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# ON RATIONAL RADII COIN REPRESENTATIONS OF THE WHEEL GRAPH

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**Abstract**

A *flower* is a coin graph representation of the wheel graph. A *petal* of a flower is an outer coin connected to the center coin. The results of this paper are twofold. First we derive a parametrization of all the rational (and hence integer) radii coins of the 3-petal flower, also known as Apollonian circles or Soddy circles. Secondly we consider a general $n$-petal flower and show there is a unique irreducible polynomial $P_n$ in $n$ variables over the rationals $\mathbb{Q}$, the affine variety of which contains the cosinus of the internal angles formed by the center coin and two consecutive petals of the flower. In that process we also derive a recursion that these irreducible polynomials satisfy.

**Keywords:** planar graph, coin graph, flower, polynomial ring, Galois theory.

**2010 Mathematics Subject Classification:** 05C10, 05C25, 05C31, 05C35.

## 1. Introduction

By a *coin graph* we mean a graph whose vertices can be represented as closed, non-overlapping disks in the Euclidean plane such that two vertices are adjacent if and only if their corresponding disks intersect at their boundaries, i.e., they touch. For $n \in \mathbb{N}$ the *wheel graph* $W_n$ on $n+1$ vertices is the simple graph obtained by connecting an additional center vertex to all the vertices of the cycle $C_n$ on $n$ vertices. A coin graph representation of a wheel graph is called a *flower*. The coins of a flower connected to the center vertex are called *petals*, so a coin representation of $W_n$ has $n$ petals and is called an *$n$-petal flower*. In Figure 1 we see an example of a flower on the left, and a configuration of coins that does not form a flower on the right.

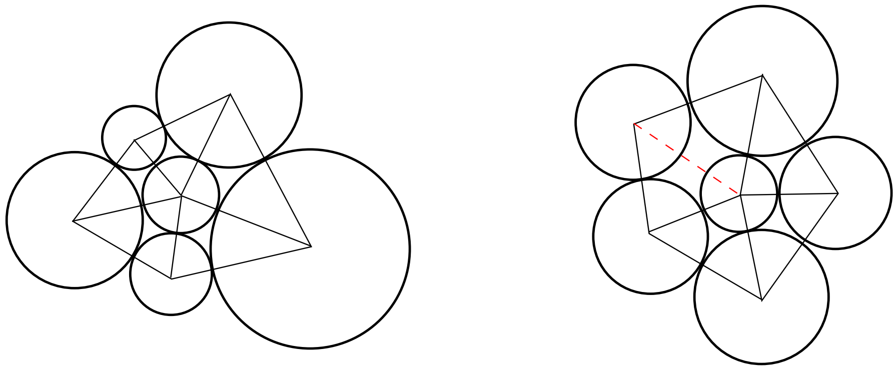

Figure 1. Examples of a flower and a non-flower.

The study of flowers is central in many discrete geometrical settings, in particular in circle packings [14] and also in the study of planar graphs in general, since every planar graph has a coin graph representation. That a coin graph is planar is clear, but that the converse is true is a nontrivial topological result, usually credited to Thurston [15], but is also due to both Koebe [11] and Andreev [1]. For a brief history of this result we refer to [16, p. 118]. Numerous simply stated but extremely hard problems involving coin graphs can be found in a recent and excellent collection of research problems in discrete geometry [3]. Also, Brightwell and Scheinerman [4] explored integral representations of coin graphs, where the radii of the coins can take arbitrary positive integer values.

In Section 2 we start by considering the case of $n = 3$, and we obtain all rational (and hence integer) radii of four mutually tangent circles, sometimes called *Soddy circles* after Frederick Soddy, an English chemist who rediscovered Descartes' Circle Theorem in 1936 [2] or *Apollonian circles* after Apollonius of Perga, who first studied mutually tangent circles more than two thousand years ago. This will be obtained in elementary ways, involving only Euclidean geometry, high school algebra and elementary number theory. Still our parametrization is new and differs from the one obtained by Graham *et al.* in [5] as we will see in in the last subsection of Section 2.

As a first step toward the study of the general case when $n \geq 3$, we will in the remaining sections then study an algebraic relation the radii of general $n$-petal flowers must satisfy, and show that for every $n \geq 3$ the cosines of the central angles of an $n$-petal flower are contained in the affine variety of a uniquely determined symmetric and irreducible polynomial $P_n$ in $n$ variables over the rationals. We note that the cosines are more interesting than the sines in this investigation, for the mere reason that cosines of the angles of an integer-sided triangle are all rational. Hence, a necessary requirement for the radii to be integers is that the mentioned cosines are all rational. Note that, unlike for the case $n = 3$, the radii are *not* determined by the central angles alone for $n \geq 4$, where a fixed set of central angle values can yield a wide range of radii, something we shall see in Section 3.

The rest of the paper is organized as follows:

In Section 2 we consider the special case of a 3-petal flower and we derive our first main result: a free parametrization of all rational radii of the outer circles when the inner circle has radius one, in an elementary way. This will then yield a new free parametrization of all integer radii of four mutually tangent Soddy circles.

In Section 3 we use Galois theory to formally define the polynomials $P_n(x_1, \dots, x_n)$, whose affine variety contain $(\cos\theta_1, \dots, \cos\theta_n)$ where $\theta_1, \dots, \theta_n$ are the internal angles of an $n$-petal flower for $n \geq 3$, and prove some useful lemmas.

In Section 4 we first obtain a recursive presentation of $P_n(x_1, \dots, x_n)$, and use that to prove our second main result of this paper, that each symmetric $P_n$ is an irreducible polynomial over $\mathbb{Q}$.

## 2. Rational radii of Soddy circles

Here we deal with the special case of $n = 3$, and we characterize all rational solutions for 3-petal flowers, thereby obtaining all rational radii of four mutually tangent Soddy circles. This section uses only elementary mathematics, but serves as a motivation (or justification) for our discussion in the sections to follow.

In general, to find all integer-radii coins forming an $n$-petal flower in the Euclidean plane, it is equivalent by scaling to find all rational radii coins where the center coin is assumed to have radius one. Hence, assume we have a 3-petal flower with a center radius of one and the outer radii $r_1, r_2$ and $r_3$. If the internal angle formed by the center coin and two outer coins of radii $r_i$ and $r_{i+1}$ is $\theta_i$, then by the law of cosines we have

$$x_i := \cos\theta_i = \frac{r_i + r_{i+1} - r_i r_{i+1} + 1}{r_i + r_{i+1} + r_i r_{i+1} + 1}, \tag{1}$$

and the equation relating the three angles is then $\theta_1 + \theta_2 + \theta_3 = 2\pi$. Clearly, if each radius $r_i$ is rational, then so is each $x_i$. Isolating one angle, say $\theta_3$, and taking the cosine of both sides we obtain

$$x_3 = x_1 x_2 - \sqrt{(1 - x_1^2)(1 - x_2^2)}. \tag{2}$$

For rational $x_1$ and $x_2$ it is clear that $x_3$ will be rational if and only if the term under the radical is the square of a rational number. For $i = 1, 2$ let $x_i = \frac{p_i}{q_i}$ for with $p_i, q_i \in \mathbb{Z}$. By (2) we then obtain

$$x_3 = x_1 x_2 - \frac{1}{q_1 q_2}\sqrt{(q_1^2 - p_1^2)(q_2^2 - p_2^2)}.$$

Here $(q_1^2 - p_1^2)(q_2^2 - p_2^2)$ is a square if and only if $q_i^2 - p_i^2 = s_i^2 \beta$ and $s_i \in \mathbb{N}$ for $i = 1, 2$ where $\beta$ is a square-free integer. Here we need the following result from elementary number theory:

**Theorem 2.1.** *Let $\beta$ be a square-free integer. The integers $x, y, z$ form a primitive solution to the Diophantine equation $x^2 + \beta y^2 = z^2$ if and only if there are positive integers $m$ and $n$ and a factorization $\beta = bc$ where $bm^2$ and $cn^2$ are relatively prime such that $x = \frac{bm^2 - cn^2}{2}$, $y = mn$, $z = \frac{bm^2 + cn^2}{2}$, where both $m$ and $n$ are odd or both are even, or $x = bm^2 - cn^2$, $y = 2mn$, $z = bm^2 + cn^2$ otherwise.*

For a proof of Theorem 2.1, see Appendix A.

Since $x_i = \frac{p_i}{q_i}$ for $i = 1, 2$ we have by Theorem 2.1 that

$$x_1 = \frac{b_1 m_1^2 - c_1 n_1^2}{b_1 m_1^2 + c_1 n_1^2}, \quad x_2 = \frac{b_2 m_2^2 - c_2 n_2^2}{b_2 m_2^2 + c_2 n_2^2}, \tag{3}$$

where $\beta = b_1 c_1 = b_2 c_2$ are two (not necessarily distinct) factorizations of the square-free integer $\beta$, and where $m_i, n_i$ can be chosen from the nonnegative integers. Note that either solution from Theorem 2.1 will yield the same form of $x_1$ and $x_2$ in (3).

By (1) we have

$$x_1 = \frac{r_1 + r_2 - r_1 r_2 + 1}{r_1 + r_2 + r_1 r_2 + 1}, \quad x_2 = \frac{r_2 + r_3 - r_2 r_3 + 1}{r_2 + r_3 + r_2 r_3 + 1}, \quad x_3 = \frac{r_3 + r_1 - r_3 r_1 + 1}{r_3 + r_1 + r_3 r_1 + 1}.$$

Rewriting each equation for $x_i$ as a polynomial equation in terms of $r_i$ and $r_{i+1}$ ( where $4 \equiv 1$ modulo 3) and then factoring in terms of $r_i$ and $r_{i+1}$ we obtain

$$\begin{aligned}
\left(r_1 + \frac{x_1-1}{x_1+1}\right)\left(r_2 + \frac{x_1-1}{x_1+1}\right) &= \frac{2(1-x_1)}{(x_1+1)^2},\\
\left(r_2 + \frac{x_2-1}{x_2+1}\right)\left(r_3 + \frac{x_2-1}{x_2+1}\right) &= \frac{2(1-x_2)}{(x_2+1)^2},\\
\left(r_3 + \frac{x_3-1}{x_3+1}\right)\left(r_1 + \frac{x_3-1}{x_3+1}\right) &= \frac{2(1-x_3)}{(x_3+1)^2}.
\end{aligned}$$

Now we can solve the first and third equations for $r_2$ and $r_3$ respectively in terms of $r_1, x_1, x_3$. Substituting these into the second equation, we can then solve that for $r_1$ in terms of $x_1, x_2, x_3$ obtaining

$$r_1 = \frac{-1 - x_1x_3 + x_3 + x_1 \pm \sqrt{2(1-x_1)(1-x_2)(1-x_3)}}{2x_2 - x_1 + x_1x_3 - 1 - x_3}. \tag{4}$$

Putting $x_1$ and $x_2$ from (3) into (2) we obtain

$$x_3 = x_1x_2 - \sqrt{(1-x_1^2)(1-x_2^2)} = \frac{\left(b_1m_1^2 - c_1n_1^2\right)\left(b_2m_2^2 - c_2n_2^2\right) - 4m_1m_2n_1n_2\beta}{(b_1m_1^2 + c_1n_1^2)(b_2m_2^2 + c_2n_2^2)}.$$

Substituting this expressions for $x_3$ and those of $x_1$ and $x_2$ from (3) into (4), we get an expression for $r_1$ in terms of $b_1, b_2, c_1, c_2, m_1, m_2, n_1, n_2$:

$$\begin{aligned}
r_1 &= \frac{n_1(b_2c_1^2m_2^2n_1^3 + 2\beta c_1m_1m_2n_1^2n_2 + \beta c_2m_1^2n_1n_2^2)}{b_1c_1c_2m_1^2n_1^2n_2^2 - b_2c_1^2m_2^2n_1^4 + c_1^2c_2n_1^4n_2^2 - 2\beta c_1m_1m_2n_1^3n_2 + b_1^2c_2m_1^4n_2^2}\\
&\pm \frac{n_1n_2(b_1m_1^2 + c_1n_1^2)\sqrt{c_1c_2(b_1c_2m_1^2n_2^2 + 2\beta m_1m_2n_1n_2 + b_2c_1m_2^2n_1^2)}}{b_1c_1c_2m_1^2n_1^2n_2^2 - b_2c_1^2m_2^2n_1^4 + c_1^2c_2n_1^4n_2^2 - 2\beta c_1m_1m_2n_1^3n_2 + b_1^2c_2m_1^4n_2^2}.
\end{aligned}$$

Using the fact that $\beta = b_1c_1 = b_2c_2$, the expression under the square root can be reduced to $\beta(c_2m_1n_2 + c_1m_2n_1)^2$. Thus, this expression for $r_1$ will only yield a perfect square when $\beta = 1$. Therefore, $r_1$ is rational if and only if $q_i^2 - p_i^2 = s_i^2$ for $i = 1, 2$, or in other words when $1 - x_i^2$ is a rational square for $i = 1, 2$. This means that both $\cos\theta_i$ and $\sin\theta_i$ are rational for $i = 1, 2, 3$.

**Proposition 2.2.** *The 3-petal flower with the center coin of radius one can have the outer coins of rational radii $r_1, r_2, r_3$ if and only if the internal angles $\theta_1, \theta_2, \theta_3$ have both rational cosines and sines for $i = 1, 2, 3$.*

Proposition 2.2 shows a property that is very special for the $n$-petal flower with rational radii when $n = 3$. We now can write a "nice" parametrization for the cosines $x_i$ and the radii $r_i$ in the case when $n = 3$: namely,

$$x_1 = \frac{m_1^2 - n_1^2}{m_1^2 + n_1^2}, \quad x_2 = \frac{m_2^2 - n_2^2}{m_2^2 + n_2^2}, \quad x_3 = \frac{\left(m_1^2 - n_1^2\right)\left(m_2^2 - n_2^2\right) - 4m_1m_2n_1n_2}{(m_1^2 + n_1^2)(m_2^2 + n_2^2)}.$$

for some $m_1, n_1, m_2, n_2 \in \mathbb{N}$. Putting these into (4) and the similar equations for $r_2$ and $r_3$ we obtain the rational forms for $r_1, r_2, r_3$ that contain all rational radii for the outer coins of a 3-petal flower with center coin of radius one:

$$\begin{aligned} r_1 &= \frac{n_1(m_1n_2 + m_2n_1)}{-m_1n_1n_2 - m_2n_1^2 + (m_1^2n_2 + n_1^2n_2)} \\ r_2 &= \frac{n_1n_2}{-n_1n_2 \pm (m_2n_1 + m_1n_2)} \\ r_3 &= \frac{n_2(m_1n_2 + m_2n_1)}{-m_1n_2^2 - m_2n_1n_2 \pm (n_1n_2^2 + m_2^2n_1)}. \end{aligned}$$

We will now determine a range of the parameters that will yield meaningful solutions in the above parametrization.

