *$S_*(H)$ and $S_\circ(H)$ are closed with respect to "$*$" and "$\circ$", respectively,*

(ii) *If $S_*(H) = H$, then $x * y$ is singleton for all $x, y \in H$,*

(iii) *If $S_\circ(H) = H$, then $x \circ y$ is singleton for every $x, y \in H$,*

(iv) *If $S(H) = H$, then H is a pseudo BCK-algebra,*

(v) *If H satisfies the conditions, $(x \circ y) * (x \circ z) \ll z \circ y$ and $(x * y) \circ (x * z) \ll z * y$ for all $x, y, z \in H$, then $S(H) = H$ and so it is a pseudo BCK-algebra.*

Proof. (i) Let $x, y \in S_*(H)$. Then $x * x = \{0\}$ and so for any $a \in x \circ y$, we have $a * a \subseteq (x * y) * (x * y) \ll x * x = \{0\}$. Thus $a * a \ll \{0\}$, that is, $a * a = \{0\}$. Therefore $a \in S_*(H)$ and so $x * y \subseteq S_*(H)$. Similarly, we can show that $x \circ y \subseteq S_\circ(H)$ for all $x, y \in S_\circ(H)$.

(ii) Let $a, b \in x * y$ for some $x, y \in H$. Then $a * b \subseteq (x * y) * (x * y) \ll x * x = \{0\}$. Therefore $a * b \ll \{0\}$ and so by Proposition 2.6(v), $a * b = \{0\}$. Consequently, $a \ll b$. Similarly, we have $b \ll a$. Hence $b = a$ and so $x * y$ is singleton for all $x, y \in H$.

(iii) The proof is similar to the proof of (ii).

(iv) By (ii) and (iii), $x * y$ and $x \circ y$ are singleton for all $x, y \in H$. This implies H is a pseudo BCK-algebra.

(v) We show that $x \circ x = \{0\} = x * x$ for all $x \in H$. By putting $y = z = 0$ in the hypothesis, we get $(x \circ 0) * (x \circ 0) \ll 0 \circ 0 = \{0\}$ and so $(x \circ 0) * (x \circ 0) = \{0\}$. Hence, since $x \circ 0 = \{x\}$, we obtain $x * x = \{0\}$ for all $x \in H$. Similarly, we can show that $x \circ x = \{0\}$ for all $x \in H$. Therefore $H = S(H)$ and so H is a pseudo BCK-algebra. ∎

4. STRONG HYPER PSEUDO BCK-IDEALS

Proposition 4.1. *Let $A \subseteq H$. If I is a hyper pseudo BCK-ideal of type* $1, 2, 3, 5$ *or* 9 *such that $A \ll I$, then $A \subseteq I$.*

Proof. Let $A \subseteq H$ and I be a hyper pseudo BCK-ideal of type 1,2,3,5 such that $A \ll I$. If $*(y, I)^{\ll} \subseteq I$ and $a \in A$, then $a * 0 = \{a\} \subseteq A \ll I$. Hence $a * 0 \ll I$. Since $0 \in I$, we get $a \in *(0, I)^{\ll}$. Now, since $*(0, I)^{\ll} \subseteq I$, we have $a \in I$ and so $A \subseteq I$. If $\circ(y, I)^{\ll} \subseteq I$, then, similarly, $A \subseteq I$. Now, let I be a hyper pseudo BCK-ideal of type 9 and $a \in A$. Since $a * 0 = \{a\} \subseteq A$ and $A \ll I$, we get $a * 0 \ll I$ and so $a \in *(0, I)^{\ll}$. Similarly, $a \circ 0 = \{a\} \subseteq A$ implies $a \in \circ(0, I)^{\ll}$. Hence $a \in *(0, I)^{\ll} \cap \circ(0, I)^{\ll}$. By the definition of hyper pseudo BCK-ideal of type 9, we have $*(0, I)^{\ll} \cap \circ(0, I)^{\ll} \subseteq I$ and so $a \in I$. Therefore $A \subseteq I$. ∎

The following examples show that Proposition 4.1 is not true for any hyper pseudo BCK-ideals of type 4,6,7,8,10,11 and 12, in general.

Example 4.2.

(i) Let $H = \{0, a, b\}$. Hyperoperations " $*$ " and " $\circ$ " on H given by the following tables:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

$*$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{b\}$	$\{0, a\}$

Then $(H; \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can check that if $I := \{0, b\}$ and $A := \{0, a\}$, then I is a hyper pseudo BCK-ideal of type 4 and $A \ll I$. But $A \not\subseteq I$.

(ii) Let $H = \{0, a, b\}$. Hyperoperations "$*$" and "$\circ$"on H given by the following tables:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

$*$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,a,b\}$

Then $(H;\circ,*,0)$ is a hyper pseudo BCK-algebra, $I := \{0,b\}$ is a hyper pseudo idealof type 6 and $A := \{a\} \ll I$. But $A \nsubseteq I$.

(iii) Let $H = \{0,a,b\}$. Hyperoperations " $*$ " and " $\circ$ " on H given by the following tables:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{a,b\}$	$\{0,b\}$

$*$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0,a,b\}$

Then $(H;\circ,*,0)$ is a hyper pseudo BCK-algebra, $I := \{0,b\}$ is a hyper pseudo idealof type 7 and $A := \{a\} \ll I$. But $A \nsubseteq I$.

(iv) Let $H = \{0,a,b\}$. Hyperoperations "$*$" and "$\circ$"on H given by the following tables:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

$*$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,a\}$

Then $(H;\circ,*,0)$ is a hyper pseudo BCK-algebra, $I := \{0,b\}$ is a hyper pseudo ideal of type 8 and $A := \{0,a\} \ll I$. But $A \nsubseteq I$.

