Discussiones Mathematicae
General Algebra and Applications 33 (2013) 121–128
doi:10.7151/dmgaa.1201

SOME REMARKS ON PRÜFER MODULES

S. Ebrahimi Atani, S. Dolati Pishhesari

and

M. Khoramdel

Faculty of Mathematical Sciences
University of Guilan
P.O. Box 1914 Rasht, Iran

e-mail: ebrahimi@guilan.ac.ir
saboura_dolati@yahoo.com
mehdikhoramdel@gmail.com

Abstract

We provide several characterizations and investigate properties of Prüfer modules. In fact, we study the connections of such modules with their endomorphism rings. We also prove that for any Prüfer module M, the forcing linearity number of M, $\mathrm{fln}(M)$, belongs to $\{0, 1\}$.

Keywords: Prüfer modules, Prüfer domains, invertible submodules, duo modules, forcing linearity number.

2010 Mathematics Subject Classification: 13C10, 13F05, 13C13.

1. Introduction

Throughout this paper, R will denote a commutative domain with identity and M a unital R-module. For the sake of completeness, we state some definitions and notations used throughout. A nonzero ideal I of R is said to be *invertible* if $II^{-1} = R$, where $I^{-1} = \{x \in K : xI \subseteq R\}$ and K is the field of fractions of R. The concept of an invertible submodule was introduced by Naoum and Al-Alwan [9] as a generalization of the concept of an invertible ideal. Let M be an R-module and S the set of all nonzero divisors of R. Then

$$T = \{t \in S : tm = 0 \text{ for some } m \in M \text{ implies } m = 0\}$$

is a multiplicatively closed subset of R. It is clear that if M is torsion free, then $T = S$. Now let $T^{-1}R$ be the localization of R at T in the usual sense. Let N be a nonzero submodule of M and $N' = \{x \in T^{-1}R : \ xN \subseteq M\}$. Following Naoum and Al-Alwan [9], we say that N is *invertible*, if $NN' = M$ and M is called a Dedekind module provided that each nonzero submodule of M is invertible.

An R-module M is called a *Prüfer* module, if every nonzero finitely generated submodule of M is invertible. Clearly, Dedekind modules are Prüfer modules. But the converse is not true. Let R be a Prüfer domain which is not a Dedekind domain, then every nonzero finitely generated ideal of R is a Prüfer R-module which is not a Dedekind R-module.

In the present paper, we show that every Prüfer module is uniform (i.e., every two nonzero submodules have nonzero intersection), and also every torsion free Prüfer module has rank one. We give equivalent conditions for Prüfer modules and Prüfer domains. We also prove that a finitely generated torsion free R-module M is Prüfer module if and only if O(M) is a Prüfer domain and M is a uniform R-module. Moreover, for a Prüfer module over a commutative domain R we study the concept of a forcing linearity number which is a type of measure of how much local linearity is needed to imply global linearity.

2. Preliminaries

In order to make this paper easier to follow, we recall in this section various notions from module theory which will be used in the sequel.

Definition. (a) The *rank* of an R-module M is defined to be the maximal number of elements of M linearly independent over R (it is easy to see that $\mathrm{rank}_R(M)$ equals $\dim_K(S^{-1}M)$).

(b) An R-module M is called a *cancellation* module, if for all ideals I and J of R, $IM \subseteq JM$ implies $I \subseteq J$.

(c) An R-module M is called a *multiplication* module when for each submodule N of M, there exists an ideal I of R such that $N = IM$.

(d) A submodule N of M is called *fully invariant*, if $f(N) \subseteq N$ for each $f \in \mathrm{End}(M)$ (we denote the ring of R-endomorphism of M by $\mathrm{End}(M)$). An R-module M is called a *duo* module provided that every submodule of M is fully invariant [11].

(e) Let M be an R-module. A submodule N of M is called dense if,

$$\mathrm{Tr}_M(N) = \sum_{\gamma \in \mathrm{Hom}_R(N,M)} \gamma(N) = M.$$

Also M is called a π-module if each nonzero submodule of M is dense in M.

Remark 1. (1) We say that $xn \in M$ where $x = \frac{r}{t} \in T^{-1}R$ and $n \in M - \{0\}$ as long as there exists an element $m \in M$ such that $tm = rn$ for some $r \in R$ [9].

