Discussiones Mathematicae General Algebra and Applications 33 (2013) 49–56 doi:10.7151/dmgaa.1199

LIE IDEALS IN PRIME Γ-RINGS WITH DERIVATIONS

NISHTEMAN N. SULIMAN

Salahaddin University, Department of Mathematics, Erbil, Iraq

e-mail: vananesh@gmail.com

AND

Abdul-rahman H. Majeed

Baghdad University, Department of Mathematics, Baghdad, Iraq

e-mail: Ahmajeed@yahoo.com

Abstract

Let M be a 2 and 3-torsion free prime Γ -ring, d a nonzero derivation on M and U a nonzero Lie ideal of M. In this paper it is proved that U is a central Lie ideal of M if d satisfies one of the following

- (i) $d(U) \subset Z$,
- (ii) $d(U) \subset U$ and $d^2(U) = 0$,
- (iii) $d(U) \subset U, d^2(U) \subset Z.$

Keywords: prime Γ -rings, Lie ideals, derivations.

2010 Mathematics Subject Classification: Primary: 16Y99; Secondary: 16W25, 16N60.

1. INTRODUCTION

The concept of a Γ -ring was first introduced by Nobusawa [5], and generalized by Barnes [1] as follows: A Γ -ring is a pair (M, Γ) where M and Γ are additive abelian groups for which there exists a map from $M \times \Gamma \times M$ to M (the image of (x, α, y) was denoted by $x\alpha y$) such that

- (i) $(x+y)\alpha z = x\alpha z + y\alpha z,$ $x(\alpha + \beta)y = x\alpha y + x\beta y,$ $x\alpha(y+z) = x\alpha y + x\alpha z,$
- (ii) $(x\alpha y)\beta z = x\alpha(y\beta z),$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, then M is called a Γ -ring.

Recall that a Γ -ring M is called prime if for any two elements $x, y \in M$, $x\Gamma M\Gamma y = 0$ implies either x = 0 or y = 0, and M is called semiprime if $x\Gamma M\Gamma x =$ 0 with $x \in M$ implies x = 0. Note that every prime Γ -ring is obviously semiprime. An additive mapping $d: M \to M$ is called a derivation if $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$. An additive subgroup I of M is called a left (right) ideal of M if $M\Gamma I \subset I$ $(I\Gamma M \subset I)$. If I is both left and right ideal of M, then we say I is an ideal of M. The set $Z = \{x \in M; x\alpha y = y\alpha x \text{ for all } x, y \in M$ and $\alpha \in \Gamma\}$ is called the center of M. An additive subgroup U of M is said to be a Lie ideal of M if $[u, x]_{\alpha} \in U$, for all $u \in U, x \in M$ and $\alpha \in \Gamma$. M is n-torsion free if nx = 0, for $x \in M$ implies x = 0, where n is an integer. The commutator $x\alpha y - y\alpha x$ will be denoted by $[x, y]_{\alpha}$. We will use for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, the basic commutator identities:

$$[x\alpha y, z]_{\beta} = x\alpha[y, z]_{\beta} + [x, z]_{\beta}\alpha y + x[\alpha, \beta]_{z}y$$
, and
 $[x, y\alpha z]_{\beta} = y\alpha[x, z]_{\beta} + [x, y]_{\beta}\alpha z + y[\beta, \alpha]_{x}z.$

Throughout this paper, We consider the following assumption $x\alpha y\beta z = x\beta y\alpha z$, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ and it will be represented by property (*) is a central.

According to the assumption property (*), the above two identities reduced to

$$[x\alpha y, z]_{\beta} = x\alpha[y, z]_{\beta} + [x, z]_{\beta}\alpha y$$
, and
 $[x, y\alpha z]_{\beta} = y\alpha[x, z]_{\beta} + [x, y]_{\beta}\alpha z$.

The relationship between the derivations and Lie ideals of a prime ring has been investigated by a number of authors (see [2, 3] and [4]). In [2], Bergen, Herstien and Kerr showed that if U is a nonzero Lie ideal of a 2-torsion free prime ring R and d a nonzero derivation of R such that $d^2(U) = 0$ or $d^2(U) \subset Z$ then U is central. Our aim in this paper is generalized the above results in prime Γ -rings with Lie ideals.