(v) Let $H = \{0,a,b,c\}$. Hyperoperations " $*$ " and " $\circ$ " on H given by the following tables:

$*$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{b\}$	$\{0\}$

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0,b\}$	$\{0,b\}$
c	$\{c\}$	$\{c\}$	$\{b,c\}$	$\{0,c\}$

Then $(H;\circ,*,0)$ is a hyper pseudo BCK-algebra. We can check that $I := \{0,b\}$ is a hyper pseudo BCK-ideal of type 10 and $A := \{0,a\} \ll I$ but $A \nsubseteq I$.

(vi) Let $H = \{0, a, b, c\}$. Hyperoperations " $*$ " and " $\circ$ " on H given by the following tables:

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{b\}$	$\{0\}$

$*$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,b\}$	$\{0,b\}$
c	$\{c\}$	$\{c\}$	$\{b,c\}$	$\{0,c\}$

Then $(H; \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can check that $I := \{0, b\}$ is a hyper pseudo BCK- ideal of type 11 and $A := \{0, a\} \ll I$ but $A \nsubseteq I$.

(vii) Let $H = \{0, a, b, c\}$. Hyperoperations " $*$ " and " $\circ$ " on H given by the following tables:

$*$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	$\{0,c\}$

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0,b\}$	$\{0,b\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	$\{0,c\}$

Then $(H; \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can check that $I := \{0, c\}$ is a hyper pseudo BCK-ideal of type 12 and $A := \{a, b\} \ll I$ but $A \nsubseteq I$.

Proposition 4.3. *Every hyper pseudo BCK-ideal of type* $1, 2, 3, 5$ *or* 9 *of H is a hyper subalgebra of H.*

Proof. Let I be a hyper pseudo BCK-ideal of type $1, 2, 3, 5$ or 9 of H. It suffices to show that $x * y$, $x \circ y \subseteq I$ for any $x, y \in I$. Let $a \in x * y$. Since $x * y \ll \{x\}$, we get $a \ll x \in I$. Hence by Proposition 4.1, $a \in I$. Therefore $x * y \subseteq I$. Similarly, we can show that $x \circ y \subseteq I$. Therefore I is a hyper subalgebra of H. ■

Notation 4.4. For any subset I of H and any element $y \in H$, we denote

$$*(y, I)^{\cap} = \{x \in H | (x * y) \cap I \neq \emptyset\},\ \circ(y, I)^{\cap} = \{x \in H | (x \circ y) \cap I \neq \emptyset\}.$$

Definition 4.5. Let $I \subseteq H$ and $0 \in I$. Then I is called a *strong hyper pseudo BCK-ideal* of H if for any $y \in I$, $*(y, I)^{\cap} \subseteq I$ and $\circ(y, I)^{\cap} \subseteq I$.

Example 4.6. Let $H = \{0, a, b, c\}$. Hyperoperations " $*$ " and " $\circ$ " on H given by the following tables:

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$	$\{0, b\}$
c	$\{c\}$	$\{c\}$	$\{b, c\}$	$\{0, c\}$

$*$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{b\}$	$\{0\}$

Then $(H; \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can check that $I := \{0, a\}$ is a strong hyper pseudo BCK-ideal.

Theorem 4.7. *Let $I \subseteq H$. Then I is a strong hyper pseudo BCK-ideal of H if and only if $0 \in I$ and for any $y \in I$, $*(y, I)^{\cap} \subseteq I$ or for any $y \in I$, $\circ(y, I)^{\cap} \subseteq I$.*

Proof. $(\Rightarrow)$ The proof is clear by Definition 4.5.

$(\Leftarrow)$ W.L.G, assume that $\circ(y, I)^{\cap} \subseteq I$ for any $y \in I$. Let $x \in *(y, I)^{\cap}$ for some $y \in I$. Then $(x * y) \cap I \neq \emptyset$ and so there is $t \in (x * y) \cap I$. Thus $0 \in (x * y) \circ t$ and so by axiom (PHK2), we get $0 \in (x \circ t) * y$. Hence $0 \in a * y$ for some $a \in x \circ t$ and so $0 \in a \circ y$. It follows $a \in \circ(y, I)^{\cap}$. Since by hypothesis $\circ(y, I)^{\cap} \subseteq I$, we have $a \in I$. Hence $(x \circ t) \cap I \neq \emptyset$ and so from $t \in I$, we obtain $x \in I$, which completes the proof. ∎

Proposition 4.8. *Let $\emptyset \neq I \subseteq H$. Then I is a strong hyper pseudo BCK-ideal if and only if $0 \in I$ and*

$$((x \circ y) \cap I \neq \emptyset \text{ or } (x * y) \cap I \neq \emptyset) \text{ and } y \in I \Rightarrow x \in I.$$

Proof. The proof is straightforward. ∎

Theorem 4.9.

(i) *Every strong hyper pseudo BCK-ideal of H is a hyper pseudo BCK-ideal of type 1, and so it is all types of hyper pseudo BCK-ideals,*

(ii) *Every strong hyper pseudo BCK -ideal of H is a hyper subalgebra.*

Proof. (i) Let I be a strong hyper pseudo BCK-ideal. Assume $x * y \ll I$ and $y \in I$. We must show that $x \in I$. For any $a \in x * y$, since $x * y \ll I$, there exists $u \in I$ such that $a \ll u$. Therefore $0 \in a * u$ and so $(a * u) \cap I \neq \emptyset$. Since $u \in I$ and I is a strong hyper pseudo BCK-ideal of type 1, we get $a \in I$. Thus $(x * y) \cap I \neq \emptyset$ and $y \in I$ and so $x \in I$. Similarly, we can show that if $a \circ y \ll I$ and $y \in I$ then $x \in I$. Therefore I is a hyper pseudo BCK-ideal of type 1.