(2) If M is a torsion-free R-module, then O(M) $= \{x \in K : xM \subseteq M\}$, the *order* of M in K. Note that O(M) is a subring of K containing R, and M is an O(M)-module. We will use the notation ${}_{\mathrm{O(M)}}M$ to indicate that M is regarded as an O(M)- module.

(3) Let

$$\mathcal{M}_R(M) = \{\ f : M \to M|\ f(rx) = rf(x),\ r \in R,\ x \in M\},$$

$$\mathrm{End}_R(M) = \{\ f \in \mathcal{M}_R(M)|\ f(x+y) = f(x) + f(y),\ x, y \in M\}.$$

Following Hausen and Johnson [4] an R-module M is called *endomorphal* if $\mathrm{End}_R(M) = \mathcal{M}_R(M)$. The set $\mathcal{M}_R(M)$ is the collection of homogeneous functions determined by the R-module M. Note that $\mathcal{M}_R(M)$ contains $\mathrm{End}_R(M)$. If $\mathcal{M}_R(M) = \mathrm{End}_R(M)$, that is, if every R-homogeneous function from M to M is an endomorphism, then M is said to be endomorphal.

We need the following propositions proved in [9, Proposition 1.3(ii)], [15, Theorem A], [11, Theorem 3.7, Lemma 3.2] and [14, Lemma 2.8], respectively.

Proposition 2. *Let M be non-zero R-module and N be a submodule of M.*

(i) *If $N = Rn$, then N is invertible in M if and only if for each $m \in M$, there exist $t \in T$ and $r \in R$ such that $tm = rn$.*

(ii) *If $\mathrm{Ann}(M) = A$, then the following statements are equivalent.*
   - (a) *M is a multiplication module.*
   - (b) *M is a finitely projective (R/A)-module and every submodule of M is fully invariant.*
   - (c) *M is a finitely projective (R/A)-module and $\mathrm{End}(M)$ is a commutative ring.*

(iii) *If R is a commutative domain, then the following statements are equivalent for a non-zero finitely generated torsion-free R-module M.*
   - (d) *M is a duo module.*
   - (e) *M contains a non-zero cyclic fully invariant submodule.*
   - (f) *M is a uniform module and* O(M) $= R$.

(iv) *Let U be a torsion-free uniform R-module. Then a mapping $f : U \to U$ is an endomorphism of U if and only if there exists $k \in O(U)$ such that $f(u) = ku$ for all $u \in U$.*

(v) *If M is a torsion-free R-module, then $\mathcal{M}_R(M) = \mathrm{End}_R(M)$ if and only if M has rank one.*

3. Prüfer modules

To prove the main theorems of this paper, we need to develop some further properties of Prüfer modules. We start with the following proposition:

Proposition 3. *Let M be a Prüfer R-module. Then the following hold:*

(i) *$T^{-1}M$ is a simple module as a $T^{-1}R$-module.*

(ii) *If M is a torsion-free R-module, then* $\mathrm{rank}_R(M) = 1$.

Proof. (i) Assume that M is a Prüfer R-module; we show that $T^{-1}M$ is simple as a $T^{-1}R$-module. Let $\mathcal{N}$ be a nonzero submodule of $T^{-1}M$. Then $\mathcal{N} = T^{-1}N$ for some $N \leq M$. Let $\frac{m}{s} \in T^{-1}M$ ($m \in M,\ s \in T$). Since for each $n \in N$, Rn is an invertible submodule (see [9]), there exist $t \in T$ and $r \in R$ such that $tm = rn$ by Proposition 2(i). Therefore $\frac{m}{s} \in T^{-1}N$. Thus $T^{-1}M$ is a simple $T^{-1}R$-module.

