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# ALL COMPLETELY REGULAR ELEMENTS IN $H y p_{G}(n)$ 

Ampika Boonmee<br>Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand<br>e-mail: ampika.b.ku.src@gmail.com

AND

Sorasak Leeratanavalee ${ }^{1}$
Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand
e-mail: sorasak.l@cmu.ac.th


#### Abstract

In Universal Algebra, identities are used to classify algebras into collections, called varieties and hyperidentities are use to classify varieties into collections, called hypervarities. The concept of a hypersubstitution is a tool to study hyperidentities and hypervarieties.

Generalized hypersubstitutions and strong identities generalize the concepts of a hypersubstitution and of a hyperidentity, respectively. The set of all generalized hypersubstitutions forms a monoid. In this paper, we determine the set of all completely regular elements of this monoid of type $\tau=(n)$.


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## 1. Introduction

Hyperidentities and hypervarieties of a given type $\tau$ without nullary operations were first introduced by J. Aczèl [1], V.D. Belousov [2], W.D. Neumann [9] and W. Taylor [10]. The main tool used to study hyperidentities and hypervarieties is the concept of a hypersubstitution. The notion of a hypersubstitution originated by K. Denecke, D. Lau, R. Pöschel and D. Schweigert [4]. In 2000, S. Leeratanavalee and K. Denecke generalized the concepts of a hypersubstitution and a hyperidentity to the concepts of a generalized hypersubstitution and a strong hyperidentity, respectively [8]. The set of all generalized hypersubstitutions together with a binary operation and the identity hypersubstitution forms a monoid. There are several published papers on algebraic properties of this monoid and its submonoids. The present paper will determine the set of all completely regular elements of this monoid of type $\tau=(n)$.

## 2. Monoid of all generalized hypersubstitutions

In this section, we give the concept of the monoid of all generalized hypersubstitutions.

Let $X:=\left\{x_{1}, x_{2}, x_{3}, \ldots\right\}$ be a countably infinite set of symbols called variables. We often refer to these variables as letters, to $X$ as an alphabet, and also refer to the set $X_{n}:=\left\{x_{1}, x_{2}, x_{3}, \ldots, x_{n}\right\}$ as an $n$-element alphabet. Let $\left(f_{i}\right)_{i \in I}$ be an indexed set which is disjoint from $X$. Each $f_{i}$ is called $n_{i}$-ary operation symbol, where $n_{i} \geq 1$ is a natural number. Let $\tau$ be a function which assigns to every $f_{i}$ the number $n_{i}$ as its arity. The function $\tau$, on the values of $\tau$ written as $\left(n_{i}\right)_{i \in I}$ is called a type. An $n$-ary term of type $\tau$, for simply an $n$-ary term, is defined inductively as follows:
(i) The variables $x_{1}, x_{2}, \ldots, x_{n}$ are $n$-ary terms.
(ii) If $t_{1}, t_{2}, \ldots, t_{n_{i}}$ are $n$-ary terms then $f_{i}\left(t_{1}, t_{2}, \ldots, t_{n_{i}}\right)$ is an $n$-ary term.

The smallest set, which contains $x_{1}, x_{2}, \ldots, x_{n}$ and is closed under finite application of (ii), is denoted by $W_{\tau}\left(X_{n}\right)$. It is clear that every $n$-ary term is also an $m$-ary term for all $m \geq n$. Let $W_{\tau}(X):=\bigcup_{n=1}^{\infty} W_{\tau}\left(X_{n}\right)$. It is called the set of all terms of type $\tau$.

Example 1. Let $\tau=(2,3)$. That means we have one binary operation symbol and one ternary operation symbol, say $f$ and $g$ respectively. These are some examples of ternary terms of type $(2,3): x_{1}, x_{2}, x_{3}, f\left(x_{3}, g\left(x_{1}, x_{3}, x_{3}\right)\right), g\left(f\left(x_{2}, x_{3}\right)\right.$, $\left.x_{1}, g\left(x_{3}, x_{1}, x_{2}\right)\right)$.