(ii) This follows immediately from (i) and Proposition 4.3. ∎

The following example shows that the converse of Theorem 4.9 (i) is not true, in general.

Example 4.10. Let H={0,a,b}. Hyperoperations " $*$ " and " $\circ$ " given by the following tables:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{a,b\}$	$\{0,a,b\}$

$*$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{a,b\}$	$\{0,b\}$

Then $(H;\circ,*,0)$ is a hyper pseudo BCK-algebra. We can check that $I := \{0,a\}$ is a hyper pseudo BCK-ideal of type 1 but is not a strong hyper pseudo BCK-ideal because $(b\circ a)\cap I \neq \emptyset$ and $b \notin I$.

Definition 4.11. Let I be a subset of H. Then I is said to be *reflexive* if $x*x \subseteq I$ and $x\circ x \subseteq I$ for all $x \in H$.

Lemma 4.12. *Let I be a reflexive hyper pseudo BCK-ideal of type $1,2,3,5$ or 9 of H. Then $(x*y)\cap I \neq \emptyset$ $((x\circ y)\cap I \neq \emptyset)$ implies $x*y \subseteq I(x\circ y \subseteq I)$.*

Proof. Let $(x\circ y)\cap I \neq \emptyset$. Then there exists $t \in (x\circ y)\cap I$. If $a \in x\circ y$, then $a\circ t \subseteq (x\circ y)\circ(x\circ y) \ll x\circ x \subseteq I$ and so $a\circ t \ll I$. Therefore by Proposition 4.1, we have $a\circ t \subseteq I$. Now, if $\circ(y,I)^{\ll} \subseteq I$, then $a\circ t \ll I$ and $t \in I$ imply $a \in I$. If $\circ(y,I)^{\subseteq} \subseteq I$, then $a\circ t \subseteq I$ and $t \in I$ imply $a \in I$. Therefore $x\circ y \subseteq I$. Similarly, if $(x\circ y)\cap I \neq \emptyset$, then $x\circ y \subseteq I$. ∎

Proposition 4.13. *If I is a reflexive hyper pseudo BCK-ideal of type $1,2,3$ or 5, then I is a strong hyper pseudo BCK-ideal.*

Proof. Let I be a reflexive hyper pseudo BCK-ideal of type $1,2,3$ or 5, $(x\circ y)\cap I \neq \emptyset$ and $y \in I$. Since I is a reflexive hyper pseudo BCK-ideal of type $1,2,3$ or 5, then by Lemma 4.12, we obtain $x\circ y \subseteq I$. Now $x\circ y \subseteq I$ and $y \in I$ imply $x \in I$. Similarly, we can show that if $(x*y)\cap I \neq \emptyset$ and $y \in I$, then $x \in I$. Therefore I is a strong hyper BCK-ideal. ∎

5. QUOTIENT HYPER PSEUDO BCK-ALGEBRA

In [1] R.A. Borzooei and H. Harizavi introduced the structure of the quotient hyper BCK-algebras. In this section, we generalize this structure on the hyper pseudo BCK-algebras.

Definition 5.1. Let H be a hyper pseudo BCK-algebra, ρ be a relation on H and $A, B \subseteq H$. Then

(i) $A\rho B$ means that there exist $a \in A$ and $b \in B$ such that $a\rho b$,

(ii) $A\overline{\rho}B$ means that for any $a \in A$, there exists $b \in B$ such that $a\rho b$, and for any $b \in B$ there exists $a \in A$ such that $a\rho b$,

(iii) ρ is called a right $*$-congruence (right $\circ$-congruence) on H if $a\rho b$ implies $a * u\overline{\rho}b * u$ ($a \circ u\overline{\rho}b \circ u$) for all $u \in H$,

(iv) ρ is called a left $*$-congruence (left $\circ$-congruence) on H if $a\rho b$ implies $u * a\overline{\rho}u * b$ ($u \circ a\overline{\rho}u \circ b$) for all $u \in H$,

(v) ρ is called a $*$-congruence ($\circ$-congruence) on H if it is a right and left $*$-congruence (a right and left $\circ$-congruence),

(vi) ρ is called a left congruence on H if it is a left $*$-congruence and a left $\circ$-congruence on H,

(vii) ρ is called a right congruence on H if it is a right $*$-congruence and a right $\circ$-congruence on H,

(viii) ρ is called a congruence on H if it is a $*$-congruence and a $\circ$-congruence on H.

Lemma 5.2. *Let ρ be an equivalent relation on H, and $A, B, C \subseteq H$. If $A\overline{\rho}B$ and $B\overline{\rho}C$, then $A\overline{\rho}C$.*

Proof. Let $a \in A$. Since $A\overline{\rho}B$, there exists $b \in B$ such that $a\rho b$. Since $B\overline{\rho}C$, there exists $c \in C$ that $b\rho c$. Therefore by the transitive condition of ρ, $a\rho c$. Similarly, for all $c \in C$ there exists $a \in A$ such that $a\rho c$. Therefore $A\overline{\rho}C$. ∎

Proposition 5.3. *Let ρ be an equivalence relation on H. Then the following statements hold:*

(i) *If ρ is a left $*$-congruence (left $\circ$-congruence) on H, then $[0]_\rho$ is a hyper pseudo BCK-ideal of type* 5–12,

(ii) *If ρ is a left congruence on H, then $[0]_\rho$ is a hyper pseudo BCK-ideal of type* 1.