**Observation 2.3.** *If $\theta_1, \theta_2, \theta_3$ are the internal angles of a 3-petal flower, then $90^\circ < \theta_i < 180^\circ$ for each $i = 1, 2, 3$ and these three inequalities are all sharp.*

For a proof of Observation 2.3, see Appendix B.

By Observation 2.3 we now know that for all the angles $\theta_i$, we have $90^\circ < \theta_i < 180^\circ$, and hence $-1 < \cos\theta_i < 0$. Hence, in the parametrization of $x_1$ and $x_2$

$$x_1 = \frac{m_1^2 - n_1^2}{m_1^2 + n_1^2}, \quad x_2 = \frac{m_2^2 - n_2^2}{m_2^2 + n_2^2},$$

we must have $n_i > m_i$. In this case $x_3$ in (2) must satisfy $\left(m_1^2 - n_1^2\right)\left(m_2^2 - n_2^2\right) - 4m_1m_2n_1n_2 < 0$, which is equivalent to $(m_1n_2 + m_2n_1)^2 > (m_1m_2 - n_1n_2)^2$. Since $m_i < n_i$, this is equivalent to $m_1n_2 + m_2n_1 > n_1n_2 - m_1m_2$, or equivalently

$$m_1n_2 + m_2n_1 + m_1m_2 > n_1n_2. \tag{5}$$

Looking at the expression for $r_1$,

$$r_1 = \frac{n_1(m_1n_2 + m_2n_1)}{-m_1n_1n_2 - m_2n_1^2 \pm (m_1^2n_2 + n_1^2n_2)}$$

we see that in order for $r_1 > 0$ to hold we must have $n_2(m_1^2 + n_1^2) > n_1(m_1n_2 + m_2n_1)$. Re-solving for the radii $r_2$ and $r_3$ using the positive term in the expression for $r_1$, we obtain:

$$r_2 = \frac{n_1 n_2}{m_2 n_1 + m_1 n_2 - n_1 n_2}, \quad r_3 = \frac{n_2(m_1 n_2 + m_2 n_1)}{n_1 n_2^2 + m_2^2 n_1 - m_1 n_2^2 - m_2 n_1 n_2},$$

which give us two additional constraints in order to ensure positive radii: $m_2 n_1 + m_1 n_2 > n_1 n_2$ and $n_2(m_1 n_2 + m_2 n_1) > n_1(m_2^2 + n_2^2)$. Note that the first of these constraints is stronger than (5). Letting $t_i = m_i/n_i$ for each $i = 1, 2, 3$, we can express our first main theorem that characterizes all rational radii (and hence integer radii by scaling) Soddy circles as follows.

**Theorem 2.4.** *If a 3-petal flower has rational radii and the innermost coin has radius of one, then the cosines $x_1, x_2$ and $x_3$ of the angles at the innermost coin are given by*

$$x_1 = \frac{t_1^2 - 1}{t_1^2 + 1}, \quad x_2 = \frac{t_2^2 - 1}{t_2^2 + 1}, \quad x_3 = \frac{(t_1^2 - 1)(t_2^2 - 1) - 4t_1 t_2}{(t_1^2 + 1)(t_2^2 + 1)}$$

*and the corresponding radii by*

$$\begin{aligned} r_1 &= \frac{t_1 + t_2}{t_1^2 - t_1 - t_2 + 1}, \\ r_2 &= \frac{1}{t_1 + t_2 - 1}, \\ r_3 &= \frac{t_1 + t_2}{t_2^2 - t_1 - t_2 + 1}, \end{aligned}$$

*for some $t_1, t_2 \in \mathbb{Q}$ such that $1 < t_1 + t_2$, $t_1 + t_2 < t_1^2 + 1$, $t_1 + t_2 < t_2^2 + 1$. Further, all rational 3-petal flowers have rational radii and internal cosines parametrized as above in terms of such $t_1, t_2 \in \mathbb{Q}$. This parametrization characterizes all sets of four mutually tangent Soddy circles of rational radius in the plane.*

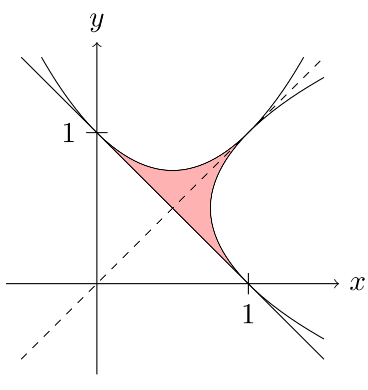


Figure 2. The region $R$.

We note that the conditions the rational parameters $t_1$ and $t_2$ satisfy in the above Theorem 2.4 is simply $(t_1, t_2) \in R \cap \mathbb{Q}^2$ where

$$R = \{(x, y) \in \mathbb{R}^2 : 1 < x + y, \quad y < x^2 - x + 1, \quad x < y^2 - y + 1\},$$

is an open region symmetric about the line $x = y$, which looks like a "hyperbolic triangle" with vertices $(0, 1), (1, 0)$ and $(1, 1)$, and where the internal angle at each of the vertices is 0 (see Figure 2.) By the 2-dimensionality of the region $R$, we note that the parametrization of the radii and the internal cosines in the above Theorem 2.4 in terms of $t_1, t_2 \in \mathbb{Q}$ is free in the sense that they don't satisfy any equation.

**Example.** Consider $t_1 = 2/3$ and $t_2 = 1/2$. We see that $(t_1, t_2) = (2/3, 1/2) \in R$, and are hence legitimate rational parameters by Theorem 2.4. Here we have $x_1 = -\frac{5}{13}$, $x_2 = -\frac{3}{5}$, $x_3 = -\frac{33}{65}$, and the corresponding radii $r_1 = \frac{21}{5}$, $r_2 = 6$, $r_3 = 14$. By scaling by the factor of 5 we obtain an integral flower with center radius of $r = 5$ and the outer radii $r_1 = 21$, $r_2 = 30$ and $r_3 = 70$. See Figure 3.

**Remark.** Consider a 3-flower with the center coin of radius one. Assume further the center coin is centered at the origin of the complex plane $\mathbb{C}$. The inversion $z \mapsto 1/\overline{z}$ of the center circle will map all the outer disks to corresponding disks inside the center disk. This configuration has three non-overlapping disks completely inside the center disk, and does therefore not represent a 3-flower. However, this inversion will keep all the angles $\theta_1, \theta_2$ and $\theta_3$ and hence their cosines $x_1, x_2$ and $x_3$ intact. The radii $r_1, r_2$ and $r_3$ will however change to $r_1', r_2'$ and $r_3'$ respectively, where $(1 + r_i)(1 - r_i') = 1$ or $r_i' = \frac{r_i}{1+r_i}$ for each $i$. By Theorem 2.4 this will automatically yield a free parametrization of three rational radii, non-overlapping mutually touching coins, all three touching the unit circle on its *inside* where $(t_1, t_2) \in R \cap \mathbb{Q}^2$ as before.

### 2.1. Descartes' circle theorem and another parametrization

A nice relation connecting the radii of four mutually tangent Soddy circles in the Euclidean plane is given by Descartes' circle theorem [2].

**Theorem 2.5** (Descartes)**.** *A collection of four mutually tangent circles in the plane, where $b_i = 1/r_i$ denotes the curvatures of the circles, satisfies the relation*

$$b_1^2 + b_2^2 + b_3^2 + b_4^2 = \frac{1}{2}(b_1 + b_2 + b_3 + b_4)^2.$$

**Remark.** Theorem 2.5 has been generalized to higher dimensions.

It is straightforward to check that our rational parametrization from Theorem 2.4 satisfies Descartes' circle theorem. Another elegant parametrization of integer Soddy circles are given by Graham et al. in [5] in the following theorem.

**Theorem 2.6** (Graham, Lagarias, Mallows, Wilks, Yan)**.** *The following parametrization characterizes the integral curvatures of a set of Soddy circles:*

$$b_1 = x, \quad b_2 = d_1 - x, \quad b_3 = d_2 - x, \quad b_4 = -2m + d_1 + d_2 - x,$$

*where* $x^2 + m^2 = d_1 d_2$ *and* $0 \leq 2m \leq d_1 \leq d_2$.

We conclude this section by briefly comparing our rational parametrization to the one given by Theorem 2.6. Suppose we have a 3-petal flower, the coins of which have integer radii. Further, assume the center coin is the coin with curvature $b_1$. By conveniently permuting indices, the remaining outer coins have radii $r_1, r_2, r_3$ given by

$$r_1 = \frac{b_1}{b_2}, \quad r_2 = \frac{b_1}{b_4}, \quad r_3 = \frac{b_1}{b_3}.$$

Using the rational parametrization of $r_1, r_2$ and $r_3$ given by Theorem 2.4 and replacing each $b_i$ with the integer parametrization from Theorem 2.6, we can solve for $d_1/x, d_2/x$ and $m/x$ in terms of $t_1$ and $t_2$ and obtain

$$\frac{m}{x} = \frac{1 - t_1 t_2}{t_1 + t_2}, \quad \frac{d_1}{x} = \frac{t_1^2 + 1}{t_1 + t_2}, \quad \frac{d_2}{x} = \frac{t_2^2 + 1}{t_1 + t_2}.$$

From this it is clear that $1 + (m/x)^2 = (d_1/x)(d_2/x)$, so the quadratic equation relating the parameters in Theorem 2.6 is automatically satisfied. Secondly, the assumption $0 \leq 2m \leq d_1 \leq d_2$ in Theorem 2.6 translates to the natural assumption that the radii are ordered by $r_2 \leq r_3 \leq r_1$.

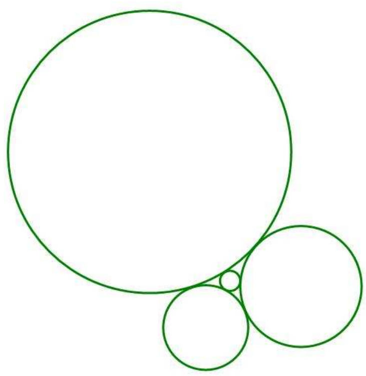

Figure 3. An example Soddy circle obtained from the parametrization.

## 3. THE INNER COSINES OF THE GENERAL $n$-PETAL FLOWER

In Section 2 we established a parametrization of all the outer radii of a 3-petal flower if the inner coin was assumed to have radius one. In doing so we first established all rational solutions of the inner cosines $x_1, x_2$ and $x_3$ from the determining relation (2). In this special case of $n = 3$, (where the inner radius is assumed to be one) the radii $r_1, r_2$ and $r_3$ are uniquely determined by the $x_i$, so next we expressed each $r_i$ in terms of these rational $x_1, x_2$ and $x_3$, thereby obtaining free parametrization of the radii $r_1, r_2$ and $r_3$. Hence, for $n \geq 4$ a necessary first step in determining all the rational radii of an $n$-petal flower with the inner coin of radius one is to determine all the rational inner cosines given by (1) where now $\theta_1, \dots, \theta_n$ form the inner angles of the $n$-petal flower in a clockwise order. When $n \geq 4$, however, the radii $r_i$ are not uniquely determined by the inner cosines $x_i = \cos\theta_i$, $i \in \{1, \dots, n\}$.

**Example.** Consider the case of a 4-petal flower with innerradius one and $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \pi/2$. In this case we have by Pythagoras's Theorem $(r_i + r_{i+1})^2 = (r_i + 1)^2 + (r_{i+1} + 1)^2$ or $r_{i+1} = (r_i + 1)/(r_i - 1)$ for each $i = 1, 2, 3, 4$, and so a complete rational parametrization of the outer radii are given by

$$r_1 = r_3 = r \in \mathbb{Q}, \quad r_2 = r_4 = \frac{r+1}{r-1}.$$

Whether a similar game can be played for any *given* inner angles $\theta_1, \dots, \theta_n$ of an $n$-petal flower with rational cosines $x_i = \cos\theta_i \in \mathbb{Q}$ is not what we will attempt to answer, but rather determining a "minimal" relation the rational inner cosines must satisfy in general. This is the topic for the remainder of this paper. We will show that for $n \geq 3$, the $x_i$ lie in an affine variety of a symmetric polynomial $P_n(x_1, \dots, x_n)$ (see [10, p. 252] for more information and general algebraic properties of symmetric polynomials.) The polynomial $P_n$ in $x_1, \dots, x_n$ will be the minimal polynomial of $\cos\theta_1, \dots, \cos\theta_n$ over $\mathbb{Q}$ where

$$\sum_{i=1}^{n} \theta_i = 2\pi. \tag{6}$$

Note that not all sets of $n$ angles $\theta_1, \dots, \theta_n$ satisfying (6) can be internal angles of an $n$-petal flower, not even when $n = 3$, as we saw in Observation 2.3. We will however see, by the way we define $P_n$ here below, that $P_n$ is symmetric when $n \geq 3$. This will be the main task of this section. We then turn to our second main contribution of the article: to prove that $P_n$ is irreducible over $\mathbb{Q}$. That we will do in the following Section 4.

**Remark.** As we remarked after Theorem 2.4, we can assume a setup where the center coin of radius one is centered at the origin of the complex plane $\mathbb{C}$. The inversion $z \mapsto 1/\overline{z}$ of this center circle will map all the outer disks to corresponding non-overlapping disks inside the center coin. Again, this configuration does not represent an $n$-flower, but the inversion will keep all the angles $\theta_1, \ldots, \theta_n$ and their cosines $x_1, \ldots, x_n$ intact. Hence, all the discussion here below about an $n$-flower, and the cosines $x_1, \ldots, x_n$ of the internal angles, also applies to the inverted configuration where the outer coins/disks are all contained inside the center unit disk.