A generalized hypersubstitution of type $\tau$ is a mapping $\sigma:\left\{f_{i} \mid i \in I\right\} \rightarrow W_{\tau}(X)$ which does not necessarily preserve the arity. We denote the set of all generalized hypersubstitutions of type $\tau$ by $\operatorname{Hyp}_{G}(\tau)$. To define a binary operation on this set, we need the concept of generalized superposition of terms $S^{m}: W_{\tau}(X)^{m+1} \rightarrow$ $W_{\tau}(X)$ which is defined by the following steps:
(i) If $t=x_{j}, 1 \leq j \leq m$, then $S^{m}\left(t, t_{1}, \ldots, t_{m}\right)=S^{m}\left(x_{j}, t_{1}, \ldots, t_{m}\right):=t_{j}$.
(ii) If $t=x_{j}, m<j \in \mathbb{N}$, then $S^{m}\left(t, t_{1}, \ldots, t_{m}\right)=S^{m}\left(x_{j}, t_{1}, \ldots, t_{m}\right):=x_{j}$.
(iii) If $t=f_{i}\left(s_{1}, s_{2}, \ldots, s_{n_{i}}\right)$, then $S^{m}\left(t, t_{1}, \ldots, t_{m}\right):=f_{i}\left(S^{m}\left(s_{1}, t_{1}, \ldots, t_{m}\right)\right.$, $\left.\ldots, S^{m}\left(s_{n_{i}}, t_{1}, \ldots, t_{m}\right)\right)$.

Every generalized hypersubstitution $\sigma$ can be extended to a mapping $\widehat{\sigma}: W_{\tau}(X) \rightarrow$ $W_{\tau}(X)$ defined as follows:
(i) $\widehat{\sigma}[x]:=x \in X$,
(ii) $\widehat{\sigma}\left[f_{i}\left(t_{1}, t_{2}, \ldots, t_{n_{i}}\right)\right]:=S^{n_{i}}\left(\sigma\left(f_{i}\right), \widehat{\sigma}\left[t_{1}\right], \ldots, \widehat{\sigma}\left[t_{n_{i}}\right]\right)$, for any $n_{i}$-ary operation symbol $f_{i}$ and supposed that $\widehat{\sigma}\left[t_{j}\right], 1 \leq j \leq n_{i}$ are already defined.

Example 2. Let $\tau=(2)$, i.e., there is only one binary operation symbol, say $f$. Let $\sigma \in H y p_{G}(2)$ where $\sigma(f)=f\left(x_{2}, f\left(x_{3}, x_{2}\right)\right)$. Then

$$
\begin{aligned}
\widehat{\sigma}\left[f\left(f\left(x_{1}, x_{5}\right), x_{3}\right)\right] & =S^{2}\left(\sigma(f), \widehat{\sigma}\left[f\left(x_{1}, x_{5}\right)\right], \widehat{\sigma}\left[x_{3}\right]\right) \\
& =S^{2}\left(f\left(x_{2}, f\left(x_{3}, x_{2}\right)\right), S^{2}\left(\sigma(f), \widehat{\sigma}\left[x_{1}\right], \widehat{\sigma}\left[x_{5}\right]\right), x_{3}\right) \\
& =S^{2}\left(f\left(x_{2}, f\left(x_{3}, x_{2}\right)\right), S^{2}\left(f\left(x_{2}, f\left(x_{3}, x_{2}\right)\right), x_{1}, x_{5}\right), x_{3}\right) \\
& =S^{2}\left(f\left(x_{2}, f\left(x_{3}, x_{2}\right)\right), f\left(x_{5}, f\left(x_{3}, x_{5}\right)\right), x_{3}\right) \\
& =f\left(x_{3}, f\left(x_{3}, x_{3}\right)\right) .
\end{aligned}
$$

We define a binary operation $\circ_{G}$ on $H y p_{G}(\tau)$ by $\sigma_{1} \circ_{G} \sigma_{2}:=\widehat{\sigma}_{1} \circ \sigma_{2}$ where $\circ$ denotes the usual composition of mappings. Let $\sigma_{i d}$ be the hypersubstitution which maps each $n_{i}$-ary operation symbol $f_{i}$ to the term $f_{i}\left(x_{1}, x_{2}, \ldots, x_{n_{i}}\right)$. In [8], S. Leeratanavalee and K. Denecke proved that:

For arbitrary terms $t, t_{1}, \ldots, t_{n} \in W_{\tau}(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_{1}, \sigma_{2}$ we have
(i) $S^{n}\left(\widehat{\sigma}[t], \widehat{\sigma}\left[t_{1}\right], \ldots, \widehat{\sigma}\left[t_{n}\right]\right)=\widehat{\sigma}\left[S^{n}\left(t, t_{1}, \ldots, t_{n}\right)\right]$,
(ii) $\left(\widehat{\sigma}_{1} \circ \sigma_{2}\right)=\widehat{\sigma}_{1} \circ \widehat{\sigma}_{2}$.

Then $\operatorname{Hyp}_{G}(\tau)=\left(\operatorname{Hyp}_{G}(\tau) ; \circ_{G}, \sigma_{i d}\right)$ is a monoid and the set of all hypersubstitutions of type $\tau$ forms a submonoid of $\underline{H y p_{G}(\tau)}$.