Proof. (i) It suffices to prove that $[0]_\rho$ is a hyper pseudo BCK-ideal of type 5. Let $x * y \ll [0]_\rho$ and $y \in [0]_\rho$. Thus, for all $a \in x * y$, there is $b \in [0]_\rho$ such that $0 \in a * b$. Since ρ is a left $*$- congruence, $b\rho 0$ implies that $(a * b)\overline{\rho}a * 0 = \{a\}$ and so $(a * b)\overline{\rho}a$. Now, $0 \in a * b$ and $(a * b)\overline{\rho}a$ imply $0\rho a$ and so $x * y \subseteq [0]_\rho$. Since $y\rho 0$ and ρ is a left $*$-congruence, $(x * y)\overline{\rho}x * 0 = \{x\}$. Thus, for all $t \in x * y$, $t\rho x$. Since $x * y \subseteq [0]_\rho$, we have $t\rho 0$. Now from $t\rho 0$ and $t\rho x$ we get $x\rho 0$. Hence $x \in [0]_\rho$ and therefore $[0]_\rho$ is a hyper pseudo BCK- ideal of type 5. The proof of the left $\circ$-congruence is similar to the proof of the left $*$-congruence.

(ii) The proof is similar to the proof of (i). ∎

Definition 5.4. Let ρ be a congruence on H and $\frac{H}{\rho} = \{[x]_\rho \,|\, x \in H\}$. We define the hyperoperations " $*$ " and " $\circ$ " and the relation $\ll$ on $\frac{H}{\rho}$ as follows:

$$[x]_\rho * [y]_\rho = \{[z]_\rho | z \in x * y\}, \quad [x]_\rho \circ [y]_\rho = \{[z]_\rho | z \in x \circ y\},$$

$$[x]_\rho \ll [y]_\rho \Leftrightarrow [0]_\rho \in [x]_\rho \circ [y]_\rho \Leftrightarrow [0]_\rho \in [x]_\rho * [y]_\rho.$$

Lemma 5.5. *The hyperoperations "$\circ$" and "$*$" on $\frac{H}{\rho}$ as Definition 5.4, are well defined.*

Proof. Let $x, x', y, y' \in H$ such that $[x]_\rho = [x']_\rho$ and $[y]_\rho = [y']_\rho$. Let $[z]_\rho \in [x]_\rho \circ [y]_\rho$. Hence, there is $u \in x \circ y$ such that $[u]_\rho = [z]_\rho$. Since $x\rho x'$, $y\rho y'$ and ρ is a congruence on H, $x \circ y \overline{\rho} x' \circ y'$. Hence, there exists $z' \in x' \circ y'$ such that $u\rho z'$ and thus $[z']_\rho = [u]_\rho$. Since $[z']_\rho \in [x']_\rho \circ [y']_\rho$ and $[z]_\rho = [u]_\rho = [z']_\rho$, $[z]_\rho \in [x']_\rho \circ [y']_\rho$. Therefore $[x]_\rho \circ [y]_\rho \subseteq [x']_\rho \circ [y']_\rho$. Similarly, we can show that $[x']_\rho \circ [y']_\rho \subseteq [x]_\rho \circ [y]_\rho$ and so $[x]_\rho \circ [y]_\rho = [x']_\rho \circ [y']_\rho$. Therefore the hyperoperation " $\circ$ " is well defined. Similarly, we can show that the hyperoperation " $*$ " is well defined, too. ∎

Theorem 5.6. *Let ρ be a congruence on H. Then the following are equivalent:*

(i) *$(x * y)\rho 0$ and $(y * x)\rho 0$ imply $x\rho y$,*

(ii) *$(x \circ y)\rho 0$ and $(y \circ x)\rho 0$ imply $x\rho y$,*

(iii) *$(\frac{H}{\rho}; *, \circ, [0]_\rho)$ is a hyper pseudo BCK-algebra.*

Proof. (i)$\Rightarrow$(ii) Let $(x \circ y)\rho 0$. Thus there exists $t \in x \circ y$ such that $t\rho 0$. Since $t \in x \circ y$, $[t]_\rho \in [x]_\rho \circ [y]_\rho$, and so $[0]_\rho \in [x]_\rho \circ [y]_\rho$. Hence $[0]_\rho \in [x]_\rho * [y]_\rho$ and so there exists $u \in x * y$ such that $u\rho 0$. Hence $(x * y)\rho 0$. Similarly, $(y \circ x)\rho 0$ implies $(y * x)\rho 0$ and so by (i), $x\rho y$.

(ii)$\Rightarrow$(iii) By Lemma 5.5, the hyperoperations " $\circ$ " and " $*$ " are well defined. Now, we show that $\frac{H}{\rho}$ satisfies the axioms of a hyper pseudo BCK-algebra.