Before we continue, we must first make some conventions and set forth some definitions: In what follows $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of natural numbers, and for $n \in \mathbb{N}$ we let $[n] = \{1, \ldots, n\}$.

Consider an $n$-petal flower with internal angles of $\theta_1, \ldots, \theta_n$. If $x_i = \cos\theta_i$ for each $i \in [n]$, then $y_i = \sin\theta_i$ satisfies the equation $x_i^2 + y_i^2 = 1$ and hence $y_i = \pm\sqrt{1 - x_i^2}$. The geometric properties of the coin graph determine that for the interior angles $\theta_i$ we have $0 \leq \theta_i < \pi$ and so $\sin\theta_i \geq 0$. Hence we have $y_i = \sqrt{1 - x_i^2}$ and so both $\cos\theta_i$ and $\sin\theta_i$ are algebraic in terms of $x_i$. We will initially view $x_1, \ldots, x_n$ on one hand and $y_1, \ldots, y_n$ on the other both as sets of $n$ independent variables respectively, related only by $y_i = \sqrt{1 - x_i^2}$ for each $i \in [n]$.

**Definition 3.1.** We define the algebraic expressions $\mathrm{EC}_n$ and $\mathrm{ES}_n$ in terms of $x_1, \ldots, x_n$, where $x_i = \cos\theta_i$ for $i \in [n]$, by taking the cosine and sine of $\sum_{i=1}^n \theta_i$ and expanding using the classical addition formulae for cosine and sine:

$$\mathrm{EC}_n(x_1, \ldots, x_n) = \cos\left(\sum_{i=1}^n \theta_i\right), \quad \mathrm{ES}_n(x_1, \ldots, x_n) = \sin\left(\sum_{i=1}^n \theta_i\right).$$

**Example.** For $n = 1$ we have $\mathrm{EC}_1(x_1) = x_1$ and $\mathrm{ES}_1(x_1) = y_1 = \sqrt{1 - x_1^2}$. For $n = 2$ we have $\mathrm{EC}_2(x_1, x_2) = x_1 x_2 - \sqrt{1 - x_1^2}\sqrt{1 - x_2^2}$ and $\mathrm{ES}_2(x_1, x_2) = x_2\sqrt{1 - x_1^2} + x_1\sqrt{1 - x_2^2}$. Directly by the addition formulae for cosine and sine we have the following recursive property of these expressions.

**Lemma 3.2.** *For each* $i \in [n]$ *we have*

$$\begin{aligned} \mathrm{EC}_n(x_1, \ldots, x_n) &= x_i \mathrm{EC}_{n-1}(\widehat{x_i}) - y_i \mathrm{ES}_{n-1}(\widehat{x_i}), \\ \mathrm{ES}_n(x_1, \ldots, x_n) &= y_i \mathrm{EC}_{n-1}(\widehat{x_i}) + x_i \mathrm{ES}_{n-1}(\widehat{x_i}). \end{aligned}$$

*where* $y_i = \sqrt{1 - x_i^2}$ *and* $(\widehat{x_i}) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. *In particular for* $i = 1$ *we have*

$$\begin{aligned} \mathrm{EC}_n(x_1, \ldots, x_n) &= x_1 \mathrm{EC}_{n-1}(\widehat{x_1}) - y_1 \mathrm{ES}_{n-1}(\widehat{x_1}), \\ \mathrm{ES}_n(x_1, \ldots, x_n) &= y_1 \mathrm{EC}_{n-1}(\widehat{x_1}) + x_1 \mathrm{ES}_{n-1}(\widehat{x_1}). \end{aligned}$$

Note that the algebraic expressions $\mathrm{EC}_n$ and $\mathrm{ES}_n$ are symmetric in $x_1, \dots, x_n$.

Recall that for the case $n = 3$ we have by (2) that $x_3 = x_1x_2 - y_1y_2$ and hence, multiplying by the conjugate, we obtain

$$0 = (x_3 - x_1x_2 - y_1y_2)(x_3 - x_1x_2 + y_1y_2) = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2x_3 - 1,$$

a symmetric polynomial equation relating the inner cosines $x_1, x_2$ and $x_3$ of the 3-petal flower, or the Soddy circles. Our objective is to obtain such a uniquely determined symmetric (and irreducible) polynomial $P_n$ for the general $n$-petal flower. Namely, by (6) we get by isolating $\theta_n$ and taking cosines of both sides that

$$x_n = EC_{n-1}(x_1, \dots, x_{n-1}). \tag{7}$$

This we will use to define $P_n$ in general for $n \in \mathbb{N}$. Before we define $P_n$ however, we need some preliminary definitions and results.

**Definition 3.3.** For $n \in \mathbb{N}$ let $G_n^*$ be the Galois group of automorphisms on $\mathbb{Q}(x_1, \dots, x_n, y_1, \dots, y_n)$ that fixes the field $\mathbb{Q}(x_1, \dots, x_n)$. Also, let $G_n$ be the Galois group of automorphisms on $\mathbb{Q}(x_1, \dots, x_n, y_iy_j : i < j)$ that fixes the field $\mathbb{Q}(x_1, \dots, x_n)$. That is,

$$\begin{aligned} G_n^* &= \mathrm{Gal}(\mathbb{Q}(x_1, x_2, \dots, x_n, y_1, \dots, y_n)/\mathbb{Q}(x_1, \dots, x_n)), \\ G_n &= \mathrm{Gal}(\mathbb{Q}(x_1, x_2, \dots, x_n, y_iy_j : i < j)/\mathbb{Q}(x_1, \dots, x_n)), \end{aligned}$$

where $x_1, \dots, x_n$ are algebraically independent indeterminates and $y_i = \sqrt{1 - x_i^2}$ for each $i$, that is $y_i$ is one root of $X^2 + x_i^2 - 1 = 0 \in \mathbb{Q}(x_1, \dots, x_n)[X]$.

**Lemma 3.4.** *For $n \geq 1$ we have $G_n^* \cong \mathbb{Z}_2^n$ and $G_n \cong \mathbb{Z}_2^{n-1}$.*

***Proof.*** For $G_n^*$, each $y_i$ is the root of an irreducible quadratic polynomial $X^2 - (1 - x_i^2)$ from the ring $\mathbb{Q}(x_1, \dots, x_n, y_1, \dots, y_{i-1})[X]$, which is the minimum polynomial of $y_i$ for each $i$. Hence we have $G_n^* \cong \mathbb{Z}_2^n$.

For $G_n$, each $y_iy_j$ with $i < j$ is also the root of an irreducible quadratic polynomial $X^2 - (1 - x_i^2)(1 - x_j^2) \in \mathbb{Q}(x_1, \dots, x_n)[X]$. However, every element of $\mathbb{Q}(x_1, x_2, \dots, x_n, y_iy_j : i < j)$ can be written as a rational function in terms of only elements of the form $y_iy_{i+1}$ as follows:

$$y_iy_j = \frac{(y_iy_{i+1})(y_{i+1}y_{i+2}) \cdots (y_{j-1}y_j)}{y_{i+1}^2 \cdots y_{j-1}^2} = \frac{(y_iy_{i+1})(y_{i+1}y_{i+2}) \cdots (y_{j-1}y_j)}{(1 - x_{i+1}^2) \cdots (1 - x_{j-1}^2)}.$$

So we have that

$$\mathbb{Q}(x_1, x_2, \ldots, x_n, y_i y_j : i < j) = \mathbb{Q}(x_1, x_2, \ldots, x_n, y_i y_{i+1} : i \in [n-1]).$$

Each term $y_i y_{i+1}$ is a root of an irreducible quadratic polynomial $X^2 - (1 - x_i^2)(1 - x_{i+1}^2)$ from the ring $\mathbb{Q}(x_1, \ldots, x_n, y_1 y_2, \ldots, y_{i-1} y_i)[X]$, which is the minimal polynomial of $y_i y_{i+1}$ for each $i \in [n-1]$. Therefore we have that $G_n \cong \mathbb{Z}_2^{n-1}$. ∎

**Lemma 3.5.** *For $n \in \mathbb{N}$, the group $G_n \cong \mathbb{Z}_2^{n-1}$ can be presented as*

$$G_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} : \sigma_i^2 = e, \sigma_i \sigma_j = \sigma_j \sigma_i \right\rangle,$$

*where each $\sigma_i$ is an automorphism fixing $\mathbb{Q}(x_1, \ldots x_n)$ and*

$$\sigma_i(y_j y_{j+1}) = \begin{cases} -y_j y_{j+1} & \text{if } i = j \\ y_j y_{j+1} & \text{if } i \neq j. \end{cases}$$

***Proof.*** Since $(y_i y_{i+1})^2 = (1 - x_i^2)(1 - x_{i+1}^2)$ and the Galois group $G_n$ is fixing the $x_i$, the only possible automorphisms are $\sigma(y_i y_{i+1}) = -y_i y_{i+1}$ and $\sigma(y_i y_{i+1}) = y_i y_{i+1}$. We can then generate the group as in the statement of the theorem with $n-1$ generators $\sigma_i$. ∎

**Corollary 3.6.** *For every $\sigma \in G_n$, let $s_{\sigma;j} \in \{-1, 1\}$ be such that $\sigma(y_j y_{j+1}) = s_{\sigma;j} y_j y_{j+1}$. Then for every $i < j$ we have $\sigma(y_i y_j) = s_{\sigma;i} s_{\sigma;i+1} \cdots s_{\sigma;j} y_i y_j$. In particular, if $i < n$ then $\sigma_{n-1}(y_i y_n) = -y_i y_n$ and if $i > 1$ then $\sigma_1(y_1 y_i) = -y_1 y_i$.*

We can now give a precise definition of $P_n$ for each $n \in \mathbb{N}$.

**Definition 3.7.** For $n \in \mathbb{N}$, define the polynomial $P_n \in \mathbb{Q}[x_1, \ldots, x_n]$ by

$$P_n(x_1, \ldots, x_n) := \prod_{\sigma \in G_{n-1}} (x_n - \sigma(\mathrm{EC}_{n-1})).$$

**Remark.** From Definition 3.7 it is not clear a priori that $P_n$ is symmetric in $x_1, \ldots, x_n$.

Our first task is to prove the symmetry of $P_n$, which we now embark on. By Definition 3.1 each of the $2^{n-2}$ terms of $\mathrm{ES}_{n-1}$ in terms of $x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}$ contains positive odd factors of $y_i$ for $i \leq n-1$. Hence $\sigma_{n-1} \in G_n$ fixes $\mathbb{Q}(x_1, \ldots, x_n, y_1 y_2, \ldots, y_{n-2} y_{n-1})$ and $\sigma_{n-1}(y_{n-1} y_n) = -y_{n-1} y_n$. Noting this, we then have by Corollary 3.6 the following:

**Claim 3.8.** *For $n \geq 2$ we have $G_n = G_{n-1} \cup G_{n-1} \sigma_{n-1} = G_{n-1} \cup \sigma_{n-1} G_{n-1}$ and $\sigma_{n-1}(y_n \mathrm{ES}_{n-1}) = -y_n \mathrm{ES}_{n-1}$.*

If $G_n$ is presented as in Lemma 3.5, then $\sigma_{n-1} \in G_n$ fixes $\mathbb{Q}(x_1, \ldots, x_n, y_1y_2, \ldots, y_{n-2}y_{n-1})$ and $\sigma_{n-1}(y_{n-1}y_n) = -y_{n-1}y_n$.

**Lemma 3.9.** *For $n \in \mathbb{N}$ let $G_n$ be presented as in Lemma 3.5. Then*

$$(\mathrm{EC}_n - 1)\left(\sigma_{n-1}\left(\mathrm{EC}_n\right) - 1\right) = \left(x_n - \mathrm{EC}_{n-1}\right)^2.$$

***Proof.*** By Claim 3.8 we have $\sigma_{n-1}(y_n\mathrm{ES}_{n-1}) = -y_n\mathrm{ES}_{n-1}$ and hence

$$\begin{aligned}
&(\mathrm{EC}_n - 1)(\sigma_{n-1}(\mathrm{EC}_n) - 1) \\
&= (x_n\mathrm{EC}_{n-1} - y_n\mathrm{ES}_{n-1} - 1)(\sigma_{n-1}(x_n\mathrm{EC}_{n-1} - y_n\mathrm{ES}_{n-1}) - 1) \\
&= (x_n\mathrm{EC}_{n-1} - y_n\mathrm{ES}_{n-1} - 1)(x_n\mathrm{EC}_{n-1} + y_n\mathrm{ES}_{n-1} - 1) \\
&= (x_n\mathrm{EC}_{n-1} - 1)^2 - y_n^2\mathrm{ES}_{n-1}^2 \\
&= (x_n\mathrm{EC}_{n-1} - 1)^2 - (1 - x_n^2)(1 - \mathrm{EC}_{n-1}^2) \\
&= (x_n - \mathrm{EC}_{n-1})^2.
\end{aligned}$$

■

**Corollary 3.10.** *For $n \geq 2$ we have $P_n^2 = C_n$ where*

$$C_n = \prod_{\sigma \in G_n} (\sigma(\mathrm{EC}_n) - 1).$$

**Remark.** It is clear that $C_n$ is symmetric in $x_1, \ldots, x_n$.

***Proof.*** By Lemma 3.9 we obtain:

$$\begin{aligned}
C_n &= \prod_{\sigma \in G_n} (\sigma(\mathrm{EC}_n) - 1) \\
&= \prod_{\sigma \in \sigma_{n-1}G_{n-1} \cup G_{n-1}} (\sigma(\mathrm{EC}_n) - 1) \\
&= \prod_{\sigma \in G_{n-1}} (\sigma(\mathrm{EC}_n) - 1)(\sigma_{n-1}\sigma(\mathrm{EC}_n) - 1) \\
&= \prod_{\sigma \in G_{n-1}} \sigma((\mathrm{EC}_n - 1)(\sigma_{n-1}(\mathrm{EC}_n) - 1)) \\
&= \prod_{\sigma \in G_{n-1}} \sigma((x_n - \mathrm{EC}_{n-1})^2) \\
&= \prod_{\sigma \in G_{n-1}} (x_n - \sigma(\mathrm{EC}_{n-1}))^2 \\
&= P_n^2.
\end{aligned}$$

■

By exactly the same token as Claim 3.8, Lemma 3.9, and Corollary 3.10, we obtain analogous results by reordering the variables $y_1, \dots, y_n$ in the reverse order: $y_n, y_{n-1}, \dots, y_1$. Namely, if $\sigma_i \in G_n$ is the field automorphism of $\mathbb{Q}(x_1, \dots, x_n, y_1y_2, y_2y_3, \dots, y_{n-1}y_n)$ with $\sigma_i(y_iy_{i+1}) = -y_iy_{i+1}$ fixing $\mathbb{Q}(x_1, \dots, x_n)$ and each $y_jy_{j+1}$ for $j \neq i$ (as in Lemma 3.5) then we have the following:

**Claim 3.11.** *If $n \geq 2$ then $G_n = G'_{n-1} \cup \sigma_1 G'_{n-1} = G'_{n-1} \cup G'_{n-1}\sigma_1$ where $G'_{n-1} = \langle \sigma_2, \dots, \sigma_{n-1} \rangle$, a subgroup of $G_n = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$, and $\sigma_1(y_1 \mathrm{ES}_{n-1}(x_2, \dots, x_n)) = -y_1 \mathrm{ES}_{n-1}(x_2, \dots, x_n)$.*

***Proof.*** By Definition 3.1, each of the $2^{n-2}$ terms of $\mathrm{ES}_{n-1}(x_2, \dots, x_n) = \mathrm{ES}_{n-1}(x_2, \dots, x_n, y_2, \dots, y_n)$ (by substituting $y_i = \sqrt{1 - x_i^2}$ for each $i = 2, \dots, n$) has positive odd factors of $y_i$ for $i \geq 2$. Hence the claim follows by Corollary 3.6. ∎

Similarly to Lemma 3.9 we now have the following.