## 3. All Completely Regular elements in $\operatorname{Hyp}_{G}(n)$

To determine the set of all completely regular elements of $\underline{H y p_{G}(n)}$, we first introduce some notations which will be used throughout this paper.

For a type $\tau=(n)$ with $n$-ary operation symbol $f$ and $t \in W_{(n)}(X)$, we denote $\sigma_{t}:=$ the generalized hypersubstitution $\sigma$ of type $\tau=(n)$ which maps $f$ to the term $t$,
$\operatorname{var}(t):=$ the set of all variables occurring in the term $t$.
A subterm of $t$ is defined inductively by the following steps.
(i) Every variable $x \in \operatorname{var}(t)$ is a subterm of $t$.
(ii) If $t=f\left(t_{1}, \ldots, t_{n}\right)$ then $t_{1}, \ldots, t_{n}$ and $t$ itself are subterms of $t$.

We denote the set of all subterms of $t$ by $\operatorname{sub}(t)$.
Lemma 3. Let $\sigma_{s}, \sigma_{t} \in \operatorname{Hyp}_{G}(n)$ where $t=f\left(t_{1}, \ldots, t_{n}\right)$ such that $t_{i_{1}}=x_{j_{1}}, \ldots$, $t_{i_{m}}=x_{j_{m}}$ for some $i_{1}, \ldots, i_{m}$ and for distinct $j_{1}, \ldots, j_{m} \in\{1, \ldots, n\}$ and $\operatorname{var}(t) \cap$ $X_{n}=\left\{x_{j_{1}}, \ldots, x_{j_{m}}\right\}$. Then $\sigma_{t} \circ_{G} \sigma_{s} \circ_{G} \sigma_{t}=\sigma_{t}$ if and only if $s=f\left(s_{1}, \ldots, s_{n}\right)$ where $s_{j_{l}}=x_{i_{l}}$ for all $l \in\{1, \ldots, m\}$.

Proof. Assume that $\sigma_{t}{ }^{\circ}{ }_{G} \sigma_{s}{ }^{\circ}{ }_{G} \sigma_{t}=\sigma_{t}$ and let $s=f\left(s_{1}, \ldots, s_{n}\right)$. Suppose that, there exists $s_{j_{l}}$ such that $s_{j_{l}} \in W_{n}(X) \backslash\left\{x_{i_{l}}\right\}$ for some $l \in\{1, \ldots, m\}$. Then

$$
\begin{aligned}
\left(\sigma_{t} \circ_{G} \sigma_{s} \circ_{G} \sigma_{t}\right)(f)= & \hat{\sigma}_{t}\left[\hat{\sigma}_{s}[t]\right] \\
= & \hat{\sigma}_{t}\left[S^{n}\left(f\left(s_{1}, \ldots, s_{n}\right), \hat{\sigma}_{s}\left[t_{1}\right], \ldots, \hat{\sigma}_{s}\left[t_{n}\right]\right)\right] \\
= & \hat{\sigma}_{t}\left[f\left(w_{1}, \ldots, w_{n}\right)\right]\left(\text { where } w_{i}=S^{n}\left(s_{i}, \hat{\sigma}_{s}\left[t_{1}\right], \ldots, \hat{\sigma}_{s}\left[t_{n}\right]\right)\right. \\
& \text { for all } i \in\{1, \ldots, n\}) \\
= & S^{n}\left(f\left(t_{1}, \ldots, t_{n}\right), \hat{\sigma}_{t}\left[w_{1}\right], \ldots, \hat{\sigma}_{t}\left[w_{n}\right]\right) \\
= & f\left(u_{1}, \ldots, u_{n}\right)\left(\text { where } u_{i}=S^{n}\left(t_{i}, \hat{\sigma}_{t}\left[w_{1}\right], \ldots, \hat{\sigma}_{t}\left[w_{n}\right]\right)\right. \\
& \quad \text { for all } i \in\{1, \ldots, n\}) .
\end{aligned}
$$

Since $t_{i_{l}}=x_{j_{l}}$ for all $l \in\{1, \ldots, m\}$, thus $u_{i_{l}}=S^{n}\left(t_{i_{l}}, \hat{\sigma}_{t}\left[w_{1}\right], \ldots, \hat{\sigma}_{t}\left[w_{n}\right]\right)=$ $\hat{\sigma}_{t}\left[w_{j_{l}}\right]$. Since $w_{j_{l}}=S^{n}\left(s_{j_{l}}, \hat{\sigma_{s}}\left[t_{1}\right], \ldots, \hat{\sigma}_{s}\left[t_{n}\right]\right)$ and $s_{j_{l}} \neq x_{i_{l}}, w_{j_{l}} \neq \hat{\sigma}_{s}\left[t_{i_{l}}\right]=x_{j_{l}}$, we get $u_{i_{l}}=\hat{\sigma}_{t}\left[w_{j_{l}}\right] \neq x_{j_{l}}$, and then $f\left(u_{1}, \ldots, u_{n}\right) \neq t$. This is a contradiction. Hence $s_{j_{l}}=x_{i_{l}}$ for all $l \in\{1, \ldots, m\}$.