$\underline{(PHK1)}$: Let $[w]_\rho \in ([x]_\rho \circ [z]_\rho) \circ ([y]_\rho \circ [z]_\rho)$ for some $[y]_\rho, [z]_\rho, [x]_\rho \in \frac{H}{\rho}$. Then there are $[u]_\rho \in [x]_\rho \circ [z]_\rho$ and $[v]_\rho \in [y]_\rho \circ [z]_\rho$ such that $[w]_\rho \in [u]_\rho \circ [v]_\rho$. Hence, there exist $u' \in x \circ z, v' \in y \circ z$ and $w' \in u \circ v$ such that $[u]_\rho = [u']_\rho, [v]_\rho = [v']_\rho$ and $[w]_\rho = [w']_\rho$. Therefore $u\rho u', v\rho v'$ and $w'\rho w$. Since ρ is a congruence on H, $u \circ v \overline{\rho} u' \circ v'$. Then from $w' \in u \circ v$, there exists $a \in u' \circ v'$ such that $w'\rho a$ and hence $[w]_\rho = [a]_\rho$. Thus $[w]_\rho = [w']_\rho = [a]_\rho$. By $(PHK1)$ of H, $a \in u' \circ v' \subseteq (x \circ z) \circ (y \circ z) \ll x \circ y$. Therefore, there is $b \in x \circ y$ such that $0 \in a \circ b$, i.e., $a \ll b$. Hence $[b]_\rho \in [x]_\rho \circ [y]_\rho$ and $[0]_\rho \in [a]_\rho \circ [b]_\rho$. Since $[w]_\rho = [w']_\rho = [a]_\rho$, $[0]_\rho \in [w]_\rho \circ [b]_\rho$ and so $[w]_\rho \ll [b]_\rho$. This implies $([x]_\rho \circ [z]_\rho) \circ ([y]_\rho \circ [z]_\rho) \ll [x]_\rho \circ [y]_\rho$. Similarly, we can show that $([x]_\rho * [z]_\rho) * ([y]_\rho * [z]_\rho) \ll [x]_\rho * [y]_\rho$. Therefore $(PHK1)$ holds.

$\underline{(PHK2)}$: Let $[w]_\rho \in ([x]_\rho \circ [y]_\rho) * [z]_\rho$. Then there exists $[u]_\rho \in [x]_\rho \circ [y]_\rho$ such that $[w]_\rho \in [u]_\rho * [z]_\rho$. Since $[u]_\rho \in [x]_\rho \circ [y]_\rho$, there exists $u' \in x \circ y$ such that $u\rho u'$, that is , $[u]_\rho = [u']_\rho$. Hence $[u']_\rho \in [x]_\rho \circ [y]_\rho$. Since $[w]_\rho \in [u]_\rho * [z]_\rho = [u']_\rho * [z]_\rho$, there exists $w' \in u' * z$ such that $w'\rho w$. Now, $w' \in u' * z \subseteq (x \circ y) * z = (x * z) \circ y$ by $(PHK2)$ of H. Therefore $w' \in (x * z) \circ y$ and $u' * z \subseteq (x * z) \circ y$, and thus there exists $b \in x * z$ such that $w' \in b \circ y$. Since $b \in x * z$, $[b]_\rho \in [x]_\rho * [z]_\rho$ and $[w']_\rho \in [b] \circ [y]_\rho$, and thus $[w]_\rho = [w']_\rho \in [b] \circ [y]_\rho \subseteq ([x]_\rho * [z]_\rho) \circ [y]_\rho$. Hence $([x]_\rho \circ [y]_\rho) * [z]_\rho \subseteq ([x]_\rho * [z]_\rho) \circ [y]_\rho$. Similarly, we can show the converse inclusion. Hence $([x]_\rho * [z]_\rho) \circ [y]_\rho = ([x]_\rho \circ [y]_\rho) * [z]_\rho$. Therefore $(PHK2)$ holds.

$\underline{(PHK3)}$: Let $[z]_\rho \in [x]_\rho \circ [y]_\rho$ for some $x, y \in H$. Thus there exists $u \in x \circ y$ such that $[u]_\rho = [z]_\rho$. Since $x \circ y \ll x$, we get $u \ll x$ and so $0 \in u \circ x$. Hence $[0]_\rho \in [u]_\rho \circ [x]_\rho = [z]_\rho \circ [x]_\rho$, that is, $[z]_\rho \ll [x]_\rho$. This implies $[x]_\rho \circ [y]_\rho \ll [x]_\rho$. Similarly, we can show that $[x]_\rho * [y]_\rho \ll [x]_\rho$ for all $x \in H$. Therefore $(PHK3)$ holds.

$\underline{(PHK4)}$: Let $[x]_\rho \ll [y]_\rho$ and $[y]_\rho \ll [x]_\rho$. Then $[0]_\rho \in [x]_\rho \circ [y]_\rho$ and $[0]_\rho \in [y]_\rho \circ [x]_\rho$. Therefore there exist $u \in x \circ y$ and $u' \in y \circ x$ such that $[0]_\rho = [u]_\rho = [u']_\rho$. Thus $u', u \in [0]_\rho$. Hence $u'\rho 0$ and $u\rho 0$, and so $(x \circ y)\rho 0$ and $(y \circ x)\rho 0$. Then from (ii), we get $x\rho y$ and so $[x]_\rho = [y]_\rho$. Therefore $(PHK4)$ holds.

(iii)$\Rightarrow$(i) Let $(x * y)\rho 0$ and $(y * x)\rho 0$. Then there exists $u \in x * y$ such that $u\rho 0$ and $[u]_\rho \in [x]_\rho * [y]_\rho$. Hence $[0]_\rho \in [x]_\rho * [y]_\rho$, that is, $[x]_\rho \ll [y]_\rho$. Similarly, $(y * x)\rho 0$ implies $[y]_\rho \ll [x]_\rho$ and so we conclude $[x]_\rho = [y]_\rho$, i.e., $x\rho y$. ■

Definition 5.7. Let ρ be an equivalence relation on H. Then ρ is called regular on H if it satisfies the conditions (i) and (ii) of Theorem 5.6, which are equivalent.