**Lemma 3.12.** *If $\sigma_1 \in G_n$ is as above then*

$$(\mathrm{EC}_n - 1)(\sigma_1(\mathrm{EC}_n - 1)) = (x_1 - \mathrm{EC}_{n-1}(x_2, \dots, x_n))^2.$$

***Proof.*** By Claim 3.11 we obtain

$$\begin{aligned}
&(\mathrm{EC}_n - 1)(\sigma_1(\mathrm{EC}_n - 1)) \\
=\ & (x_1 \mathrm{EC}_{n-1}(x_2, \dots, x_n) - y_1 \mathrm{ES}_{n-1}(x_2, \dots, x_n) - 1) \\
& \cdot(\sigma_1(x_1 \mathrm{EC}_{n-1}(x_2, \dots, x_n) - y_1 \mathrm{ES}_{n-1}(x_2, \dots, x_n)) - 1) \\
=\ & (x_1 \mathrm{EC}_{n-1}(\widehat{x}_1) - y_1 \mathrm{ES}_{n-1}(\widehat{x}_1) - 1) \\
& \cdot(x_1 \mathrm{EC}_{n-1}(\widehat{x}_1) + y_1 \mathrm{ES}_{n-1}(\widehat{x}_1) - 1) \\
=\ & (x_1 \mathrm{EC}_{n-1}(\widehat{x}_1) - 1)^2 - y_1^2 \mathrm{ES}_{n-1}(\widehat{x}_1)^2 \\
=\ & (x_1 \mathrm{EC}_{n-1}(\widehat{x}_1) - 1)^2 - (1 - x_1^2)(1 - \mathrm{EC}_{n-1}(\widehat{x}_1)^2 \\
=\ & (x_1 - \mathrm{EC}_{n-1}(\widehat{x}_1))^2,
\end{aligned}$$

where $(\widehat{x_1}) = (x_2, \dots, x_n)$ as above. ∎

**Corollary 3.13.** *For $n \geq 3$ we have*

$$C_n = \prod_{\sigma \in G'_{n-1}} (x_1 - \sigma(\mathrm{EC}_{n-1}(\widehat{x}_1)))^2.$$

***Proof.*** By Lemma 3.12 we obtain as in the proof of Corollary 3.10

$$\begin{aligned}
C_n &= \prod_{\sigma\in G_n} (\sigma(\mathrm{EC}_n) - 1) \\
&= \prod_{\sigma\in G'_{n-1}\cup\sigma_1 G'_{n-1}} (\sigma(\mathrm{EC}_n) - 1) \\
&= \prod_{\sigma\in G'_{n-1}} (\sigma(\mathrm{EC}_n) - 1)(\sigma\sigma_1(\mathrm{EC}_n) - 1) \\
&= \prod_{\sigma\in G'_{n-1}} \sigma\left(((\mathrm{EC}_n) - 1)(\sigma_1(\mathrm{EC}_n) - 1)\right) \\
&= \prod_{\sigma\in G'_{n-1}} \sigma((x_1 - \mathrm{EC}_{n-1}(\widehat{x}_1))^2) \\
&= \prod_{\sigma\in G'_{n-1}} (x_1 - \sigma(\mathrm{EC}_{n-1}(\widehat{x}_1)))^2.
\end{aligned}$$

■

**Remark.** For $n = 1, 2$ we have from (7) and definition of $C_n$ in Corollary 3.10 that $P_1 = C_1 = x_1 - 1$ and $P_2 = x_2 - x_1$. However, the latter is a matter of taste, since we could have set $P_2 = x_1 - x_2$. The case $n = 2$ is the only one where $C_2(x_1, x_2)$ is symmetric while $P_2$ is not.

By Corollary 3.13 we obtain $C_n = Q_n^2$ where

$$Q_n = \prod_{\sigma\in G'_{n-1}} (x_1 - \sigma(\mathrm{EC}_{n-1}(\widehat{x}_1))) \in \mathbb{Q}[x_1, \ldots, x_n].$$

Since $P_n^2 = C_n = Q_n^2$, then as elements in a polynomial ring over a field, an integral domain, we get $0 = P_n^2 - Q_n^2 = (P_n - Q_n)(P_n + Q_n)$ and hence for each $n \geq 2$ we have $Q_n = P_n$ or $Q_n = -P_n$.

For $n = 2$ we obtain $P_2 = x_2 - x_1$ and $Q_2 = x_1 - x_2$ so $Q_2 = -P_2$.

For $n \geq 3$ we first note that by evaluating $\mathrm{EC}_{n-1}(\widehat{x}_n)$ and $\mathrm{EC}_{n-1}(\widehat{x}_1)$ at $x_2 = \cdots = x_{n-1} = 1$ yields $\mathrm{EC}_{n-1}(\widehat{x}_n)|_{x_2=\cdots=x_{n-1}=1} = x_1$ and $\mathrm{EC}_{n-1}(\widehat{x}_1)|_{x_2=\cdots=x_{n-1}=1} = x_n$ and hence we obtain

$$\begin{aligned}
P_n(x_1, 1, \ldots, 1, x_n) &= \prod_{\sigma\in G_{n-1}} (x_n - x_1) = (x_n - x_1)^{2^{n-2}} \\
Q_n(x_1, 1, \ldots, 1, x_n) &= \prod_{\sigma\in G'_{n-1}} (x_1 - x_n) = (x_1 - x_n)^{2^{n-2}}.
\end{aligned}$$

As $n \geq 3$, we have $2^{n-2}$ is even and so $(x_n - x_1)^{2^{n-2}} = (x_1 - x_n)^{2^{n-2}}$ and hence $P_n(x_1, 1, \ldots, 1, x_n) = Q_n(x_1, 1, \ldots, 1, x_n)$. Therefore we obtain the following:

**Corollary 3.14.** *For $n \geq 3$ we have $Q_n = P_n$, and hence*

$$P_n = \prod_{\sigma \in G_{n-1}} (x_1 - \sigma(\mathrm{EC}_{n-1}(\widehat{x}_1))).$$

We now show that for $n \geq 3$ the polynomial $P_n$ is symmetric. Let $n \geq 3$. If $\pi \in S_n$ is a permutation on $\{1, \ldots, n\}$ then $\pi$ acts naturally on $(x_1, \ldots, x_n)$ by $\pi(x_1, \ldots, x_n) = (x_{\pi(1)}, \ldots, x_{\pi(n)})$. By definition of $P_n$ in Corollary 3.10 we have

$$(P_n \circ \pi)(x_1, \ldots, x_n) = P_n(x_{\pi(1)}, \ldots, x_{\pi(n)}) = P_n(x_1, \ldots, x_n)$$

or $P_n \circ \pi = P_n \pi = P_n$ for all $\pi \in S_n$ with $\pi(n) = n$. Likewise by Corollary 3.14 we have $P_n \pi = P_n$ for all $\pi \in S_n$ with $\pi(1) = 1$.

Let $\tau \in S_n$ be an arbitrary transposition $\tau = (i, j)$. If $\{i, j\} \subseteq \{1, \ldots, n-1\}$ or $\{i, j\} \subseteq \{2, \ldots, n\}$ then by the above, $P_n \tau = P_n$. Otherwise if $\tau = (1, n)$ then since $n \geq 3$ there is an $l \in \{2, \ldots, n-1\}$ such that we can write $\tau = (1, n) = (1, l)(l, n)(1, l)$ where $\{1, l\} \subseteq \{2, \ldots, n\}$. From the above, we therefore have

$$P_n \tau = P_n(1, n) = P_n(1, l)(l, n)(1, l) = P_n(l, n)(1, l) = P_n(1, l) = P_n.$$

Since each permutation $\pi \in S_n$ is a composition of transpositions then we have $P_n \pi = P_n$ for each $\pi \in S_n$.

**Theorem 3.15.** *For $n \geq 3$ the polynomial $P_n = P_n(x_1, \ldots, x_n)$ is symmetric.*

**Corollary 3.16.** *For $n \geq 3$ and $i \in [n]$ we have for $G''_{n-1} = \langle \sigma_1, \ldots, \widehat{\sigma_i}, \ldots, \sigma_n \rangle$,*

$$P_n = \prod_{\sigma \in G''_{n-1}} (x_i - \sigma(\mathrm{EC}_{n-1}(\widehat{x_i}))).$$

*In particular, as a polynomial in $x_i$, then $P_n$ is monic of degree $2^{n-2}$ in each $x_i$.*

By Corollary 3.16 and definition of $C_{n-1}$ we obtain by letting $x_i = 1$ the following:

**Observation 3.17.** *For $n \geq 3$ then*

$$\begin{aligned} &P_n(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) \\ &= \prod_{\sigma \in G''_{n-1}} (1 - \sigma(\mathrm{EC}_{n-1}(\widehat{x_i}))) = C_{n-1}(\widehat{x_i}) = P_{n-1}(\widehat{x_i})^2. \end{aligned}$$

Other more general equations and formulae hold as well. Let $n \in \mathbb{N}$ and $n_1 + \cdots + n_k = n$. If $\sum_{i=1}^{n} \theta_i = 2\pi$ and $x_i = \cos \theta_i$ for each $i \in \{1, \ldots, n\}$, then for each $l \in \{1, \ldots, k\}$ let $\phi_l = \theta_{n_1+\cdots+n_{l-1}+1} + \cdots + \theta_{n_1+\cdots+n_l}$. Then $\sum_{l=1}^{k} \phi_l = 2\pi$ and hence if $\tau_l = \cos(\phi_l)$ then by Corollary 3.16 we get

$$0 = P_k(\tau_1, \ldots, \tau_l) = P_k(\mathrm{EC}_{n_1}, \ldots, \mathrm{EC}_{n_k}),$$

where for each $l \in \{1, \ldots, k\}$ we have $\mathrm{EC}_{n_l} = \mathrm{EC}_{n_l}(x_{n_1+\cdots+n_{l-1}+1}, \ldots, x_{n_1+\cdots+n_l})$. In particular for $k = n-1$ and $n_1 = \cdots = n_{n-2} = 1$ and $n_{n-1} = 2$, we have $P_{n-1}(x_1, \ldots, x_{n-2}, \mathrm{EC}_2(x_{n-1}, x_n)) = 0$, something we can use to compute $P_n$ recursively, as we will see in the next section.

## 4. The polynomial of the general flower and its irreducibility

In this section we deduce our second main result of this article: $P_n$ is irreducible over $\mathbb{Q}$. We first show that $P_n$ can be presented by a recursion, which we will then use to prove its irreducibility.

Recall that by Claim 3.8 we have for $n-1 \geq 2$ that

$$G_{n-1} = G_{n-2} \cup \sigma_{n-2} G_{n-2} = G_{n-2} \cup G_{n-2}\sigma_{n-2}$$

and $\sigma_{n-2}(y_{n-1}\mathrm{ES}_{n-2}) = -y_{n-1}\mathrm{ES}_{n-2}$.