Conversely, let $s=f\left(s_{1}, \ldots, s_{n}\right)$ where $s_{j_{l}}=x_{i_{l}}$ for all $l \in\{1, \ldots, m\}$. Then $\left(\sigma_{t} \circ_{G} \sigma_{s} \circ_{G} \sigma_{t}\right)(f)=\hat{\sigma_{t}}\left[f\left(w_{1}, \ldots, w_{n}\right)\right]$ where $w_{i}=S^{n}\left(s_{i}, \hat{\sigma_{s}}\left[t_{1}\right], \ldots, \hat{\sigma_{s}}\left[t_{n}\right]\right)$ for all $i \in\{1, \ldots, n\}$. Since $s_{j_{l}}=x_{i_{l}}$ for all $l \in\{1, \ldots, m\}, w_{i_{l}}=S^{n}\left(x_{i_{l}}, \hat{\sigma}_{s}\left[t_{1}\right], \ldots\right.$, $\left.\hat{\sigma_{s}}\left[t_{n}\right]\right)=\hat{\sigma_{s}}\left[t_{i_{l}}\right]=x_{j_{l}}$, we get

$$
\hat{\sigma}_{t}\left[f\left(w_{1}, \ldots, w_{n}\right)\right]=S^{n}\left(f\left(t_{1}, \ldots, t_{n}\right), \hat{\sigma}_{t}\left[w_{1}\right], \ldots, \hat{\sigma}_{t}\left[w_{n}\right]\right)=f\left(t_{1}, \ldots, t_{n}\right)=t
$$

Hence $\sigma_{t}{ }^{\circ} G \sigma_{s}{ }^{\circ}{ }_{G} \sigma_{t}=\sigma_{t}$.
Definition [6]. An element $a$ of a semigroup $S$ is called regular if there exists $x \in S$ such that $a x a=a$.

Let $\sigma_{t} \in \operatorname{Hyp}_{G}(n)$, we denote
$R_{1}:=\left\{\sigma_{x_{i}} \mid x_{i} \in X\right\} ;$
$R_{2}:=\left\{\sigma_{t} \mid \operatorname{var}(t) \cap X_{n}=\emptyset\right\} ;$
$R_{3}:=\left\{\sigma_{t} \mid t=f\left(t_{1}, \ldots, t_{n}\right)\right.$ where $t_{i_{1}}=x_{j_{1}}, \ldots, t_{i_{m}}=x_{j_{m}}$ for some $i_{1}, \ldots, i_{m}$ and for distinct $j_{1}, \ldots, j_{m} \in\{1, \ldots, n\}$ and $\left.\operatorname{var}(t) \cap X_{n}=\left\{x_{j_{1}}, \ldots, x_{j_{m}}\right\}\right\}$.

In 2010, W. Puninagool and S. Leeratanavalee [5] showed that $\bigcup_{i=1}^{3} R_{i}$ is the set of all regular elements in $H y p_{G}(n)$.

Definition [6]. For any monoid $S$, an element $u \in S$ is called unit if there exists $u^{-1} \in S$ such that $u u^{-1}=e=u^{-1} u$ where $e$ is the identity element of $S$, and $U(S)$ denote the set of all unit elements of $S$.

Let $S_{n}$ be the set of all permutations of $\{1,2, \ldots, n\}$. In 2013, A. Boonmee and S. Leeratanavalee [3] showed that $U\left(\operatorname{Hyp}_{G}(n)\right):=\left\{\sigma_{t} \in H y p_{G}(n) \mid t=\right.$ $f\left(x_{\pi(1)}, \ldots, x_{\pi(n)}\right)$ where $\left.\pi \in S_{n}\right\}$ is the set of all unit elements in $H y p_{G}(n)$. And we see that $U\left(\operatorname{Hyp}_{G}(n)\right) \subset R_{3}$.

Definition [6]. An element $e$ of a semigroup $S$ is called idempotent if $e^{2}=e e=e$, and we denote the set of all idempotents in $S$ by $E(S)$.