Corollary 2. *If ρ is a regular congruence on H, then $(\frac{H}{\rho}; \circ\ , *\ , [0]_\rho)$ is a hyper pseudo BCK-algebra.*

Proof. This follows immediately from Theorem 5.6 and Definition 5.7. ■

Theorem 5.8. *Let ρ and ρ' be two regular congruences on H such that $[0]_\rho = [0]_{\rho'}$. Then $\rho = \rho'$.*

Proof. It suffices to show that for all $x, y \in H$

$$x\rho y \Leftrightarrow x\rho' y.$$

If $x\rho y$, then $x \circ x\overline{\rho} x \circ y$. Since $0 \in x \circ x$ and ρ is a congruence on H, there exists $t \in x \circ y$ such that $0\rho t$. Then $t \in [0]_\rho = [0]_{\rho'}$ and thus $t \in [0]_{\rho'}$, that is, $0\rho' t$. Therefore $0\rho' x \circ y$. Similarly, we can show that $0\rho' y \circ x$. Thus $x\rho' y$ because ρ' is regular. By the same argument, ρ' implies $x\rho y$. Hence $\rho = \rho'$. ■

Lemma 5.9. *Let ρ be a regular congruence on H. Then $[0]_\rho$ is a strong hyper pseudo BCK-ideal.*

Proof. Let $(x * y) \cap [0]_\rho \neq \emptyset$ and $y \in [0]_\rho$. Then there exists $t \in x * y$ such that $t\rho 0$ and $y\rho 0$. Therefore by Definition 5.1(i), we get $(x * y)\rho\{0\}$. Since $y\rho 0$, $y * x \overline{\rho} 0 * x = \{0\}$. Hence by regularity of ρ, we have $x\rho y$. Now, $x\rho y$ and $y\rho 0$ imply $x\rho 0$, that is, $x \in [0]_\rho$. Similarly, we can show that, if $(x \circ y) \cap [0]_\rho \neq \emptyset$ and $y \in [0]_\rho$, then $x \in [0]_\rho$. Hence $[0]_\rho$ is a strong hyper pseudo BCK-ideal. ■

Definition 5.10. Let I be a reflexive hyper pseudo BCK-ideal of type 1,2 or 3. Define the binary relations $\rho_{*(I)}$, $\rho_{\circ(I)}$ and $\rho_{(I)}$ on H by

$$x\rho_{*(I)}y \Leftrightarrow x * y \subseteq I \text{ and } y * x \subseteq I,$$

$$x\rho_{\circ(I)}y \Leftrightarrow x \circ y \subseteq I \text{ and } y \circ x \subseteq I,$$

$$x\rho_{(I)}y \Leftrightarrow x\rho_{*(I)}y \text{ and } x\rho_{\circ(I)}y.$$

Proposition 5.11. *$\rho_{*(I)}$, $\rho_{\circ(I)}$ and $\rho_{(I)}$ are equivalence relations on H.*

Proof. It is clear $\rho_{*(I)}$ is reflexive and symmetric. Now, we will show that it is transitive. Let $x, y, z \in H$ such that $x\rho_{*(I)}y$ and $y\rho_{*(I)}z$. Then $x * y, y * x \subseteq I$ and $y * z, z * y \subseteq I$. By (PHK1), we have $(x * z) * (y * z) \ll x * y \subseteq I$. Therefore $(x * z) * (y * z) \ll I$. Since $y * z \subseteq I$, we get $x * z \subseteq I$ and so $\rho_{*(I)}$ is transitive. Therefore $\rho_{*(I)}$ is an equivalence relation on H. Similarly, we can show that, $\rho_{\circ(I)}$ is an equivalence relation on H. Therefore by Definition 5.10, $\rho_{(I)}$ is an equivalence relation on H. ■

Proposition 5.12. *Let I be a reflexive hyper pseudo BCK-ideal of type $1, 2$ or 3. Then*

(i) $[0]_{\rho_{*(I)}} = I = [0]_{\rho_{\circ(I)}}$.

(ii) *$\rho_{*(I)}$ ($\rho_{\circ(I)}$) is a right $*$-congruence (right $\circ$-congruence), respectively.*

(iii) *Let $\frac{H}{\rho_{*(I)}}$ ($\frac{H}{\rho_{\circ(I)}}$) be the quotient set with respect to $\rho_{*(I)}$ ($\rho_{\circ(I)}$), respectively, and let $[x]_*$ ($[x]_\circ$) be the equivalence class of $x \in H$ with respect to $\rho_{*(I)}$ ($\rho_{\circ(I)}$), respectively. Then*

   (1) $[x]_* \ll [y]_* \Leftrightarrow x * y \subseteq I$,

   (2) $[x]_\circ \ll [y]_\circ \Leftrightarrow x \circ y \subseteq I$.

Proof. (i) We have $x\rho_{*(I)}0 \Leftrightarrow (x * 0 \subseteq I \text{ and } 0 * x \subseteq I) \Leftrightarrow x \in I$. Therefore $I = [0]_{\rho_{*(I)}}$. By the same argument, we obtain $I = [0]_{\rho_{\circ(I)}}$.

(ii) Let $x\rho_{\circ(I)}y$ and $a \in H$. Then $x \circ y, y \circ x \subseteq I$. By (PHK1), we have $(x \circ a) \circ (y \circ a) \ll x \circ y$. Now, $x \circ y \subseteq I$ implies $(x \circ a) \circ (y \circ a) \ll I$. Since I is a hyper pseudo BCK-ideal of type 1, it follows from Proposition 4.1 that $(x \circ a) \circ (y \circ a) \subseteq I$. Thus $u \circ v \subseteq I$ for all $u \in x \circ a$ *and* $v \in y \circ a$. Similarly, it follows from $(y \circ a) \circ (x \circ a) \ll y \circ x \subseteq I$ that $(y \circ a) \circ (x \circ a) \subseteq I$. Thus for all

$v \in y \circ a$ *and* $u \in x \circ a$, $v \circ u \subseteq I$. Therefore $u\rho_{\circ(I)}v$ and this implies $x \circ a\overline{\rho}_{\circ(I)}y \circ a$ and so $\rho_{\circ(I)}$ is a right $\circ$-congruence on H. Similarly, we can show that $\rho_{*(I)}$ is a right $*$-congruence on H.