2. The results

For proving the main results, we have needed some important lemmas. So we start as follows:

Remark 1. Let M be 2-torsion free prime Γ -ring and d a derivation of M. Then for all $x, y \in M$ and $\alpha \in \Gamma$, we have the followings:

- (i) If $d^2 = 0$ on M, then d = 0,
- (ii) $d([x,y]_{\alpha}) = [d(x),y]_{\alpha} + [x,d(y)]_{\alpha}$,
- (iii) $d^2(x\alpha y) = d^2(x)\alpha y + 2d(x)\alpha d(y) + x\alpha d^2(y),$
- (iv) $d^{3}(x\alpha y) = d^{3}(x)\alpha y + 3d^{2}(x)\alpha d(y) + 3d(x)\alpha d^{2}(y) + x\alpha d^{3}(y).$

Lemma 2 ([6], Lemma 1). Let M be 2-torsion free prime Γ -ring and Z the center of M. Then the following are satisfied:

- (i) If $x \in Z$, and $x\Gamma y = 0$, then either x = 0 or y = 0.
- (ii) If $x \in Z$, and $x \Gamma y \subset Z$, then either x = 0 or $y \in Z$.

Lemma 3 ([3], Lemma 2). Let $0 \neq U$ be a Lie ideal of a 2-torsion free prime Γ -ring M and $U \not\subseteq Z$. If for $a, b \in M$ such that $a\Gamma U\Gamma b = 0$, then a = 0 or b = 0.

Lemma 4. Let U be a nonzero Lie ideal of prime Γ -ring M. If $[M, U]_{\alpha} \subset Z$, then $U \subset Z$.

Proof. For all $x \in M, u \in U$ and $\alpha \in \Gamma$, we have $[x, u]_{\alpha} \in [M, U]_{\alpha}$. Replacing x by $x\beta u$, we get

 $[x\beta u, u]_{\alpha} = [x, u]_{\alpha}\beta u \in \mathbb{Z}$, for all $x \in M$, $u \in U$ and $\alpha, \beta \in \Gamma$.

Since $[x, u]_{\alpha} \in \mathbb{Z}$, then by Lemma 2(ii) we obtain $[x, u]_{\alpha} = 0$ or $u \in \mathbb{Z}$, then the result required.

Lemma 5. Let $0 \neq U$ be a Lie ideal of 2-torsion free prime Γ -ring M satisfying property (*). If $[U, U]_{\Gamma} = 0$, then $U \subset Z$ (If U is a commutative Lie ideal, then U is central).

Proof. For all $x \in M, u \in U$ and $\alpha \in \Gamma$, we have $[u, x]_{\alpha} \in U$. Hence by hypothesis we have

 $[u, [u, x]_{\alpha}]_{\beta} = 0$, for all $x \in M$, $u \in U$ and $\alpha, \beta \in \Gamma$.

Equivalently

(1) $u\beta[u,x]_{\alpha} = [u,x]_{\alpha}\beta u$, for all $x \in M, u \in U$ and $\alpha, \beta \in \Gamma$.

Replacing x by $x \alpha y$, for $y \in M$ and $\alpha \in \Gamma$, we get

(2) $u\beta x\alpha[u,y]_{\alpha} + u\beta[u,x]_{\alpha}\alpha y = x\alpha[u,y]_{\alpha}\beta u + [u,x]_{\alpha}\alpha y\beta u.$

Using (1) for $u\beta[u, x]_{\alpha} = [u, x]_{\alpha}\beta u$ and $[u, y]_{\alpha}\beta u = u\beta[u, y]_{\alpha}$ in (2) we obtain

 $u\beta x\alpha[u,y]_{\alpha} + [u,x]_{\alpha}\beta u\alpha y = x\alpha u\beta[u,y]_{\alpha} + [u,x]_{\alpha}\alpha y\beta u.$

Using property (*) we get $2[u, x]_{\alpha}\beta[u, y]_{\alpha} = 0$. Since M is 2-torsion free, this leads to

$$[u, x]_{\alpha}\beta[u, y]_{\alpha} = 0$$
, for all $x, y \in M, u \in U$ and $\alpha, \beta \in \Gamma$.

Replacing y by $y\gamma x$, we find that

$$[u, x]_{\alpha}\beta y\gamma[u, x]_{\alpha} = 0, \text{ for all } x, y \in M, u \in U \text{ and } \alpha, \beta, \gamma \in \Gamma.$$

Thus $[u, x]_{\Gamma} \Gamma M \Gamma[u, x]_{\Gamma} = 0$, for all $x \in M, u \in U$. By primeness of M, we conclude $[u, x]_{\Gamma} = 0$, yields $U \subset Z$.

Lemma 6. Let U be a nonzero Lie ideal of 2-torsion free prime Γ -ring M and d a nonzero derivation of M. If $a \in U$ such that $[a, d(x)]_{\alpha} = 0$, for all $x \in M$ and $\alpha \in \Gamma$, then $a \in Z$.