Let $\sigma_{t} \in \operatorname{Hyp}_{G}(n)$, we denote $E:=\left\{\sigma_{t} \mid t=f\left(t_{1}, \ldots, t_{n}\right)\right.$ where $t_{i_{1}}=x_{i_{1}}, \ldots, t_{i_{m}}=$ $x_{i_{m}}$ for some $i_{1}, \ldots, i_{m} \in\{1, \ldots, n\}$ and $\left.\operatorname{var}(t) \cap X_{n}=\left\{x_{i_{1}}, \ldots, x_{i_{m}}\right\}\right\} \subset R_{3}$.

In 2010, W. Puninagool and S. Leeratanavalee [5] showed that $E\left(\operatorname{Hyp}_{G}(n)\right)=$ $\left(R_{1} \cup R_{2} \cup E\right)$ is the set of all idempotent elements in $H y p_{G}(n)$.

Definition [7]. An element $a$ of a semigroup $S$ is called completely regular element if there exists $b \in S$ such that $a=a b a$ and $a b=b a$.

Proposition 4. For each $\sigma_{t} \in U\left(H y p_{G}(n)\right)$, $\sigma_{t}$ is a completely regular element in $H y p_{G}(n)$.

Proof. Let $\sigma_{t} \in U\left(H y p_{G}(n)\right)$. Then there exists $\sigma_{t^{-1}} \in U\left(H y p_{G}(n)\right) \subseteq H y p_{G}(n)$ such that $\sigma_{t} \circ_{G} \sigma_{t^{-1}}=\sigma_{i d}=\sigma_{t^{-1}} \circ_{G} \sigma_{t}$ and $\sigma_{t} \circ_{G} \sigma_{t^{-1}} \circ_{G} \sigma_{t}=\sigma_{t}$.

Proposition 5. For each $\sigma_{t} \in E\left(H y p_{G}(n)\right)$, $\sigma_{t}$ is completely regular element in $H y p_{G}(n)$.

Proof. The proof is obvious.
Let $\sigma_{t} \in \operatorname{Hyp}_{G}(n)$, we denote $C R\left(R_{3}\right):=\left\{\sigma_{t} \mid t=f\left(t_{1}, \ldots, t_{n}\right)\right.$ where $t_{i_{1}}=x_{\pi\left(i_{1}\right)}$, $\ldots, t_{i_{m}}=x_{\pi\left(i_{m}\right)}$ and $\pi$ is a bijective map on $\left\{i_{1}, \ldots, i_{m}\right\}$ for some $i_{1}, \ldots, i_{m} \in$ $\{1, \ldots, n\}$ and $\left.\operatorname{var}(t) \cap X_{n}=\left\{x_{\pi\left(i_{1}\right)}, \ldots, x_{\pi\left(i_{m}\right)}\right\}\right\}$.

Then we have $\left(E \cup U\left(\operatorname{Hyp}_{G}(n)\right)\right) \subseteq C R\left(R_{3}\right) \subset R_{3}$.
Proposition 6. For each $\sigma_{t} \in C R\left(R_{3}\right)$, $\sigma_{t}$ is completely regular element in $H_{y p}{ }_{G}(n)$.

Proof. Let $\sigma_{t} \in C R\left(R_{3}\right)$. Then $t=f\left(t_{1}, \ldots, t_{n}\right)$ where $t_{i_{1}}=x_{\pi\left(i_{1}\right)}, \ldots, t_{i_{m}}=$ $x_{\pi\left(i_{m}\right)}$ and $\pi$ is a bijective map on $\left\{i_{1}, \ldots, i_{m}\right\}$ for some $i_{1}, \ldots, i_{m} \in\{1, \ldots, n\}$ and $\operatorname{var}(t) \cap X_{n}=\left\{x_{\pi\left(i_{1}\right)}, \ldots, x_{\pi\left(i_{m}\right)}\right\}$. Let $s \in W_{(n)}(X)$ where $s=f\left(s_{1}, \ldots, s_{n}\right)$ such that $s_{\pi\left(i_{1}\right)}=x_{i_{1}}, \ldots, s_{\pi\left(i_{m}\right)}=x_{i_{m}}$. Let $t_{k} \in \operatorname{sub}\left(t_{j}\right)$ and $s_{k} \in \operatorname{sub}\left(s_{j}\right)$ for all $j \in\{1, \ldots, n\} \backslash\left\{i_{1}, \ldots, i_{m}\right\}$ and $k \in\{1, \ldots, n\}$. If $\operatorname{var}\left(t_{k}\right) \cap X_{n}=\emptyset$ then we choose $s_{k}=t_{k}$. And, if $t_{k}=x_{\pi\left(i_{l}\right)}$ and $\pi\left(i_{p}\right)=i_{l}$ for some $i_{p}, i_{l} \in\left\{i_{1}, \ldots, i_{m}\right\}$ we choose $s_{k}=x_{i_{p}}$. By Lemma 3, we have $\sigma_{t}{ }^{\circ}{ }_{G} \sigma_{s}{ }^{\circ}{ }_{G} \sigma_{t}=\sigma_{t}$. Next, we will show that $\sigma_{t}{ }^{\circ}{ }_{G} \sigma_{s}=\sigma_{s} \circ_{G} \sigma_{t}$. Consider