(iii) (1) ($\Rightarrow$) Let $[x]_* \ll [y]_*$. Then by Definition 5.4, $[0]_* \in [x]_* * [y]_*$ and so $I \in [x]_* * [y]_*$. Therefore, there exists $w \in x * y$ such that $[w]_* = I$. Since $w \in [w]_*$, we get $w \in I$ and so $(x*y) \cap I \neq \emptyset$. Thus by Proposition 4.12, $x*y \subseteq I$.

($\Leftarrow$) Let $x * y \subseteq I$ and $[a]_* \in [x]_* * [y]_*$. Therefore, there exists $u \in x * y$ such that $[a]_* = [u]_*$. Since $x * y \subseteq I$, $u \in I$ and so by (i), $[u]_* = I$. Therefore $[x]_* * [y]_* = \{I\}$ and so $[x]_* \ll [y]_*$.

(2) The proof is similar to the proof of (1). ■

Definition 5.13. Let $\emptyset \neq I \subseteq H$. Then I is called a *normal hyper pseudo BCK-ideal of type i* for $1 \leq i \leq 12$, if I is a pseudo hyper BCK-ideal of type i for $1 \leq i \leq 12$, and $x \circ y \subseteq I \Leftrightarrow x * y \subseteq I$ for any $x, y \in H$.

Example 5.14. Let $H = \{0, a, b\}$. Hyperoperations "$*$" and "$\circ$" on H given by the following tables:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

$*$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

Then $(H; \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can check that $I := \{0, a\}$ is a normal hyper pseudo BCK-ideal of type 1 of H.

Proposition 5.15. *Let ρ be a regular congruence on H. If $[0]_\rho$ is reflexive, then $[0]_\rho$ is a normal hyper pseudo BCK-ideal of type* 1.

Proof. By Lemma 5.9, $[0]_\rho$ is a strong hyper pseudo BCK-ideal. Assume that $x * y \subseteq [0]_\rho$. Then $[x]_\rho * [y]_\rho = \{[a]_\rho | a \in x * y\} \subseteq \{[a]_\rho | a \in [0]_\rho\} = \{[0]_\rho\}$. Therefore $[0]_\rho \in [x]_\rho * [y]_\rho$. Thus, since $\frac{H}{\rho}$ is a hyper pseudo BCK-algebra, we get $[0]_\rho \in [x]_\rho \circ [y]_\rho$. Hence, there exists $t \in x \circ y$ such that $[t]_\rho = [0]_\rho$. That is, $t \in [0]_\rho$ and so $(x \circ y) \cap [0]_\rho \neq \emptyset$. Hence by Lemma 4.12, since $[0]_\rho$ is reflexive, we get $x \circ y \subseteq [0]_\rho$. Similarly, if $x \circ y \subseteq [0]_\rho$ then $x * y \subseteq [0]_\rho$. Thus we have shown that $x \circ y \subseteq [0]_\rho$ if and only if $x * y \subseteq [0]_\rho$. Therefore $[0]_\rho$ is a normal hyper pseudo BCK-ideal of type 1. ■

Theorem 5.16. *Let ρ be a regular congruence on H. Then the following are equivalent:*

(i) $[0]_\rho$ *is a reflexive hyper pseudo BCK-ideal of type* 1.

(ii) $\frac{H}{\rho}$ *is a pseudo BCK-algebra.*

Proof. Let $\frac{H}{\rho}$ be a pseudo BCK-algebra. By Proposition 5.3 (ii), $[0]_\rho$ is a hyper pseudo BCK- ideal of type 1. Now we must show that for all $x \in H$, $x \circ x$, $x * x \subseteq [0]_\rho$. Let $z \in x \circ x$. Then $[z]_\rho \in [x]_\rho \circ [x]_\rho$. Since $\frac{H}{\rho}$ is a pseudo BCK-algebra and $[0]_\rho \in [x]_\rho \circ [x]_\rho$, we get $[0]_\rho = [z]_\rho$, that is, $z\rho 0$. Hence $z \in [0]_\rho$ and so $x \circ x \subseteq [0]_\rho$ for all $x \in H$. Similarly, we can show that, $x * x \subseteq [0]_\rho$. Therefore $[0]_\rho$ is a reflexive hyper pseudo BCK-ideal of H. Conversely, let $[a]_\rho \in \frac{H}{\rho}$. Since $a \circ a \subseteq [0]_\rho$ for all $a \in H$, it follows

$$[a]_\rho \circ [a]_\rho = \{[z]_\rho | z \in a \circ a\} \subseteq \{[z]_\rho | z \in [0]_\rho\} = \{[0]_\rho\}.$$

Thus $[a]_\rho \circ [a]_\rho = [0]_\rho$ for all $a \in H$. Similarly, since $a * a \subseteq [0]_\rho$ for all $a \in H$, we get $[a]_\rho * [a]_\rho = [0]_\rho$ for all $a \in H$, and so $S(\frac{H}{\rho}) = \frac{H}{\rho}$. Therefore by Theorem 3.6 (iv), $\frac{H}{\rho}$ is a pseudo BCK-algebra. ■

Proposition 5.17. *Let I be a reflexive hyper pseudo BCK-ideal of type 1. Then the following are equivalent:*

(i) *I is normal,*

(ii) *For any $x, y \in H, [x]_* \ll [y]_*$ if and only if $[x]_\circ \ll [y]_\circ$.*