Proof. By hypothesis we have $[a, d(x)]_{\alpha} = 0$, for all $x \in M$ and $\alpha \in \Gamma$. Replacing x by $x\beta y$, we get

$$0 = [a, d(x\beta y)]_{\alpha}$$

= $[a, d(x)]_{\alpha}\beta y + d(x)\beta[a, y]_{\alpha} + x\beta[a, d(y)]_{\alpha} + [a, x]_{\alpha}\beta d(y)$
= $d(x)\beta[a, y]_{\alpha}a + [a, x]_{\alpha}\beta d(y).$

Replacing x by d(x), we obtain

$$d^2(x)\beta[a,y]_{\alpha} = 0$$
, for all $x, y \in M$ and $\alpha, \beta \in \Gamma$.

Replacing y by $z\gamma y$, we get

$$d^2(x)\beta z\gamma[a,y]_\alpha=0,\,\text{for all }x,y,z\in M\,\,\text{and }\alpha,\beta,\gamma\in\Gamma.$$

By primeness we get $d^2(x) = 0$ or $[a, y]_{\alpha} = 0$, since $d \neq 0$, therefore $a \in \mathbb{Z}$.

Theorem 7. Let U be a nonzero Lie ideals of a 2-torsion free prime Γ -ring M and d a nonzero derivation of M. If $d(U) \subset Z$, then $U \subset Z$.

Proof. suppose that $U \nsubseteq Z$, then by Lemma 5 we have $V = [U,U] \nsubseteq Z$. Let $u, w \in U$, hence from

$$d([u, w]_{\alpha}) = [d(u), w]_{\alpha} + [u, d(w)]_{\alpha} = 0.$$

Since $d(u), d(w) \in Z$. It follows that d(V) = 0.

Let $v \in V, m \in M$ and $\alpha \in \Gamma$, since d(v) = 0 and $d([v, m]_{\alpha}) = 0$, we get

 $[v, d(m)]_{\alpha} = 0$, for all $v \in V, m \in M$ and $\alpha \in \Gamma$.

Therefore by Lemma 6 we get $v \in Z$, contradiction. Accordingly, $U \subset Z$.

Lemma 8. Let $U \nsubseteq Z$ be a Lie ideal of 2-torsion free prime Γ -ring M and d a nonzero derivation of M. If $a \in M$ and $a\Gamma d(U) = 0$ $(d(U)\Gamma a = 0)$, then a = 0.

Proof. For all $u \in U$, $x \in M$ and $\alpha \in \Gamma$ we have $[u, x]_{\beta} \gamma u \in U$. By hypothesis we have

$$0 = a\alpha d([u, x]_{\beta}\gamma u)$$

= $a\alpha[u, x]_{\beta}\gamma d(u), forall x \in M, u \in Uand\alpha, \beta, \gamma \in \Gamma.$

Replacing x by $d(v)\lambda x$, we get

$$a\alpha u\beta d(v)\lambda x\gamma d(u) = 0$$
, for all $u, v \in U, x \in M$ and $\alpha, \beta, \gamma, \lambda \in \Gamma$.

By primeness we obtain $a\alpha u\beta d(v) = 0$ or d(u) = 0.

Now let $K = \{u \in Ua\alpha u\beta d(v) = 0\}$ and $L = \{u \in U | d(u) = 0\}$. Since K and L are additive subgroups of U and $U = K \cup L$, but a group can't be union of its two proper subgroups and hence U = K or U = L.

According to Theorem 7, $d(U) \neq 0$, which proves that U = K. Hence we get $a\Gamma U\Gamma d(v) = 0$, for all $v \in U$. By Lemma 3 we get a = 0 or d(v) = 0, again by Theorem 7 $d(U) \neq 0$, therefore a = 0.

Theorem 9. Let M be a 2-torsion free prime Γ -ring, U be a nonzero Lie ideal of M and d be a nonzero derivation of M. If $d^2(U) = 0$ and $d(U) \subset U$, then $U \subset Z$.

Proof. Suppose that $U \nsubseteq Z$, for all $x \in M$, $u \in U$ and $\alpha \in \Gamma$ we have $[x, u]_{\alpha} \in U$. Since $d^2(U) = 0$, then by using Remark 1 we get

$$0 = d^{2}([x\beta u, u]_{\alpha})$$

= $d^{2}([x, u]_{\alpha})\beta u + 2d([x, u]_{\alpha})\beta d(u) + [x, u]_{\alpha}\beta d^{2}(u).$

Since M is 2-torsion free and $d^2(U) = 0$, then we get

$$d([x, u]_{\alpha})\beta d(u) = 0$$
, for all $x \in M$, $u \in U$ and $\alpha, \beta \in \Gamma$.