**Lemma 4.1.** *For $n \geq 3$ we have*

$$(x_n - \mathrm{EC}_{n-1})(x_n - \sigma_{n-2}(\mathrm{EC}_{n-1})) = x_{n-1}^2 + x_n^2 - 1 - 2x_{n-1}x_n\mathrm{EC}_{n-2}^2 + \mathrm{EC}_{n-2}^2.$$

***Proof.*** Since $\mathrm{EC}_{n-1} = x_{n-1}\mathrm{EC}_{n-2} - y_{n-1}\mathrm{ES}_{n-2}$, we obtain by above

$$\begin{aligned}
&(x_n - \mathrm{EC}_{n-1})(x_n - \sigma_{n-2}(\mathrm{EC}_{n-1})) \\
&= (x_n - x_{n-1}\mathrm{EC}_{n-2} + y_{n-1}\mathrm{ES}_{n-2}) \\
&\quad \cdot (x_n - x_{n-1}\mathrm{EC}_{n-2} - y_{n-1}\mathrm{ES}_{n-2}) \\
&= (x_n - x_{n-1}\mathrm{EC}_{n-2})^2 - y_{n-1}^2\mathrm{ES}_{n-2}^2 \\
&= (x_n - x_{n-1}\mathrm{EC}_{n-2})^2 - (1 - x_{n-1}^2)(1 - \mathrm{EC}_{n-2}^2) \\
&= x_{n-1}^2 + x_n^2 - 1 - 2x_{n-1}x_n\mathrm{EC}_{n-2}^2 + \mathrm{EC}_{n-2}^2.
\end{aligned}$$

■

Let $\overline{\mathrm{EC}_2}(x_i, x_{i+1}) = x_i x_{i+1} + y_i y_{i+1}$ be the conjugate of $\mathrm{EC}_2(x_i, x_{i+1})$. By direct computation and the definition of $P_{n-1}$ we get

$$\begin{aligned}
&P_{n-1}(x_1,\dots,x_{n-2},\mathrm{EC}_2(x_{n-1}x_n))P_{n-1}(x_1,\dots,x_{n-2},\overline{\mathrm{EC}_2}(x_{n-1}x_n))\\
&=\prod_{\sigma\in G_{n-2}}(\mathrm{EC}_2(x_{n-1},x_n)-\sigma(\mathrm{EC}_{n-2}))\prod_{\sigma\in G_{n-2}}(\overline{\mathrm{EC}_2}(x_{n-1},x_n)-\sigma(\mathrm{EC}_{n-2}))\\
&=\prod_{\sigma\in G_{n-2}}(x_{n-1}x_n-y_{n-1}y_n-\sigma(\mathrm{EC}_{n-2}))\prod_{\sigma\in G_{n-2}}(x_{n-1}x_n+y_{n-1}y_n-\sigma(\mathrm{EC}_{n-2}))\\
&=\prod_{\sigma\in G_{n-2}}\left((x_{n-1}x_n-\sigma(\mathrm{EC}_{n-2}))^2-y_{n-1}^2y_n^2\right)\\
&=\prod_{\sigma\in G_{n-2}}\left((x_{n-1}x_n-\sigma(\mathrm{EC}_{n-2}))^2-(1-x_{n-1}^2)(1-x_n^2)\right)\\
&=\prod_{\sigma\in G_{n-2}}\left(x_{n-1}^2+x_n^2-1-2x_{n-1}x_n\sigma(\mathrm{EC}_{n-2}^2)+\sigma(\mathrm{EC}_{n-2}^2)\right)\\
&=\prod_{\sigma\in G_{n-2}}\sigma\left(x_{n-1}^2+x_n^2-1-2x_{n-1}x_n\mathrm{EC}_{n-2}^2+\mathrm{EC}_{n-2}^2\right).
\end{aligned}$$

From this we can prove the following:

**Theorem 4.2.** *The polynomials $P_n$ are completely determined by the following recursion: $P_1 = x_1 - 1$, $P_2 = x_2 - x_1$ and for $n \geq 3$*

$$P_n = P_{n-1}(x_1,\dots,x_{n-2},\mathrm{EC}_2(x_{n-1},x_n))P_{n-1}(x_1,\dots,x_{n-2},\overline{\mathrm{EC}_2}(x_{n-1},x_n)).$$

***Proof.*** By Lemma 4.1 and the preceding paragraph we get

$$\begin{aligned}
P_n &= \prod_{\sigma\in G_{n-1}}(x_n-\sigma(\mathrm{EC}_{n-1}))\\
&= \prod_{\sigma\in G_{n-2}\cup\sigma_{n-2}G_{n-2}}(x_n-\sigma(\mathrm{EC}_{n-1}))\\
&= \prod_{\sigma\in G_{n-2}}(x_n-\sigma(\mathrm{EC}_{n-1}))(x_n-\sigma\sigma_{n-2}(\mathrm{EC}_{n-1}))\\
&= \prod_{\sigma\in G_{n-2}}\sigma\left((x_n-(\mathrm{EC}_{n-1}))(x_n-\sigma_{n-2}(\mathrm{EC}_{n-1}))\right)\\
&= \prod_{\sigma\in G_{n-2}}\sigma\left(x_{n-1}^2+x_n^2-1-2x_{n-1}x_n\mathrm{EC}_{n-2}^2+\mathrm{EC}_{n-2}^2\right)\\
&= P_{n-1}(x_1,\dots,x_{n-2},\mathrm{EC}_2(x_{n-1},x_n))\cdot P_{n-1}(x_1,\dots,x_{n-2},\overline{\mathrm{EC}_2}(x_{n-1},x_n)).
\end{aligned}$$

∎

**Example.** With the help of MAPLE [12] the first 5 polynomials $P_n$ can now be computed quickly and efficiently by the recursion in Theorem 4.2:

$$\begin{aligned}
P_1 &= x_1 - 1.\\
P_2 &= x_2 - x_1.\\
P_3 &= P_2(x_1, \mathrm{EC}_2(x_2,x_3))P_2(x_1, \overline{\mathrm{EC}_2}(x_2,x_3))\\
&= (x_2x_3 - y_2y_3 - x_1)(x_2x_3 + y_2y_3 - x_1)\\
&= x_1^2 + x_2^2 + x_3^2 - 2x_1x_2x_3 - 1.\\
P_4 &= P_3(x_1, x_2, \mathrm{EC}_2(x_3,x_4))P_3(x_1, x_2, \overline{\mathrm{EC}_2}(x_3,x_4))\\
&= (x_1^2 + x_2^2 + (x_3x_4 - y_3y_4)^2 - 2x_1x_2(x_3x_4 - y_3y_4) - 1)\\
&\quad \cdot (x_1^2 + x_2^2 + (x_3x_4 + y_3y_4)^2 - 2x_1x_2(x_3x_4 + y_3y_4) - 1)\\
&= x_1^4 + x_2^4 + x_3^4 + x_4^4 - 2(x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_4^2 + x_1^2x_4^2 + x_1^2x_3^2 + x_2^2x_4^2)\\
&\quad + 4(x_1^2x_2^2x_3^2 + x_2^2x_3^2x_4^2 + x_1^2x_3^2x_4^2 + x_1^2x_2^2x_4^2)\\
&\quad + 4x_1x_2x_3x_4(2 - x_1^2 - x_2^2 - x_3^2 - x_4^2).\\
P_5 &= \text{a display of terms on two letter size pages, see Appendix C.}
\end{aligned}$$

The recursion given in Theorem 4.2, although fundamental for computation, is a special case of a more general recursion that $P_n$ satisfies:

**Claim 4.3.** *Let $n, k \geq 2$ and $n_1 + \cdots + n_k = n$. By the right interpretation of $\sigma_i$ for each $i \in [k]$ (and with some abuse of notation) then $P_n = P_n(x_1, \ldots, x_n)$ satisfies the following general recursion*

$$\begin{aligned}
P_n(x_1, \ldots, x_n) = \prod_{\substack{\sigma_i \in G_{n_i - 1}\\ i\in[k]}} & P_k\left(\sigma_1(\mathrm{EC}_{n_1}(x_1, \ldots, x_{n_1})), \sigma_2(\mathrm{EC}_{n_2}(x_{n_1+1}, \ldots, x_{n_1+n_2})),\right.\\
&\left. \ldots, \sigma_k(\mathrm{EC}_{n_k}(x_{n-n_k+1}, \ldots, x_n))\right).
\end{aligned}$$

This more general recursion of Claim 4.3, whose proof will be omitted, will not be used to obtain our main result Theorem 4.5 here below. A proof of Claim 4.3 can be obtained by using induction and Theorem 4.2 as a stepping stone.

**Example.** We demonstrate how Claim 4.3 works by using it to compute $P_5$ here below:

$$
\begin{aligned}
&P_5(x_1,\ldots,x_5)\\
&= \prod_{\sigma_3'\in G_3=\langle\sigma_3\rangle,\, \sigma_1'\in G_1=\langle\sigma_1\rangle,\, \sigma_2'\in G_2=\{e\}} P_3(\sigma_1'(\mathrm{EC}_2(x_1,x_2)),\sigma_2'(\mathrm{EC}_1(x_3)),\sigma_3'(\mathrm{EC}_2(x_4,x_5)))\\
&= \prod_{\sigma_3'\in G_3=\langle\sigma_3\rangle,\, \sigma_1'\in G_1=\langle\sigma_1\rangle,\, \sigma_2'\in G_2=\{e\}} P_3(\sigma_1'(x_1x_2-y_1y_2),\sigma_2'(x_3),\sigma_3'(x_4x_5-y_4y_5))\\
&= P_3(x_1x_2-y_1y_2,x_3,x_4x_5-y_4y_5)\cdot P_3(x_1x_2+y_1y_2,x_3,x_4x_5-y_4y_5)\\
&\quad\cdot P_3(x_1x_2-y_1y_2,x_3,x_4x_5+y_4y_5)\cdot P_3(x_1x_2+y_1y_2,x_3,x_4x_5+y_4y_5)\\
&= ((x_1x_2-y_1y_2)^2+x_3^2+(x_4x_5-y_4y_5)^2-2(x_1x_2-y_1y_2)x_3(x_4x_5-y_4y_5)-1)\\
&\quad\cdot((x_1x_2+y_1y_2)^2+x_3^2+(x_4x_5-y_4y_5)^2-2(x_1x_2+y_1y_2)x_3(x_4x_5-y_4y_5)-1)\\
&\quad\cdot((x_1x_2-y_1y_2)^2+x_3^2+(x_4x_5+y_4y_5)^2-2(x_1x_2-y_1y_2)x_3(x_4x_5+y_4y_5)-1)\\
&\quad\cdot((x_1x_2+y_1y_2)^2+x_3^2+(x_4x_5+y_4y_5)^2-2(x_1x_2+y_1y_2)x_3(x_4x_5+y_4y_5)-1).
\end{aligned}
$$

Expanded, this last product yields the same expression for $P_5$ as given in Appendix C.

Our final goal in this section, and our second main result of the paper, is to prove the irreducibility of $P_n$. To illuminate our approach we state and prove the following simplest case, that $P_3 = P_3(x_1,x_2,x_3)$ is irreducible.

Suppose $P_3 = fg$ with $f,g \in \mathbb{Q}[x_1,x_2,x_3]$. Since $P_3$ is monic in $x_3$, both $f$ and $g$ contain the variable $x_3$, and hence both $f$ and $g$ are of degree 1 in $x_3$ (unless $f$ or $g = P_3$.) Since $P_3$ factors in $\mathbb{Q}(x_1,x_2,y_1y_2)[x_3]$ as $P_3 = (x_3 - x_1x_2 - y_1y_2)(x_3 - x_1x_2 + y_1y_2)$ by definition of $P_3$, then since $\mathbb{Q}(x_1,x_2,y_1y_2)[x_3]$ is a unique factorization domain (UFD) we must have

$$\{f,g\} = \{x_3 - x_1x_2 - y_1y_2, x_3 - x_1x_2 + y_1y_2\}$$

which contradicts the assumption that $f,g \in \mathbb{Q}[x_1,x_2,x_3]$. Hence we have the following observation:

**Observation 4.4.** *The polynomial $P_3(x_1,x_2,x_3)$ is irreducible over $\mathbb{Q}$.*

Note that the same argument holds if $\mathbb{Q}$ is replaced with the complex field $\mathbb{C}$ in the above.

We now use this same approach to prove the following:

**Theorem 4.5.** *For each $n \geq 3$ the polynomial $P_n(x_1, \ldots, x_n)$ is irreducible over $\mathbb{Q}$.*

We will prove Theorem 4.5 by induction on $n$, assuming that $P_{n-1}$ is irreducible over $\mathbb{Q}$. But before we can delve into that, we need to prove the following:

**Lemma 4.6.** *Let $n \geq 3$. If $P_{n-1}$ is irreducible over $\mathbb{Q}$ then $P_{n-1}(\mathrm{EC}_2(x_1, x_2), x_3, \ldots, x_n) = P_{n-1}(x_1x_2 - y_1y_2, x_3, \ldots, x_n)$ and $P_{n-1}(\overline{\mathrm{EC}_2}(x_1, x_2), x_3, \ldots, x_n) = P_{n-1}(x_1x_2 + y_1y_2, x_3, \ldots, x_n)$ are irreducible in $\mathbb{Q}(x_1, x_2, y_1y_2)[x_3, \ldots, x_n]$.*

***Proof.*** Let $P^*_{n-1} := P_{n-1}(x_1x_2 - y_1y_2, x_3, \ldots, x_n)$ and assume it factors as $P^*_{n-1} = h^*k^*$ in the ring $\mathbb{Q}(x_1, x_2, y_1y_2)[x_3, \ldots, x_n]$, where both $h^*$ and $k^*$ involve $x_n$. Since $P_{n-1} = \prod_{\sigma \in G_{n-2}} (x_n - \sigma(\mathrm{EC}_{n-2}))$, we see that

$$P^*_{n-1} = \prod_{\sigma \in G_{n-2}} (x_n - \sigma(\mathrm{EC}_{n-2}(x_1x_2 - y_1y_2, x_3, \ldots, x_n))),$$

and hence both $h^*$ and $k^*$ must be products of these linear factors. In particular, we can evaluate $P^*_{n-1} = h^*k^*$ at $x_1 = 1$ and obtain

$$P_{n-1}(x_2, \ldots, x_n) = \left(P^*_{n-1}\right)|_{x_1=1} = (h^*|_{x_1-1})\,(k^*|_{x_1-1}) = hk$$

in $\mathbb{Q}(x_2)[x_3, \ldots, x_n]$, which is a UFD. By assumption $P_{n-1}(x_2, \ldots, x_n)$ is irreducible in the ring $\mathbb{Q}[x_2, \ldots, x_n] = \mathbb{Q}[x_2][x_3, \ldots, x_n]$ and hence also in $\mathbb{Q}(x_2)[x_3, \ldots, x_n]$ (as a monic polynomial in $x_n$ [10, Lemma 6.13 p. 163]). Therefore either $h$ or $k$ equals $P_{n-1}(x_2, \ldots, x_n)$, which contradicts the fact that both $h^*$ and $k^*$ involve $x_n$. Hence $P^*_{n-1}$ is irreducible. In the same way we obtain that $P_{n-1}(x_1x_2 + y_1y_2, x_3, \ldots, x_n)$ is irreducible. ∎

***Proof of Theorem* 4.5.** Let $n \geq 3$ and assume that $P_{n-1}$ is irreducible over $\mathbb{Q}$. Assume $P_n = fg$ with $f, g \in \mathbb{Q}[x_1, \ldots, x_n]$. We may assume $f$ is irreducible. Let $\phi_i : \mathbb{Q}[x_1, \ldots, x_n] \longrightarrow \mathbb{Q}[\widehat{x_i}]$ be the evaluation at $x_i = 1$, that is $\phi_i(F) = F(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$. Since $\phi_i$ is a $\mathbb{Q}$-algebra homomorphism for each $i \in [n]$ we have for $i = 1$ that

$$\phi_1(P_n) = \phi_1(fg) = \phi_1(f)\phi_1(g) \in \mathbb{Q}[x_2, \ldots, x_n].$$

But $\phi_1(P_n) = P_{n-1}(x_2, \ldots, x_n)^2 \in \mathbb{Q}[x_2, \ldots, x_n]$, which is a UFD. By the inductive hypothesis, $P_{n-1}$ is irreducible in $\mathbb{Q}[x_2, \ldots, x_n]$. Therefore, $\phi_1(f) = P_{n-1} = \phi_1(g)$ (unless $f = P_n$, in which case we are done since $f$ is irreducible).