Proof. By Proposition 5.12, we have

$$I \text{ is normal} \Leftrightarrow (x * y \subseteq I \text{ iff } x \circ y \subseteq I) \Leftrightarrow ([x]_* \ll [y]_* \text{ iff } [x]_\circ \ll [y]_\circ)$$

for any $x, y \in H$, which completes the proof. ■

Theorem 5.18. *Let I be a reflexive and normal hyper BCK-ideal of type 1. Then*

(i) *$\rho_{(I)}$ is a congruence on H,*

(ii) *$I = [0]_{\rho_{(I)}}$,*

(iii) *$\rho_{(I)}$ is regular on H.*

Proof. (i) By Proposition 5.11, $\rho_{(I)}$ is an equivalence relation on H. By Proposition 5.12, $\rho_{*(I)}$ and $\rho_{\circ(I)}$ are right $*$-congruence and right $\circ$-congruences on H, respectively. Therefore $\rho_{(I)}$ is a right congruence on H. Now we show that $\rho_{(I)}$ is a left congruence on H, that is, if $x\rho_{(I)}y$ and $a \in H$ then $a \circ x\overline{\rho}_{(I)}a \circ y$. Let $x\rho_{(I)}y$, $a \in H$ and $u \in a \circ x$. Since $(a \circ x) \circ (y \circ x) \ll a \circ y$, there exists $t \in y \circ x$ such that $u \circ t \ll a \circ y$ and so for any $w \in u \circ t$ there is $v' \in a \circ y$ such that $w \ll v'$. Since $0 \in w * v' \subseteq (u \circ t) * v' = (u * v') \circ t$, there exists $c \in u * v'$ such that $0 \in c \circ t$ and so $(c \circ t) \cap I \neq \emptyset$. Hence by Lemma 4.12, $(c \circ t) \subseteq I$. Since $t \in y \circ x \subseteq I$ and I is a hyper pseudo BCK-ideal of type 1, $c \in I$. Hence $(u * v') \cap I \neq \emptyset$ and so by Lemma 4.12, $u * v' \subseteq I$. Therefore $u \circ v' \subseteq I$ because I is a normal hyper pseudo BCK- ideal of type 1. Moreover, since $(a \circ y) \circ (x \circ y) \ll a \circ x$ and

$v' \in a \circ y$, we can similarly show that there is $u' \in a \circ x$ such that $v' \circ u' \subseteq I$. Since $u, u' \in a \circ x$ *and* I is a reflexive hyper pseudo BCK-ideal of H, we have $u' \circ u \subseteq (a \circ x) \circ (a \circ x) \ll (a \circ a) \subseteq I$ and so $u' \circ u \ll I$. Hence by Proposition 4.1, $u' \circ u \subseteq I$. Now, since $(v' \circ u) \circ (u' \circ u) \ll v' \circ u' \subseteq I$ *and* I is a hyper pseudo BCK-ideal of type 1, we get $v' \circ u \subseteq I$. From $u \circ v' \subseteq I$ and $v' \circ u \subseteq I$, we have $u\rho_{(I)}v'$. Similarly, we can prove that for any $v \in a \circ y$, there exists $u' \in a \circ x$ such that $u'\rho_{(I)}v$. Hence $a \circ x\overline{\rho}_{(I)}a \circ y$ for all $a \in H$. Therefore $\rho_{(I)}$ is a left $\circ$-congruence on H. Similarly, we can show that $\rho_{(I)}$ is a left $*$-congruence on H. Consequently, $\rho_{(I)}$ is a congruence on H.

(ii) This follows from Proposition 5.12.

(iii) Let $x, y \in H$ such that $(x \circ y)\rho_{(I)}0$ and $(y \circ x)\rho_{(I)}0$. Then there are $a \in x \circ y$ and $b \in y \circ x$ such that $a, b\rho_{(I)}0$. Hence $a \circ 0 \subseteq I$ and $b \circ 0 \subseteq I$ and so $a, b \in I$. Therefore $(x \circ y) \cap I \neq \emptyset$ *and* $(y \circ x) \cap I \neq \emptyset$. Then by Lemma 4.12, $x \circ y \subseteq I$ and $y \circ x \subseteq I$. This implies $x\rho y$ and so ρ is regular on H. ■

Theorem 5.19. *Let I be a reflexive and normal hyper pseudo BCK-ideal of type 1 of H. Then $(\frac{H}{\rho_{(I)}}; \ll, *, \circ, [0]_{\rho_{(I)}})$ is a pseudo BCK-algebra.*

Proof. The proof follows from Theorems 5.18 and 5.16. ■

Theorem 5.20. *Let φ be a regular congruence on H. If $(\frac{H}{\varphi}; \ll, *, \circ, [0]_\varphi)$ is a pseudo BCK-algebra, then $\varphi = \rho_{([0]_\varphi)}$, that is, $\rho_{([0]_\varphi)}$ is only regular congruence on H such that $\frac{H}{\rho_{([0]_\varphi)}}$ is a pseudo BCK-algebra.*

Proof. Since $\frac{H}{\varphi}$ is a pseudo BCK-algebra, it follows from Theorem 5.16 that $[0]_\varphi$ is a reflexive hyper pseudo BCK-ideal of type 1. Hence by Proposition 5.17, $[0]_\varphi$ is normal and so by Theorem 5.18, $[0]_\varphi = [0]_{\rho_{([0]_\varphi)}}$. Moreover by Theorem 5.18, since $[0]_\varphi$ is reflexive and normal, $\rho_{([0]_\varphi)}$ is a regular congruence on H. Using Theorem 5.8, we obtain $\varphi = \rho_{([0]_\varphi)}$, which completes the proof. ■

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