(ii) Since M is a torsion free module, $T = S$; hence the module $S^{-1}M$ is a simple K-module by (i). Therefore $\mathrm{rank}_R(M) = 1$. ■

Lemma 4. *Every finitely generated torsion free Prüfer R-module is isomorphic to an ideal of R.*

Proof. We will prove this lemma similar to [1, Corollary 3.7]. Let $M = Rm_1 + \ldots + Rm_s$, where s is a positive integer, and choose a nonzero element $x \in M$. Then by Proposition 3, $S^{-1}M = Kx$. Therefore, $m_i = \frac{a_i}{t}x$ for some $a_i \in R$ and $0 \neq t \in R$. Thus $M \subseteq R(\frac{x}{t})$ and for each element m of M, there exists $a \in R$ such that $m = a(\frac{x}{t})$. The element a is uniquely determined by m. Consequently, we can define an R-monomorphism f from M to R with $f(m) = a$, where $m = a(\frac{x}{t})$. An inspection will show that f is a monomorphism. Hence M is isomorphic to an ideal of R. ■

Now we give the following proposition which will be very useful in proving our aims.

Proposition 5. *If M is a Prüfer R-module, then M is a uniform module.*

Proof. Let N and N' be two nonzero submodules of M, and $N \cap N' = \{0\}$. Let $n \in N$ such that $n \notin N'$. Since Rn is invertible, for each $0 \neq n' \in N'$, there exist $t \in T$ and $r \in R$ such that $tn' = rn$, by Proposition 2(i). Therefore $tn' = rn \in N \cap N' = \{0\}$, whence $n' = 0$, a contradiction. Thus for each nonzero submodules $N,\ N'$ of M, $N \cap N' \neq 0$ and so M is a uniform R-module. ■

Saraç *et al.* used π-modules to characterize Dedekind modules, [13, Proposition 12]. In the following theorem, we characterize Prüfer modules by their finitely generated dense submodules.

Theorem 6. *A torsion free R-module M is a Prüfer module if and only if M is a uniform module and each finitely generated submodule of M is dense in M.*

Proof. Assume that M is a Prüfer R-module and let N be a nonzero finitely generated submodule of M such that f is an R-homomorphism from N to M. Let $n \in N$. Since M is a uniform module (by Proposition 5), there exist $a, b \in R$ such that $af(n) = bn$. By uniformity of M, it is easy to see that $f(n') = \frac{b}{a}n'$ for each $n' \in N$. Since $f(N) \subseteq M$, $\frac{b}{a} \in N'$. Thus $NN' = M$, because $\mathrm{Tr}_M(N) = M$. Therefore M is a Prüfer module. The other implication is clear. ■

Let $\mathcal{W}$ be a set of submodules of M. Then $\mathcal{W}$ is forcing if an R-homogenous map $f : M \to M$, that is linear on each submodule $W \in \mathcal{W}$, is linear on M [8]. As in [14], to each non-zero R-module M we assign a number $\mathrm{fln(M)} \in \mathbb{N} \cup \{0, \infty\}$, call the forcing linearity number of M, as follows:

(i) If $\mathcal{M}_R(M) = \mathrm{End}_R(M)$, then $\mathrm{fln(M)} = 0$;

(ii) If $\mathcal{M}_R(M) \neq \mathrm{End}_R(M)$, then

$$\mathrm{fln}(M) = \min\{|\mathcal{W}| : \mathcal{W} \text{ is forcing}\}.$$

(iii) If neither of the above conditions holds, we say $\mathrm{fln(M)} = \infty$.

In the next theorem, we characterize the forcing linearity of Prüfer modules.

Theorem 7. *If M is a Prüfer module, then* $\mathrm{fln(M)}$ *is either* 0 *or* 1.

Proof. Let $0 \neq m \in M$. Set $\mathcal{W} = \{Rm\}$. We claim that $\mathcal{W}$ forces linearity on M. Let f be a homogeneous on M and linear on Rm. Let $x, y \in M$. Put $q = f(x + y) - f(x) - f(y)$. Since Rm is invertible, there exist $t_1, t_2 \in T$ such that $t_1x,\ t_2x \in Rm$. Thus we have $t_1t_2q = f(t_1t_2x + t_1t_2y) - f(t_1t_2x) - f(t_1t_2y)$. Since f is linear on Rm, $t_1t_2q = 0$. So $q = 0$, as desired. ■

Now we give a condition on prüfer modules to satisfy the property "every homogeneous map is linear".

Proposition 8. *If M is a torsion free prüfer module, then* $\mathrm{fln(M)} = 0$.