Viewing $f, g \in \mathbb{Q}[x_1, \ldots, x_{n-1}][x_n]$, then since $P_n$ and $P_{n-1}$ are monic in every variable $x_i$ (and hence also in $x_n$,) we have

$$\deg_{x_n}(f) = \deg_{x_n}(g) = \frac{\deg_{x_n}(P_n)}{2} = 2^{n-3}.$$

By symmetry of $P_n$ for $n \geq 3$ from Theorem 3.15, and by Theorem 4.2 we have

$$P_n = P_{n-1}(\mathrm{EC}_2(x_1, x_2), x_3, \dots, x_n) P_{n-1}(\overline{\mathrm{EC}_2}(x_1, x_2), x_3, \dots, x_n)$$

in $\mathbb{Q}(x_1, x_2, y_1 y_2)[x_3, \dots, x_n]$, which is a UFD. By assumption we have $P_n = fg$ where $f \in \mathbb{Q}[x_1, \dots, x_n]$ is irreducible, and $f|_{x_1=1} = \phi_1(f) = P_{n-1}(x_2, \dots, x_n)$ which also by assumption is irreducible over $\mathbb{Q}$. That $f$ is also irreducible in $\mathbb{Q}(x_1, x_2, y_1 y_2)[x_3, \dots, x_n]$ can now be seen in the same way as in the proof of Lemma 4.6: namely, by evaluating at $x_1 = 1$ and obtain a factorization of $P_{n-1}(x_2, \dots, x_n)$.

So we have

$$P_n = fg = P_{n-1}(\mathrm{EC}_2, x_3, \dots, x_n) P_{n-1}(\overline{\mathrm{EC}_2}, x_3, \dots, x_n)$$

in $\mathbb{Q}(x_1, x_2, y_1 y_2)[x_3, \dots, x_n]$, which is a UFD. Therefore we have

$$f \in \{P_{n-1}(\mathrm{EC}_2, x_3, \dots, x_n), P_{n-1}(\overline{\mathrm{EC}_2}, x_3, \dots, x_n)\},$$

By repeated application of Observation 3.17 we obtain

$$P_{n-1}(\mathrm{EC}_2, 1, \dots, 1) = P_1(\mathrm{EC}_2)^{2^{n-2}} = (\mathrm{EC}_2 - 1)^{2^{n-2}}$$

which is not contained in $\mathbb{Q}[x_1, \dots, x_n]$. Similarly $P_{n-1}(\overline{\mathrm{EC}_2}, x_3, \dots, x_n) \notin \mathbb{Q}[x_1, \dots, x_n]$ and hence we have a contradiction, since $f \in \mathbb{Q}[x_1, \dots, x_n]$. ■

**Remark.** Replacing $\mathbb{Q}$ with $\mathbb{C}$ in the previous proofs will yield the same result.

As a corollary we obtain the following, which in fact equivalent to Theorem 4.5:

**Corollary 4.7.** *For $n \in \mathbb{N}$ we have $[\mathbb{Q}(x_1, \dots, x_n, \mathrm{EC}_n) : \mathbb{Q}(x_1, \dots, x_n)] = 2^{n-1}$. In fact, for any $m \leq n$ we have $[\mathbb{Q}(x_1, \dots, x_n, \mathrm{EC}_m) : \mathbb{Q}(x_1, \dots, x_n)] = 2^{m-1}$.*

We conclude this section with a summarizing result:

**Corollary 4.8.** *For $1 \leq m \leq n$ we have*

- $\mathbb{Q}(x_1, \dots, x_n, \mathrm{EC}_m) = \mathbb{Q}(x_1, \dots, x_n, y_1 y_2, \dots, y_{m-1} y_m)$
- $\mathrm{Gal}(\mathbb{Q}(x_1, \dots, x_n, \mathrm{EC}_m)/\mathbb{Q}(x_1, \dots, x_n))$
  $= \mathrm{Gal}(\mathbb{Q}(x_1, \dots, x_n, y_1 y_2, \dots, y_{m-1} y_m)/\mathbb{Q}(x_1, \dots, x_n))$
  $\cong \mathbb{Z}_2^{m-1}$.
- $P_{m+1}(x_1, \dots, x_m, X) \in \mathbb{Q}(x_1, \dots, x_m)[X]$ *is the minimal polynomial of* $\mathrm{EC}_m = \mathrm{EC}_m(x_1, \dots, x_m)$ *over* $\mathbb{Q}(x_1, \dots, x_m)$.

## 5. Conclusion

Four mutually tangent circles, also known as Apollonian circles or Soddy circles, have been studied for over two thousand years, starting with Apollonius of Perga and continuing through Descartes, whose Theorem 2.5 is a formula relating the curvatures of four mutually tangent circles in the Euclidean plane. In any such Soddy circle configuration in the plane, where the disks are non-overlapping, there is always a center circle, or an innermost circle. Needless to say, using an appropriate inversion, which corresponds to a Möbius transformation of the complex plane, one can always make any of the disks the inner circle. Of great interest in number theory, as demonstrated in [5], are such Soddy circles where each curvature is an integer. Any inversion of Soddy circles with rational curvatures, and hence rational radii, yields Soddy circles with rational curvature/radii. Since Soddy circles with rational radii also yield, by scaling, Soddy circles with integer radii, the case studied in Section 2 covers all integer radii and integer curvature Soddy circles. In particular, by scaling one can obtain all integer radii and integer curvature Soddy circles from the parametrization in Theorem 2.4. What is new about this given parametrization is that the rational parameters $t_1$ and $t_2$ are free, as long as $(t_1, t_2)$ is contained within the region $R$ in Figure 2. The importance of integer curvature Soddy circles has further been demonstrated by the comprehensive sequence of articles [6, 7], and [8] in which the Möbius transformations yielding integral curvatures are studied from a group theoretic perspective. Integral curvature Soddy circles, and hence integral radii Soddy circles, are an active area of research and most is still unknown, as demonstrated in [9] in which heuristic data is used to back up various conjectures on Soddy circles of prime curvature less than a given number.

Since any Soddy configuration of non-overlapping disks does have a center circle and three outer circles tangent to it, it is quite natural to consider a generalization of this, namely a configuration of a center disk, which by scaling can be assumed to have radius one, and then $n \geq 3$ disks tangentially touching the center disk in a cyclic fashion. Hence a flower, or a coin graph representation of the wheel graph $W_n$, is a natural generalization of a Soddy configuration. Further knowledge about such general disk configurations would be valuable for studying general planar circle packings where each circle is tangent to an arbitrary number of other circles instead of only three, and in particular those with integral curvature. To the best of the our knowledge, this article is the first to address the case of rational/integer radii, and hence rational/integer curvature circles for $n \geq 4$. As we saw in Section 3, an immediate difficulty is encountered when $n \geq 4$: the mere fact that the outer radii $r_1, \ldots, r_n$ are not uniquely determined by the inner cosines $x_i = \cos\theta_i$ for $i = 1, \ldots, n$. It is, however, clear that all the $x_i$ are rational if the $r_i$ are rational.

To determine all possible rational $x_i$ one must determine all rational points in the affine variety of the polynomial $P_n(x_1, \ldots, x_n)$ from Definition 3.7. This is easy enough for Soddy circles when $n = 3$, but for $n \geq 4$ this becomes progressively more difficult as $P_n$ is by Theorem 4.5 irreducible, and by Corollary 3.16 monic of degree $2^{n-2}$ in each variable $x_i$. For $n = 4$, we have $2^{n-2} = 4$, and so we think finding all rational points in the variety of $P_4$ is within reach, especially having the recursion from Theorem 4.2 in mind.

**Problem 5.1.** What does a free parametrization $(x_1, x_2, x_3, x_4)$ of all rational points in the affine variety of $P_4(x_1, x_2, x_3, x_4)$ look like?

We expect, and conjecture below, that each $x_i$ will have the form $x_i = p(t_1, t_2, t_3)/q(t_1, t_2, t_3)$ where $p$ and $q$ are rational polynomials with integer coefficients and $t_1, t_2, t_3$ are free rational variables contained in a bounded open region. In general we would like to see the following conjecture solved:

**Conjecture 5.2.** *The free parametrization $(x_i, \ldots, x_n)$ of all rational points in the affine variety of $P_n$ has the form*

$$x_i = \frac{p(t_1, \ldots, t_{n-1})}{q(t_1, \ldots, t_{n-1})},$$

*where $p$ and $q$ are rational polynomials with integer coefficients, and $(t_1, \ldots, t_{n-1}) \in R \cap \mathbb{Q}^{n-1}$ where $R$ is some open and bounded $(n-1)$-dimensional region in $\mathbb{R}^{n-1}$.*

Given a fixed rational point $(\rho_1, \rho_2, \rho_3, \rho_4)$ satisfying $P_4(\rho_1, \rho_2, \rho_3, \rho_4) = 0$, one can then attempt to determine all the rational radii $r_i$ with $\cos\theta_i = \rho_i$ for each $i$.

**Problem 5.3.** For each fixed rational point $(\rho_1, \rho_2, \rho_3, \rho_4)$ in the affine variety of $P_4$, what are the corresponding rational radii $r_1, r_2, r_3, r_4$ yielding the rational $\cos\theta_i = \rho_i$?

And more generally we would like an answer to the following general version:

**Problem 5.4.** For each fixed rational point $(\rho_1, \ldots, \rho_n)$ in the affine variety of $P_n$, what are the corresponding rational radii $r_1, \ldots, r_n$ yielding the rational $\cos\theta_i = \rho_i$?

With Theorems 4.2 and 4.5 in mind, a casual conclusion of the paper can perhaps be summed up as follows: *the "difficulty", measured by the degree of the irreducible polynomial $P_n$ in each of the variables $x_i$, in determining all the rational radii, and hence integer curvature, of a coin graph representation of the wheel graph $W_n$ for $n \geq 4$, grows "exponentially" as $n$ tends to infinity.* Although determining all the rational cosines $x_1, \ldots, x_n$ satisfying $P_n$ is hard, it certainly seems easier than determining all the rational radii $r_1, \dots, r_n$, especially since we have (1) for each $i \in \{1, \dots, n\}$.

**Remarks.** (i) It is evident that this paper has dealt exclusively with Soddy disks in the Euclidean plane. It is, however, natural to ask whether this can be generalized to higher dimension, in particular to three dimensions, and consider arrangements of touching spheres in $\mathbb{R}^3$. After all, Descartes' Theorem 2.5 does have a natural generalization, namely the *Soddy-Gosset Theorem*:

$$\sum_{i=1}^{n+2} b_i^2 = \frac{1}{n}\left(\sum_{i=1}^{n+2} b_i\right)^2,$$

where $b_i = 1/r_i$ are the oriented curvatures of $n+2$ mutually tangent spheres in $\mathbb{R}^n$ for every $n \geq 2$. However, the mere case $n = 3$ has been shown to be notoriously difficult; even the *kissing number* for $n = 3$, the maximum number of non-overlapping radius one spheres that can touch a given radius one sphere, was proved to be equal to 12, but not until 1953 [3, p. 93]. But unlike dimensions $n = 1, 2$ there is quite a bit of "wiggle room" when attaching 12 radius one spheres to a given radius one sphere for $n = 3$. Hence it seems difficult at best to obtain algebraic conditions for a set of general radii spheres all touching a given sphere of radius one, if they are all mutually touching. To the best of our knowledge, no one has yet ventured into an investigation of algebraic relations of the radii of more than four non-overlapping spheres all touching a given sphere, let alone rational radii spheres in $\mathbb{R}^3$.

(ii) Finally, Soddy circles in the Euclidean plane yield a 3-petal flower, a coin representation of the wheel graph $W_3$. In such a coin graph, all the disks are non-overlapping and hence no disk properly contains another. This setting corresponds to all curvatures being oriented with the same sign, which we can assume to be positive. However, embedding the $n$-flower into the complex plane $\mathbb{C}$ in such a way that the center unit disk is given by $|z| \leq 1$, then the inversion $z \mapsto 1/\overline{z}$ of the unit circle yields a configuration where we have $n$ circles touching the unit disk on the *inside* of it. This corresponds to the curvature of the unit disk having a different sign from the curvatures of the other disks inside it. By an appropriate Möbius transformation, we can choose any of the disks of an $n$-flower to have a curvature of different sign from the others. However, we have for the sake of consistency focused our attention on $n$-flowers, for which the disks are non-overlapping and hence all the curvatures oriented with the same sign.

## A. GENERALIZATIONS OF THE PYTHAGOREAN TRIPLES

In the following, a *primitive solution* is a solution where $x, y$, and $z$ are pairwise relatively prime. To prove Theorem 2.1, we need the following:

**Claim A.1.** *If r,s,t are positive integers such that r and s are relatively prime and $rs = t^2$ then there are relatively prime integers m and n such that $r = m^2$ and $s = n^2$.*

***Proof of Theorem* 2.1.** Note that $\gcd(b, c) = 1$. This proof follows and extends the exposition in [13].

Assume $x, y, z$ form a primitive solution. In this case, $x$ and $y$ cannot both be even.