$$
\left(\sigma_{t} \circ_{G} \sigma_{s}\right)(f)=S^{n}\left(f\left(t_{1}, \ldots, t_{n}\right), \hat{\sigma}_{t}\left[s_{1}\right], \ldots, \hat{\sigma}_{t}\left[s_{n}\right]\right)=f\left(w_{1}, \ldots, w_{n}\right)
$$

where $w_{i}=S^{n}\left(t_{i}, \hat{\sigma}_{t}\left[s_{1}\right], \ldots, \hat{\sigma}_{t}\left[s_{n}\right]\right)$ for all $i \in\{1, \ldots, n\}$. And consider

$$
\left(\sigma_{s} \circ_{G} \sigma_{t}\right)(f)=S^{n}\left(f\left(s_{1}, \ldots, s_{n}\right), \hat{\sigma_{s}}\left[t_{1}\right], \ldots, \hat{\sigma_{s}}\left[t_{n}\right]\right)=f\left(u_{1}, \ldots, u_{n}\right)
$$

where $u_{i}=S^{n}\left(s_{i}, \hat{\sigma_{s}}\left[t_{1}\right], \ldots, \hat{\sigma_{s}}\left[t_{n}\right]\right)$ for all $i \in\{1, \ldots, n\}$.
Case 1. $i_{l} \in\left\{i_{1}, \ldots, i_{m}\right\}$.
Since $\pi$ is a bijective map on $\left\{i_{1}, \ldots, i_{m}\right\}$, there exists $i_{p} \in\left\{i_{1}, \ldots, i_{m}\right\}$ such that $\pi\left(i_{p}\right)=i_{l}$. Then
$u_{i_{l}}=S^{n}\left(s_{i_{l}}, \hat{\sigma_{s}}\left[t_{1}\right], \ldots, \hat{\sigma_{s}}\left[t_{n}\right]\right)=S^{n}\left(x_{i_{p}}, \hat{\sigma_{s}}\left[t_{1}\right], \ldots, \hat{\sigma}_{s}\left[t_{n}\right]\right)=\hat{\sigma_{s}}\left[t_{i_{p}}\right]=x_{\pi\left(i_{p}\right)}=x_{i_{l}}$
and

$$
w_{i_{l}}=S^{n}\left(t_{i_{l}}, \hat{\sigma}_{t}\left[s_{1}\right], \ldots, \hat{\sigma}_{t}\left[s_{n}\right]\right)=S^{n}\left(x_{\pi\left(i_{l}\right)}, \hat{\sigma}_{t}\left[s_{1}\right], \ldots, \hat{\sigma}_{t}\left[s_{n}\right]\right)=\hat{\sigma}_{t}\left[s_{\pi\left(i_{l}\right)}\right]=x_{i_{l}}
$$