Proof. By Proposition 3, $\mathrm{rank}(M) = 1$. Thus by Proposition 2(v), $\mathcal{M}_R(M) = \mathrm{End}_R(M)$, whence $\mathrm{fln(M)} = 0$. ■

Lemma 9. *If M is a finitely generated torsion free Prüfer R-module, then M is a Prüfer* $\mathrm{O(M)}$*-module and* $\mathrm{O(M)}$ *is a Prüfer domain.*

Proof. The proof is similar to [13, Lemma 2]. ■

Lemma 10. *Let S be a Prüfer domain such that $R \subseteq S \subseteq K$. If $\mathcal{I}$ is a nonzero finitely generated ideal of S, then $\mathcal{I}$ is a torsion free Prüfer R-module.*

Proof. Let $\mathcal{J}$ be a finitely generated R-submodule of $\mathcal{I}$. Since $S\mathcal{J}$ is finitely generated, $S\mathcal{J}$ is invertible; hence $(S\mathcal{J})^{-1}\mathcal{J} = S$. Thus $[(S\mathcal{J})^{-1}\mathcal{I}]\mathcal{J} = \mathcal{I}$. ■

In the next proposition, we prove the converse of Proposition 8 by some conditions.

Proposition 11. *Let M be a finitely generated and torsion free R-module, where* O(M) *is Prüfer. If* $\mathrm{fln(M)} = 0$*, then M is Prüfer.*

Proof. By Proposition 2(v), $\mathrm{rank}_R(M) = 1$; hence $\mathrm{rank}_{O(M)}M = 1$. By an argument like that Lemma 4, $M \cong \mathcal{J}$ for some ideal $\mathcal{J}$ of O(M). Therefore by Lemma 10, M is a Prüfer module. ■

Now, we are ready to give several characterizations of Prüfer modules. In fact, the following theorems give equivalent conditions for Prüfer domains and Prüfer modules.

Theorem 12. *If M is a finitely generated torsion free Prüfer R-module, then the following are equivalent:*

(i) *R is integrally closed;*

(ii) *R is a Prüfer domain;*

(iii) *M is a multiplication module;*

(iv) *M is a duo module;*

(v) *M is a projective module;*

(vi) *M is a cancellation module.*

Proof. (i)⇒(ii) Let I be a nonzero finitely generated proper ideal of R. Since M is finitely generated, $IM \neq M$, and so IM is a finitely generated submodule of M. Therefore, $IM(IM)' = M$. As R is an integrally closed domain, we get $I(IM)' = R$. Thus I is an invertible ideal of R.

(ii)⇒(iii) Since M is a Prüfer module, by Lemma 4, $M \cong I$ for some finitely generated ideal I of R. Since R is a Prüfer domain, I is invertible, whence I is a multiplication R-module. Thus M is a multiplication module.

(iii)⇒(iv) Follows from Proposition 2(ii).

(iv)⇒(i) Since M is a duo module, O(M) $= R$ by Proposition 2(iii). Thus by Theorem 6, R is an integrally closed domain.

(iii)⇔(vi) By Lemma 4, $M \cong I$. For an ideal I of a commutative domain R, I is projective if and only if I is multiplication. Hence M a is a multiplication module if and only if it is a projective module.

(vi)⇒(ii) Let I be any nonzero finitely generated proper ideal of R. Then

$$(IM)' = \{q \in K : \ qIM \subseteq M\} = \{q \in K : \ qI \subseteq R\} = I^{-1}.$$

Therefore $I^{-1}IM = M$. Since M is a cancellation module, $I^{-1}I = R$; thus R is a Prüfer domain.

(iii)⇒(vi) Since M is a finitely generated multiplication module, it has cancellation property by [3, Theorem 3.1]. ■

In the following, we characterize the endomorphism ring of a Prüfer module M over an integral domain R and necessary and sufficient conditions for an R-module M to be a Prüfer module are given.

Theorem 13. *If M is a finitely generated torsion free R-module, then the following are equivalent:*

(i) *M is a Prüfer R-module;*

(ii) O(M) *is a Prüfer domain and M is a uniform module as an R-module;*

(iii) O(M) *is a Prüfer domain and* $\mathrm{End}_R(M) \cong$ O(M)*;*

(iv) O(M) *is a Prüfer domain and M is a duo* O(M)*-module.*

Proof. (i)⇒(ii) Follows from Proposition 5 and Lemma 9.