*Case* 1. $x, y$ are both odd. Then $x^2 \equiv 1 \pmod 4$ and $y^2 \equiv 1 \pmod 4$, giving $z^2 \equiv 1 + \beta \pmod 4$. Since $z^2 \equiv 0, 1 \pmod 4$, then $\beta \equiv 0$ or $\beta \equiv 3 \pmod 4$ must hold. However, $\beta \equiv 0 \pmod 4$ implies that 4 divides $\beta$, contradicting the assumption that $\beta$ is square-free. So the only case to consider here is the case where $z$ is even and $\beta \equiv 3 \pmod 4$.

$$\beta y^2 \quad = \quad z^2 - x^2 = (z + x)(z - x). \tag{8}$$

Letting $\gcd(z + x, z - x) = d$ we get that $d$ divides both $z + x + z - x = 2z$ and $z + x - (z - x) = 2x$. Since $x$ and $z$ are relatively prime, $d = 1$ or 2. Since both $z + x$ and $z - x$ are odd, then $d = 1$ must hold. Since now $\gcd(z + x, z - x) = 1$ we have from (8) that for some factorization $\beta = bc$ then $r = z + x$ is divisible by $b$ and $s = z - x$ is divisible by $c$. Since $\gcd\left(\frac{r}{b}, \frac{s}{c}\right) = 1$, we have by Claim A.1 that $m^2 = \frac{r}{b}$ and $n^2 = \frac{s}{c}$, and hence $y = mn$, $x = \frac{r-s}{2} = \frac{bm^2-cn^2}{2}$, and $z = \frac{r+s}{2} = \frac{bm^2+cn^2}{2}$.

*Case* 2. $x$ is even and $y$ is odd. Then $x^2 \equiv 0 \pmod 4$ and $y^2 \equiv 1 \pmod 4$, giving $z^2 \equiv \beta \pmod 4$. Therefore $\beta \equiv 0$ or $\beta \equiv 1 \pmod 4$. However, $\beta \equiv 0 \pmod 4$ implies that 4 divides $\beta$, again contradicting the assumption that $\beta$ is square-free. So the only case to consider here is the case where $z$ is odd and $\beta \equiv 1 \pmod 4$, which proceeds exactly as in case 1.

*Case* 3. $x$ is odd and $y$ is even. Then $x^2 \equiv 1 \pmod 4$ and $y^2 \equiv 0 \pmod 4$, giving $z^2 \equiv 1 \pmod 4$, and so $z$ is odd.

Unlike cases 1 and 2, $z+x$ and $z-x$ are both even. Letting $\gcd\left(\frac{z+x}{2}, \frac{z-x}{2}\right) = d$ we get that $d$ divides $\frac{z+x+z-x}{2} = z$ and $\frac{z+x-(z-x)}{2} = x$. Since $x$ and $z$ are relatively prime, $d = 1$. Now we have $\frac{\beta y^2}{4} = rs$ where $r = \frac{z+x}{2}$ and $s = \frac{z-x}{2}$. Hence $b$ divides $r$ and $c$ divides $s$ for some appropriate factorization $\beta = bc$. Since $\gcd\left(\frac{r}{b}, \frac{s}{c}\right) = 1$, so we have by Claim A.1 that $m^2 = \frac{r}{b}$ and $n^2 = \frac{s}{c}$ and hence $y = 2mn$, $x = r - s = bm^2 - cn^2$, and $z = r + s = bm^2 + cn^2$.

For the other direction, first we show that $x, y, z$ as given in Cases 1 and 2 do form a solution:

$$\begin{aligned} x^2 + \beta y^2 &= \left(\frac{bm^2 - cn^2}{2}\right)^2 + \beta(mn)^2 \\ &= \frac{(bm^2)^2 - 2bm^2cn^2 + (cn^2)^2}{4} + \beta(mn)^2 \\ &= \frac{(bm^2)^2 + 2\beta m^2n^2 + (cn^2)^2}{4} \\ &= \left(\frac{bm^2 + cn^2}{2}\right)^2. \end{aligned}$$

Also for case 3 we get:

$$\begin{aligned} x^2 + \beta y^2 &= \left(bm^2 - cn^2\right)^2 + \beta(2mn)^2 \\ &= (bm^2)^2 - 2bm^2cn^2 + (cn^2)^2 + \beta(2mn)^2 \\ &= (bm^2)^2 + 2\beta m^2n^2 + (cn^2)^2 \\ &= \left(bm^2 + cn^2\right)^2. \end{aligned}$$

To show that the triple is primitive for Cases 1 and 2, assume on the contrary that $\gcd(x, y, z) = d > 1$. Then there is a prime $p$ that divides $d$. This $p$ divides $x$ and $z$ and also their sum and difference: $x + z = \frac{bm^2 - cn^2}{2} + \frac{bm^2 + cn^2}{2} = bm^2$ and $x - z = \frac{bm^2 - cn^2}{2} - \frac{bm^2 + cn^2}{2} = cn^2$. This contradicts the assumption that $bm^2$ and $cn^2$ are relatively prime.

For Case 3, again assume on the contrary that $(x, y, z) = d > 1$. Then there is an odd prime $p$ that divides $d$. $p \neq 2$ because $x$ and $z$ are both odd. This $p$ divides $x$ and $z$ and also their sum and difference: $x + z = 2bm^2$ and $x - z = 2cn^2$. Again, this contradicts the assumption that $bm^2$ and $cn^2$ are relatively prime. ■

## B. PROOF OF OBSERVATION 2.3

***Proof.*** As each $\theta_i$ is an angle in a triangle formed by the three mutually tangent coins, we have that $\theta_i < 180°$. On the other hand, keeping the radius of the center coin fixed (say, at $r = 1$) and letting $r_i = r_{i+1} \to \infty$, we see that $\theta_i \to 180°$ from below. We also see from this scenario that the other two angles tend to $90°$ from below.

What remains to show is that $\theta_i > 90°$ for each $i$. It suffices to show this for $i = 1$. By keeping the radii $r_1$ and $r_2$ fixed and letting $r_3 \to \infty$, the radius $r$ of the central coin will increase and $\theta_1$, the angle between the first and second coins, will decrease. Figure 4 illustrates this situation.

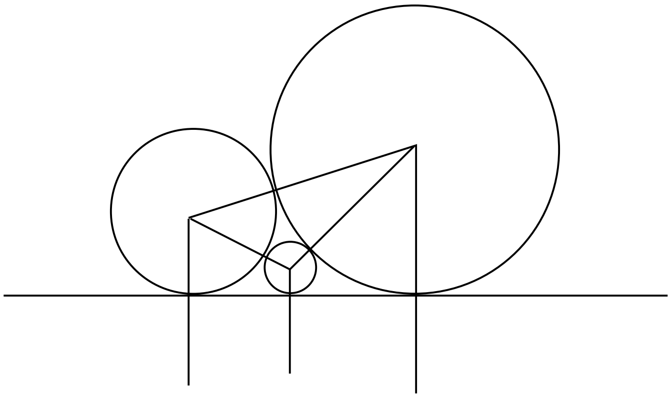

Figure 4. A 3-petal flower where the radius of one petal is increased to infinity.

It suffices to show that $\theta_1 > 90°$ for this case. If we start with Figure 4 and draw a line parallel to the infinite circle that goes through the center of the central coin, we have 2 right triangles with side lengths $r_i - r, r_i + r$ and, by the Pythagorean theorem, $2\sqrt{r_i r}$ for each $i = 1, 2$. Therefore, the length of the segment forming the bottom of the rhombus, formed by the center of the two outer circles and their touching points to the infinite circle, is $2\left(\sqrt{r_1 r} + \sqrt{r_2 r}\right)$. We can now draw a segment parallel to this segment and passing through the center of the coin with the smaller radius. Without loss of generality we may assume $r_1 \leq r_2$. Now we have a right triangle with side lengths $2\left(\sqrt{r_1 r} + \sqrt{r_2 r}\right), r_2 - r_1$ and $r_1 + r_2$ and hence by the Pythagorean theorem we have $4\left(\sqrt{r_1 r} + \sqrt{r_2 r}\right)^2 + (r_2 - r_1)^2 = (r_1 + r_2)^2$, which can be solved for $r$, obtaining

$$r = \frac{r_1 r_2}{\left(\sqrt{r_1} + \sqrt{r_2}\right)^2}.$$

With this expression for $r$, it suffices to show that

$$(r_1 + r_2)^2 > (r + r_1)^2 + (r + r_2)^2, \tag{9}$$

which implies $\theta_1 > 90°$. Letting $X = (\sqrt{r_1} + \sqrt{r_2})^2$, we get by algebraic manipulation that (9) is equivalent to $X^2 - (r_1 + r_2)X - r_1 r_2 > 0$, and since $X^2 - (r_1 + r_2)X = 2\sqrt{r_1 r_2}$, this is equivalent to $2(r_1 + r_2) + 3\sqrt{r_1 r_2} > 0$, which clearly holds. Hence, $\theta_1 > 90°$. ■