So $u_{i_{l}}=w_{i_{l}}$ for all $l \in\{1, \ldots, m\}$.
Case 2. $j \in\{1, \ldots, n\} \backslash\left\{i_{1}, \ldots, i_{m}\right\}$.
Let $t_{k} \in \operatorname{sub}\left(t_{j}\right)$ and $s_{k} \in \operatorname{sub}\left(s_{j}\right)$ for all $k \in\{1, \ldots, n\}$. Then $w_{j}=S^{n}\left(t_{j}, \hat{\sigma}_{t}\left[s_{1}\right]\right.$, $\left.\ldots, \hat{\sigma}_{t}\left[s_{n}\right]\right)$ and $u_{j}=S^{n}\left(s_{j}, \hat{\sigma}_{s}\left[t_{1}\right], \ldots, \hat{\sigma_{s}}\left[t_{n}\right]\right)$. We put $w_{k}^{\prime}=S^{n}\left(t_{k}, \hat{\sigma}_{t}\left[s_{1}\right], \ldots\right.$, $\left.\hat{\sigma_{t}}\left[s_{n}\right]\right)$ and $u_{k}^{\prime}=S^{n}\left(s_{k}, \hat{\sigma_{s}}\left[t_{1}\right], \ldots, \hat{\sigma_{s}}\left[t_{n}\right]\right)$ for all $k \in\{1, \ldots, n\}$. If $\operatorname{var}\left(t_{k}\right) \cap$ $X_{n}=\emptyset$, then $w_{k}^{\prime}=t_{k}$ and $u_{k}^{\prime}=s_{k}=t_{k}$. If $t_{k}=x_{\pi\left(i_{l}\right)}$ and $\pi\left(i_{p}\right)=i_{l}$, then
$w_{k}^{\prime}=S^{n}\left(t_{k}, \hat{\sigma}_{t}\left[s_{1}\right], \ldots, \hat{\sigma}_{t}\left[s_{n}\right]\right)=S^{n}\left(x_{\pi\left(i_{l}\right)}, \hat{\sigma}_{t}\left[s_{1}\right], \ldots, \hat{\sigma}_{t}\left[s_{n}\right]\right)=\hat{\sigma}_{t}\left[s_{\pi\left(i_{l}\right)}\right]=x_{i_{l}}$
and
$u_{k}^{\prime}=S^{n}\left(s_{k}, \hat{\sigma_{s}}\left[t_{1}\right], \ldots, \hat{\sigma}_{s}\left[t_{n}\right]\right)=S^{n}\left(x_{i_{p}}, \hat{\sigma_{s}}\left[t_{1}\right], \ldots, \hat{\sigma_{s}}\left[t_{n}\right]\right)=\hat{\sigma_{s}}\left[t_{i_{p}}\right]=x_{\pi\left(i_{p}\right)}=x_{i_{l}}$.
So $w_{j}=u_{j}$ for all $j \in\{1, \ldots, n\} \backslash\left\{i_{1}, \ldots, i_{m}\right\}$.
Hence $f\left(w_{1}, \ldots, w_{n}\right)=f\left(u_{1}, \ldots, u_{n}\right)$, so $\sigma_{t} \circ_{G} \sigma_{s}=\sigma_{s}{ }^{\circ}{ }_{G} \sigma_{t}$. Therefore $\sigma_{t}$ is completely regular element in $H y p_{G}(n)$.

Lemma 7. Let $t=f\left(t_{1}, \ldots, t_{n}\right)$ where $t_{i_{1}}=x_{j_{1}}, \ldots, t_{i_{m}}=x_{j_{m}}$ for some $i_{1}, \ldots, i_{m}$ and for distinct $j_{1}, \ldots, j_{m} \in\{1, \ldots, n\}$ and $\operatorname{var}(t) \cap X_{n}=\left\{x_{j_{1}}, \ldots, x_{j_{m}}\right\}$. If there exists $l \in\{1, \ldots, m\}$ such that $t_{i_{l}}=x_{j_{l}}$ where $i_{l} \notin\left\{j_{1}, \ldots, j_{m}\right\}$, then $\sigma_{t} \neq \sigma_{s}{ }^{\circ} \sigma_{t}^{2}$ for all $\sigma_{s} \in \operatorname{Hyp}_{G}(n)$.
Proof. Assume that the condition holds. Consider

$$
\left(\sigma_{t} \circ_{G} \sigma_{t}\right)(f)=\hat{\sigma}_{t}[t]=S^{n}\left(f\left(t_{1}, \ldots, t_{n}\right), \hat{\sigma}_{t}\left[t_{1}\right], \ldots, \hat{\sigma}_{t}\left[t_{n}\right]\right)=f\left(u_{1}, \ldots, u_{n}\right)
$$

where $u_{i}=S^{n}\left(t_{i}, \hat{\sigma}_{t}\left[t_{1}\right], \ldots, \hat{\sigma}_{t}\left[t_{n}\right]\right)$ for all $i \in\{1, \ldots, n\}$. We have $u_{i}=$ $S^{n}\left(t_{i}, \hat{\sigma_{t}}\left[t_{1}\right], \ldots, \hat{\sigma}_{t}\left[t_{n}\right]\right) \in\left\{x_{j_{1}}, \ldots, x_{j_{m}}\right\}$ if and only if $t_{i}=x_{i_{k}}$ for some $k \in$ $\{1, \ldots, m\}$. Since $i_{l} \notin\left\{j_{1}, \ldots, j_{m}\right\}, t_{i} \neq x_{i_{l}}$ for all $i \in\{1, \ldots, n\}$. So $u_{i} \neq x_{j_{l}}$. Hence $\sigma_{t}^{2}(f)=f\left(u_{1}, \ldots, u_{n}\right)$ where $u_{i} \neq x_{j_{l}}$ for all $i \in\{1, \ldots, n\}$. Let $\sigma_{s} \in$ $\operatorname{Hyp}_{G}(n)$. Next, we will show that $\sigma_{t} \neq \sigma_{s} \circ_{G} \sigma_{t}^{2}$. If $s=x_{i}$ where $x_{i} \in X$, then $\left(\sigma_{s} \circ_{G} \sigma_{t}^{2}\right)(f)=x_{j} \neq \sigma_{t}(f)$ for some $x_{j} \in X$. If $s=f\left(s_{1}, \ldots, s_{n}\right)$ where $s_{1}, \ldots, s_{n} \in W_{(n)}(X)$, then