(ii)⇒(iii) Since M is uniform, $\mathrm{End}({}_RM) \cong$ O(M) by Proposition 2(iv).

(iii)⇒(i) Since M is finitely generated torsion free module, M is a projective O(M)-module (see [12, Theorem 4.32]). It is clear that $\mathrm{End}({}_RM) = \mathrm{End}({}_{O(M)}M)$; hence $\mathrm{End}({}_{O(M)}M)$ is a commutative domain. By Proposition 2(ii), M is a multiplication O(M)- module. It follows that M is a Prüfer O(M)-module. Thus by Lemma 4, $M \cong \mathcal{I}$ for some finitely generated ideal $\mathcal{I}$ of O(M). Therefore by Lemma 9, M is a Prüfer R-module.

(iii)⇒(iv) Since O(M) is a Prüfer domain, and M is a finitely generated torsion free module, M is a projective O(M)-module by [12, Theorem 4.32]. As $\mathrm{End}({}_{O(M)}M)$ is commutative, by Proposition 2(ii), every submodule of ${}_{O(M)}M$ is fully invariant. Therefore M is a duo module.

(iv)⇒(iii) By an argument like that ((iii)⇒(iv)), M is a projective O(M)-module. Since every submodule of M is fully invariant, M is a multiplication O(M)-module by Proposition 2(ii). Thus by [10, Corollary 3.3], $\mathrm{End}({}_RM) = \mathrm{End}({}_{O(M)}M) \cong$ O(M). ■

Acknowledgment

The authors express their gratitude to Professor Marco Fontana for a number of helpful suggestions and thank the referee for several useful comments on the first draft of the manuscript.

References

[1] M. Alkan, B. Saraç and Y. Tiraş, *Dedekind Modules*, Comm. Alg. **33(5)** (2005) 1617–1626. doi:10.1081/AGB-200061007

[2] D.D. Anderson and D.F. Anderson, *Cancellation modules and related modules*, in: Lect. Notes Pure Appl. Math, 220 (Ed(s)), (Dekker, New York) 2001. 13–25

[3] Z.A. El-Bast and P.F. Smith, *Multiplication modules*, Comm. Alg. **16(4)** (1988) 755–779. doi:10.1080/00927878808823601

[4] J. Hausen and J.A. Johnson, *Centralizer near-rings that are rings*, J. Austral. Soc. (Series A) **59** (1995) 173–183. doi:10.1017/S144678870003857X

[5] I. Kaplansky, Commutative Rings (Boston: Allyn and Bacon, 1970).

[6] M. Khoramdel and S. Dolati Pish Hesari, *Some notes on Dedekind modules*, Hacettepe Journal of Mathematics and Statistics **40(5)** (2011) 627–634.

[7] H. Matsumura, Commutative Ring Theory (Cambridge: Cambridge University Press, 1989). doi:10.1017/CBO9781139171762

[8] C.J. Maxson and J.H. Meyer, *Forcing linearity numbers*, J. Algebra **223** (2000) 190–207. doi:10.1006/jabr.1999.7991

[9] A.G. Naoum and F.H. Al-Alwan, *Dedekind modules*, Comm. Alg. **24(2)** (1996) 397–412. doi:10.1080/00927879608825576

[10] A.G. Naoum, *On the ring of endomorphisms of finitely generated multiplication modules*, Period. Math. Hungar. **21(3)** (1990) 249–255. doi:10.1007/BF02651092

[11] A.Ç. Özcan, A. Harmanci and P.F. Smith, *Duo modules*, Glasg. Math. J. **48** (2006) 533–545. doi:10.1017/S0017089506003260

[12] J.J. Rotman, An Introduction to Homological Algebra (Academic Press, New York). 1979

[13] B. Saraç, P.F. Smith and Y. Tiraş, *On Dedekind Modules*, Comm. Alg. **35(5)** (2007) 1533–1538. doi:10.1080/00927870601169051

[14] J. Sanwong, *Forcing Linearity Numbers for Multiplication Modules*, Comm. Alg. **34** (2006) 4591–4596. doi:10.1080/00927870600936740

[15] P.F. Smith, *Multiplication Modules and Projective Modules*, Period. Math. Hungar. **29(2)** (1994) 163–168. doi:10.1007/BF01876873

Received 28 April 2012
Revised 16 June 2013