## C. The polynomial $P_5$

$$
\begin{aligned}
P_5 &= P_4(x_1,x_2,x_3,\mathrm{EC}_2(x_4,x_5))\cdot P_4(x_1,x_2,x_3,\overline{\mathrm{EC}_2}(x_4,x_5))\\
&= x_5^8 - 8x_1x_2x_3x_4x_5^7 - 8x_3^2x_4^2x_5^6 + 4x_2^2x_5^6 - 4x_5^6 + 4x_3^2x_5^6 + 16x_1^2x_2^2x_3^2x_5^6\\
&- 8x_2^2x_3^2x_5^6 - 8x_1^2x_4^2x_5^6 - 8x_1^2x_3^2x_5^6 - 8x_1^2x_2^2x_5^6 + 4x_4^2x_5^6 - 8x_2^2x_4^2x_5^6\\
&+ 16x_1^2x_3^2x_4^2x_5^6 + 16x_2^2x_3^2x_4^2x_5^6 + 4x_1^2x_5^6 + 16x_1^2x_2^2x_4^2x_5^6 + 40x_1x_2^3x_3x_4x_5^5\\
&+ 40x_1x_2x_3x_4^3x_5^5 - 32x_1^3x_2x_3x_4^3x_5^5 + 40x_1^3x_2x_3x_4x_5^5 - 32x_1^3x_2x_3^3x_4x_5^5\\
&- 32x_1x_2^3x_3^3x_4x_5^5 - 32x_1x_2^3x_3x_4^3x_5^5 - 24x_1x_2x_3x_4x_5^5 - 32x_1^3x_2^3x_3x_4x_5^5\\
&- 32x_1x_2x_3^3x_4^3x_5^5 + 40x_1x_2x_3^3x_4x_5^5 + 64x_1^2x_2^4x_3^2x_4^2x_5^4 - 16x_1^4x_4^2x_5^4\\
&+ 28x_2^2x_4^2x_5^4 - 16x_3^2x_4^4x_5^4 - 24x_1^2x_2^2x_3^2x_5^4 + 28x_1^2x_4^2x_5^4 - 12x_3^2x_5^4\\
&+ 28x_2^2x_3^2x_5^4 - 16x_2^2x_3^4x_5^4 - 16x_2^2x_4^4x_5^4 + 64x_1^2x_2^2x_3^4x_4^2x_5^4 - 24x_2^2x_3^2x_4^2x_5^4\\
&+ 16x_1^4x_4^4x_5^4 - 12x_4^2x_5^4 - 24x_1^2x_3^2x_4^2x_5^4 + 6x_5^4 + 6x_2^4x_5^4 - 16x_1^2x_3^4x_5^4\\
&- 16x_1^2x_4^4x_5^4 + 6x_4^4x_5^4 + 64x_1^4x_2^2x_3^2x_4^2x_5^4 + 16x_1^4x_3^4x_5^4 + 16x_2^4x_3^4x_5^4\\
&- 16x_1^4x_2^2x_5^4 - 24x_1^2x_2^2x_4^2x_5^4 + 16x_1^4x_2^4x_5^4 + 16x_3^4x_4^4x_5^4 + 6x_3^4x_5^4 - 12x_2^2x_5^4\\
&- 144x_1^2x_2^2x_3^2x_4^2x_5^4 + 64x_1^2x_2^2x_3^2x_4^4x_5^4 + 28x_1^2x_3^2x_5^4 - 16x_2^4x_4^2x_5^4\\
&- 16x_2^4x_3^2x_5^4 - 16x_3^4x_4^2x_5^4 + 28x_3^2x_4^2x_5^4 + 28x_1^2x_2^2x_5^4 - 12x_1^2x_5^4 - 16x_1^4x_3^2x_5^4\\
&+ 6x_1^4x_5^4 - 16x_1^2x_2^4x_5^4 + 16x_2^4x_4^4x_5^4 + 112x_1^3x_2x_3^3x_4x_5^3 + 112x_1^3x_2x_3x_4^3x_5^3\\
&+ 40x_1x_2x_3^5x_4x_5^3 - 32x_1^5x_2x_3x_4^3x_5^3 - 32x_1x_2^5x_3x_4^3x_5^3 + 112x_1^3x_2^3x_3x_4x_5^3\\
&+ 40x_1x_2^5x_3x_4x_5^3 + 40x_1^5x_2x_3x_4x_5^3 + 40x_1x_2x_3x_4^5x_5^3 - 32x_1^3x_2x_3x_4^5x_5^3\\
&- 112x_1x_2^3x_3x_4x_5^3 - 32x_1x_2x_3^5x_4^3x_5^3 - 32x_1^5x_2x_3^3x_4x_5^3 - 112x_1^3x_2x_3x_4x_5^3\\
&+ 112x_1x_2^3x_3^3x_4x_5^3 - 32x_1x_2^3x_3^5x_4x_5^3 - 32x_1x_2^5x_3^3x_4x_5^3 - 32x_1x_2^3x_3x_4^5x_5^3\\
&- 32x_1^3x_2^5x_3x_4x_5^3 + 112x_1x_2x_3^3x_4^3x_5^3 + 112x_1x_2^3x_3x_4^3x_5^3 - 112x_1x_2x_3^3x_4x_5^3\\
&- 112x_1x_2x_3x_4^3x_5^3 + 72x_1x_2x_3x_4x_5^3 - 128x_1^3x_2^3x_3^3x_4^3x_5^3 - 32x_1^5x_2^3x_3x_4x_5^3\\
&- 32x_1^3x_2x_3^5x_4x_5^3 - 32x_1x_2x_3^3x_4^5x_5^3 + 16x_1^2x_2^6x_4^2x_5^2 + 28x_3^2x_4^4x_5^2 + 28x_1^4x_3^2x_5^2\\
&- 8x_2^2x_4^6x_5^2 - 12x_4^4x_5^2 - 16x_1^4x_4^4x_5^2 + 16x_1^2x_3^2x_4^6x_5^2 - 24x_1^2x_2^2x_3^4x_5^2 - 8x_1^6x_3^2x_5^2\\
&- 32x_2^2x_3^2x_5^2 - 24x_2^2x_3^2x_4^4x_5^2 - 16x_1^4x_2^4x_5^2 - 8x_1^6x_4^2x_5^2 + 40x_2^2x_3^2x_4^2x_5^2\\
&- 16x_1^4x_3^4x_5^2 - 24x_1^4x_2^2x_4^2x_5^2 + 28x_3^4x_4^2x_5^2 - 8x_1^2x_2^6x_5^2 + 64x_1^2x_2^4x_3^4x_4^2x_5^2\\
&- 32x_1^2x_4^2x_5^2 - 16x_3^4x_4^4x_5^2 - 12x_3^4x_5^2 + 28x_2^2x_3^4x_5^2 - 144x_1^2x_2^2x_3^4x_4^2x_5^2\\
&+ 16x_2^2x_3^6x_4^2x_5^2 + 28x_1^2x_4^4x_5^2 + 4x_4^6x_5^2 + 64x_1^2x_2^2x_3^4x_4^4x_5^2 - 24x_2^4x_3^2x_4^2x_5^2\\
&+ 28x_2^4x_3^2x_5^2 - 8x_1^6x_2^2x_5^2 + 16x_1^6x_2^2x_3^2x_5^2 + 16x_1^2x_2^2x_4^6x_5^2 + 16x_2^2x_3^2x_4^6x_5^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 12x_4^2x_5^2 - 24x_1^4x_2^2x_3^2x_5^2 + 16x_1^2x_3^6x_4^2x_5^2 - 24x_1^2x_2^4x_3^2x_5^2 - 8x_2^2x_3^6x_5^2 \\
&+ 64x_1^4x_2^2x_3^2x_4^4x_5^2 + 192x_1^2x_2^2x_3^2x_4^2x_5^2 - 12x_2^4x_5^2 - 24x_1^2x_2^4x_4^2x_5^2 + 12x_2^2x_5^2 \\
&- 8x_1^2x_3^6x_5^2 + 40x_1^2x_3^2x_4^2x_5^2 - 24x_1^2x_2^2x_4^4x_5^2 - 32x_1^2x_2^2x_5^2 + 64x_1^4x_2^2x_3^4x_4^2x_5^2 \\
&+ 28x_2^2x_4^4x_5^2 - 8x_1^2x_4^6x_5^2 - 4x_5^2 + 4x_1^6x_5^2 + 12x_1^2x_5^2 + 28x_1^4x_4^2x_5^2 - 16x_2^4x_3^4x_5^2 \\
&+ 16x_1^6x_3^2x_4^2x_5^2 - 8x_2^6x_4^2x_5^2 + 16x_1^6x_2^2x_4^2x_5^2 + 64x_1^4x_2^4x_3^2x_4^2x_5^2 \\
&- 24x_1^2x_3^2x_4^4x_5^2 + 12x_3^2x_5^2 + 16x_1^2x_2^2x_3^6x_5^2 + 16x_2^6x_3^2x_4^2x_5^2 + 16x_1^2x_2^6x_3^2x_5^2 \\
&+ 40x_1^2x_2^2x_4^2x_5^2 - 8x_3^2x_4^6x_5^2 - 24x_1^2x_3^4x_4^2x_5^2 - 16x_2^4x_4^4x_5^2 + 28x_1^2x_3^4x_5^2 \\
&- 144x_1^2x_2^2x_3^2x_4^4x_5^2 + 28x_2^4x_4^2x_5^2 - 8x_3^6x_4^2x_5^2 - 32x_2^2x_4^2x_5^2 + 64x_1^2x_2^4x_3^2x_4^4x_5^2 \\
&+ 40x_1^2x_2^2x_3^2x_5^2 - 24x_2^2x_3^4x_4^2x_5^2 - 144x_1^2x_2^4x_3^2x_4^2x_5^2 - 12x_1^4x_5^2 + 4x_3^6x_5^2 \\
&+ 28x_1^2x_2^4x_5^2 - 144x_1^4x_2^2x_3^2x_4^2x_5^2 - 8x_2^6x_3^2x_5^2 + 28x_1^4x_2^2x_5^2 - 32x_3^2x_4^2x_5^2 \\
&- 32x_1^2x_3^2x_5^2 + 4x_2^6x_5^2 - 24x_1^4x_3^2x_4^2x_5^2 - 24x_1x_2^5x_3x_4x_5 - 112x_1^3x_2x_3x_4^3x_5 \\
&- 112x_1^3x_2^3x_3x_4x_5 + 40x_1^3x_2x_3^5x_4x_5 - 112x_1x_2x_3^3x_4^3x_5 - 32x_1x_2^3x_3^5x_4^3x_5 \\
&- 32x_1^3x_2x_3^5x_4^3x_5 - 8x_1x_2x_3^7x_4x_5 + 40x_1^3x_2x_3x_4^5x_5 - 24x_1x_2x_3x_4^5x_5 \\
&- 112x_1x_2^3x_3x_4^3x_5 + 40x_1x_2^5x_3^3x_4x_5 + 40x_1x_2^3x_3x_4^5x_5 - 32x_1^5x_2x_3^3x_4^3x_5 \\
&- 32x_1^3x_2^3x_3^5x_4x_5 + 40x_1^3x_2^5x_3x_4x_5 - 8x_1x_2^7x_3x_4x_5 - 8x_1x_2x_3x_4^7x_5 + 112x_1x_2^3x_3^3x_4^3x_5 \\
&+ 40x_1x_2^5x_3x_4^3x_5 - 112x_1^3x_2x_3^3x_4x_5 + 40x_1^5x_2x_3^3x_4x_5 + 72x_1x_2x_3^3x_4x_5 \\
&- 32x_1x_2^3x_3^3x_4^5x_5 + 72x_1x_2^3x_3x_4x_5 + 40x_1^5x_2^3x_3x_4x_5 + 40x_1^5x_2x_3x_4^3x_5 \\
&- 32x_1^3x_2^3x_3x_4^5x_5 - 24x_1x_2x_3^5x_4x_5 - 32x_1^5x_2^3x_3x_4^3x_5 - 32x_1^3x_2^5x_3x_4^3x_5 \\
&- 32x_1x_2^5x_3^3x_4^3x_5 + 112x_1^3x_2^3x_3x_4^3x_5 + 40x_1x_2x_3^5x_4^3x_5 - 40x_1x_2x_3x_4x_5 \\
&+ 40x_1x_2x_3^3x_4^5x_5 - 32x_1^5x_2^3x_3^3x_4x_5 - 112x_1x_2^3x_3^3x_4x_5 - 24x_1^5x_2x_3x_4x_5 \\
&- 32x_1^3x_2^5x_3^3x_4x_5 + 72x_1x_2x_3x_4^3x_5 - 8x_1^7x_2x_3x_4x_5 + 112x_1^3x_2^3x_3^3x_4x_5 \\
&+ 112x_1^3x_2x_3^3x_4^3x_5 - 32x_1^3x_2x_3^3x_4^5x_5 + 40x_1x_2^3x_3^5x_4x_5 + 72x_1^3x_2x_3x_4x_5 \\
&+ 28x_1^4x_3^2x_4^2 + 16x_1^2x_2^6x_3^2x_4^2 - 24x_1^2x_2^2x_3^2x_4^4 + 28x_1^2x_2^2x_4^4 - 8x_1^2x_3^2x_4^6 \\
&- 16x_1^2x_2^4x_4^4 + 28x_2^2x_3^4x_4^2 + 28x_1^4x_2^2x_4^2 - 32x_1^2x_2^2x_4^2 - 8x_1^2x_2^2x_4^6 + 4x_3^2x_4^6 \\
&- 24x_1^2x_2^2x_3^4x_4^2 - 12x_3^2x_4^4 + 16x_2^4x_3^4x_4^4 + 16x_1^2x_2^2x_3^6x_4^2 + 4x_2^2x_4^6 - 32x_1^2x_3^2x_4^2 \\
&+ 4x_2^2x_3^6 - 24x_1^2x_2^4x_3^2x_4^2 - 12x_2^4x_4^2 + 28x_1^2x_3^4x_4^2 + 6x_1^4x_4^4 - 16x_1^4x_3^2x_4^4 + 28x_1^2x_2^4x_4^2 \\
&+ 12x_1^2x_3^2 + 6x_3^4x_4^4 + 16x_1^4x_2^4x_4^4 - 8x_2^2x_3^2x_4^6 - 12x_2^2x_4^4 - 32x_2^2x_3^2x_4^2 - 8x_2^2x_3^6x_4^2 \\
&+ 12x_2^2x_4^2 + 4x_2^6x_3^2 - 16x_1^4x_2^2x_3^4 - 16x_2^4x_3^4x_4^2 + 40x_1^2x_2^2x_3^2x_4^2 + 6x_3^4 - 8x_2^6x_3^2x_4^2 \\
&- 4x_2^2 - 16x_1^2x_3^4x_4^4 + x_3^8 - 12x_1^2x_4^4 - 16x_2^4x_3^2x_4^4 + 6x_1^4x_3^4 - 16x_1^4x_2^4x_4^2 - 12x_1^4x_2^2
\end{aligned}
$$

$$+ 16x_1^4x_3^4x_4^4 - 4x_4^2 - 8x_1^2x_3^6x_4^2 + x_4^8 - 8x_1^2x_2^6x_3^2 - 16x_1^4x_3^4x_4^2 + 28x_1^2x_2^2x_3^4$$
$$+ 4x_3^6x_4^2 + 16x_1^4x_2^4x_3^4 + 16x_1^6x_2^2x_3^2x_4^2 + 12x_2^2x_3^2 + 4x_1^2x_3^6 + 4x_1^2x_4^6 + 4x_2^6x_4^2$$
$$- 8x_1^2x_2^2x_3^6 + 16x_1^2x_2^2x_3^2x_4^6 - 4x_4^6 - 8x_1^6x_2^2x_3^2 - 12x_1^2x_2^4 - 16x_1^4x_2^4x_3^2 - 12x_1^2x_3^4$$
$$- 12x_2^4x_3^2 - 16x_2^2x_3^4x_4^4 + 12x_1^2x_4^2 + x_1^8 + 4x_1^6x_4^2 - 24x_1^4x_2^2x_3^2x_4^2 - 8x_1^2x_2^6x_4^2$$
$$+ 6x_4^4 + 12x_3^2x_4^2 + 28x_2^2x_3^2x_4^4 + 6x_1^4x_2^4 + 6x_1^4 + 28x_1^4x_2^2x_3^2 + 28x_2^4x_3^2x_4^2 + 6x_2^4x_3^4$$
$$- 32x_1^2x_2^2x_3^2 + 4x_1^2x_2^6 - 4x_3^2 - 4x_1^6 - 4x_1^2 - 8x_1^6x_3^2x_4^2 + x_2^8 - 16x_1^4x_2^2x_4^4 - 16x_1^2x_2^4x_3^4$$
$$+ 4x_1^6x_2^2 + 6x_2^4x_4^4 - 4x_3^6 - 8x_1^6x_2^2x_4^2 - 12x_3^4x_4^2 + 12x_1^2x_2^2 - 12x_2^2x_3^4 + 28x_1^2x_3^2x_4^4$$
$$- 12x_1^4x_4^2 + 28x_1^2x_2^4x_3^2 + 1 - 4x_2^6 - 12x_1^4x_3^2 + 6x_2^4 + 4x_1^6x_3^2.$$


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## References


[1] E.M. Andreev, *Convex polyhedra in Lobačevskiĭ spaces*, Matematicheskiĭ Sbornik. Novaya Seriya **81** (123) (1970) 445–478.

[2] D. Austin, *When Kissing Involves Trigonometry*, AMS Features Column **9** (1999).

[3] P. Brass, W. Moser and J. Pach, Research Problems in Discrete Geometry, Springer-Verlag, New York, 2005.

[4] G.R. Brightwell and E.R. Scheinerman, *Representations of planar graphs*, SIAM J. Discrete Math. **6** (1993) 214–229. doi:10.1137/0406017

[5] R.L. Graham, J.C. Lagarias, C.L. Mallows, A.R. Wilks and C.H. Yan, *Apollonian circle packings: number theory*, J. Number Theory **100** (2003) 1–45. doi:10.1016/S0022-314X(03)00015-5

[6] R.L. Graham, J.C. Lagarias, C.L. Mallows, A.R. Wilks and C.H. Yan, *Apollonian circle packings: geometry and group theory. I. The Apollonian group*, Discrete Comput. Geom. **34** (4) (2005) 547–585. doi:10.1007/s00454-005-1196-9

[7] R.L. Graham, J.C. Lagarias, C.L. Mallows, A.R. Wilks and C.H. Yan, *Apollonian circle packings: geometry and group theory. II. Super-Apollonian group and integral packings*, Discrete Comput. Geom. **35** (1) (2006) 1–36. doi:10.1007/s00454-005-1195-x

[8] R.L. Graham, J.C. Lagarias, C.L. Mallows, A.R. Wilks and C.H. Yan, *Apollonian circle packings: geometry and group theory. III. Higher dimensions*, Discrete Comput. Geom. **35** (1) (2006) 37–72. doi:10.1007/s00454-005-1197-8

[9] E. Fuchs and K. Sanden, *Some experiments with integral Apollonian circle packings*, Exp. Math. **20** (4) (2011) 380–399. doi:10.1080/10586458.2011.565255

[10] T. Hungerford, Algebra, Graduate Texts in Mathematics, GTM-73 Springer-Verlag, 1974.

[11] P. Koebe, *Kontaktprobleme der konformen Abbildung*, Ber. Verh. Sächs, Akademie der Wissenshaften Leipzig, Math.-Phys. Klasse **88** (1936) 141–164.

[12] *MAPLE, mathematics software tool for symbolic computation*, http://www.maplesoft.com/products/Maple/academic/index.aspx

[13] K.H. Rosen, Elementary Number Theory and Its Applications, Pearson Addison Wesley, 2005.

[14] K. Stephenson, Introduction to Circle Packing: The Theory of Discrete Analytic Functions, Cambridge University Press, 2005.

[15] W. Thurston, Three-Dimensional Geometry and Topology, Princeton University Press, 1997.

[16] G.M. Ziegler, Lectures on Polytopes, Graduate Texts in Mathematics, GMT-152 Springer Verlag, 1995. doi:10.1007/978-1-4613-8431-1

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