$$
\begin{aligned}
\left(\sigma_{s} \circ_{G} \sigma_{t}^{2}\right)(f) & =\hat{\sigma_{s}}\left[f\left(u_{1}, \ldots, u_{n}\right)\right] \\
& =S^{n}\left(f\left(s_{1}, \ldots, s_{n}\right), \hat{\sigma}_{s}\left[u_{1}\right], \ldots, \hat{\sigma}_{s}\left[u_{n}\right]\right) \\
& =f\left(w_{1}, \ldots, w_{n}\right)
\end{aligned}
$$

where $w_{i}=S^{n}\left(s_{i}, \hat{\sigma}_{s}\left[u_{1}\right], \ldots, \hat{\sigma}_{s}\left[u_{n}\right]\right)$ for all $i \in\{1, \ldots, n\}$. Since $u_{i} \neq x_{j_{l}}$ for all $i \in\{1, \ldots, n\}, \hat{\sigma_{s}}\left[u_{i}\right] \neq x_{j_{l}}$. So $w_{i} \neq x_{j_{l}}$ for all $i \in\{1, \ldots, n\}$. Hence $f\left(w_{1}, \ldots, w_{n}\right) \neq f\left(t_{1}, \ldots, t_{n}\right)$, so $\sigma_{t} \neq \sigma_{s}{ }^{\circ} \sigma_{t}^{2}$.

Theorem 8. An element a of a semigroup $S$ is completely regular if and only if $a \in a^{2} S a^{2}$.

Proof. See [7].
Theorem 9. Let $C R\left(\operatorname{Hyp}_{G}(n)\right):=C R\left(R_{3}\right) \cup R_{1} \cup R_{2}$. Then $C R\left(\operatorname{Hyp}_{G}(n)\right)$ is the set of all completely regular elements in $H y p_{G}(n)$.

Proof. By Proposition 5 and Proposition 6, every element in $C R\left(\operatorname{Hyp}_{G}(n)\right)$ is completely regular element. Let $\sigma_{t}$ be a regular element where $\sigma_{t} \notin C R\left(\operatorname{Hyp}_{G}(n)\right)$. Then $\sigma_{t} \in R_{3} \backslash C R\left(R_{3}\right)$. By Lemma $7, \sigma_{t} \neq \sigma_{s}{ }^{\circ}{ }_{G} \sigma_{t}^{2}$ for all $\sigma_{s} \in H y p_{G}(n)$. Then $\sigma_{t} \neq\left(\sigma_{t}^{2} \circ_{G} \sigma_{u}\right) \circ_{G} \sigma_{t}^{2}$ where $\sigma_{t}^{2} \circ_{G} \sigma_{u} \in H y p_{G}(n)$. By Theorem $8, \sigma_{t}$ is not completely regular element in $\operatorname{Hyp}_{G}(n)$. Therefore $C R\left(\operatorname{Hyp}_{G}(n)\right)$ is the set of all completely regular elements in $H y p_{G}(n)$.

Definition [7]. An element $a$ of a semigroup $S$ is called left(right) regular if $a \in S a^{2}\left(a \in a^{2} S\right)$ and $a$ is called intra-regular if $a \in S a^{2} S$.

Theorem 10. An element a of a semigroup $S$ is completely regular if and only if $a$ is both left regular and right regular.

Proof. See [7].
Theorem 11. Let $S$ be a semigroup and $a \in S$. If $a$ is completely regular, then $a$ is intra-regular.

Proof. Let $a$ be a completely regular. Then there exists $b \in S$ such that $a=a b a$ and $a b=b a$. So $a=a b a=a(a b)=a b a(a b)=(a b) a^{2}(b) \in S a^{2} S$.

Corollary 12. Let $\sigma_{t} \in C R\left(\operatorname{Hyp}_{G}(n)\right)$. Then $\sigma_{t}$ is is both left regular and right regular element in $\operatorname{Hyp}_{G}(n)$, and $\sigma_{t}$ is intra-regular element in $\operatorname{Hyp}_{G}(n)$.

Corollary 13. If $\sigma_{t} \in R_{3} \backslash C R\left(R_{3}\right)$, then $\sigma_{t}$ is not left regular element in $H y p_{G}(n)$.

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[^0]:    ${ }^{1}$ Corresponding author.