Replacing u by u + d(u), we get $d([x, d(u)]_{\alpha}\beta d(u) = 0$, so that

$$[d(x), d(u)]_{\alpha}\beta d(u) = 0$$
, for all $x \in M$, $u \in U$ and $\alpha, \beta \in \Gamma$.

According to Lemma 8 we get $[d(x), d(u)]_{\alpha} = 0$ for all $x \in M, w \in U$ and $\alpha \in \Gamma$, therefor by Lemma 6 we conclude that $d(U) \subset Z$, which is contradicts Theorem 7, this prove the theorem.

Lemma 10. Let M be a 2 and 3-torsion free prime Γ -ring, U be a nonzero Lie ideal of M and d be a nonzero derivation of M. If $d(U) \subset U$, $d^2(U) \subset Z$ and $d^3(U) = 0$ then $U \subset Z$.

Proof. For all $x \in M$, $u \in U$ and $\alpha \in \Gamma$ we have $[x, u]_{\alpha} \in U$. Since $d^{3}(U) = 0$, then we obtain $d^{3}([x, u]_{\alpha}) = 0$. Replacing x by $x\beta u$ and using Remark 1(iv) we get

$$0 = d^{3}([x\beta u, u]_{\alpha})$$

= $3d^{2}([x, u]_{\alpha})\beta d(u) + 3d([x, u]_{\alpha})\beta d^{2}(u).$

Since M is 3-torsion free, then we get

$$d^2([x,u]_{\alpha})\beta d(u) + d([x,u]_{\alpha})\beta d^3(u) = 0$$
, for all $x \in M, u \in U$ and $\alpha, \beta \in \Gamma$.

Replacing u by d(u) and using $d^2(U) = 0$ we obtain

$$d^2([x, d(u)]_\alpha)\beta d^2(u) = 0.$$

Since $d^2(U) \subset Z$, then by Lemma 2(i) we get

(3)
$$d^2([x, d(u)]_{\alpha}) = 0 \text{ or } d^2(u) = 0.$$

If $d^2([x, d(u)]_{\alpha}) = 0$, then replacing x by $x\beta d(u)$ we obtain

$$0 = d^{2}([x\beta d(u), d(u)]_{\alpha})$$

= $d^{2}([x, d(u)]_{\alpha}\beta d(u))$
= $d^{2}([x, d(u)]_{\alpha}\beta d(u) + 2d([x, d(u)]_{\alpha}\beta d^{2}(u) + [x, d(u)]_{\alpha}\beta d^{3}(u)).$

Since $d^{3}(U) = 0$, M is a 2-torsion free and by relation (3), then the last equation reduced to

$$d([x, d(u)]_{\alpha}\beta d^2(u) = 0$$
, for all $x \in M$, $u \in U$ and $\alpha, \beta \in \Gamma$.

Since $d^2(U) \subset Z$, then by Lemma 2(i) we get $d([x, d(u)]_{\alpha} = 0 \text{ or } d^2(u) = 0$. If $d([x, d(u)]_{\alpha} = 0$, then replacing x by $x\gamma d(u)$, we obtain

$$0 = d([x\gamma d(u), d(u)]_{\alpha})$$

= $d([x, d(u)]_{\alpha}\gamma d(u))$
= $d([x, d(u)]_{\alpha})\gamma d(u) + [x, d(u)]_{\alpha}\gamma d^{2}(u)$
= $[x, d(u)]_{\alpha}\beta d^{2}(u).$

Since $d^2(U) \subset Z$ and $d(U) \subset U$, then by Lemma 2(i) we get $[x, d(u)]_{\alpha} = 0$ or $d^2(u) = 0$. If $[x, d(u)]_{\alpha} = 0$, then we have $d(u) \subset Z$. Hence from relation (3) we have either $d(u) \subset Z$ or $d^2(u) = 0$.

Now let $K = \{u \in U | d(u) \subset Z\}$ and $L = \{u \in U | d^2(u) = 0\}$. Since K and L are additive subgroups of U and $U = K \cup L$, but a group can't be union of its two proper subgroups and hence U = K or U = L. If U = K, that is $d(u) \subset Z$, then by Theorem 7 we get $U \subset Z$, or U = L, that is $d^2(u) = 0$, hence by Theorem 9 we get $U \subset Z$.

Theorem 11. Let M be a 2 and 3-torsion free prime Γ -ring, U be a nonzero Lie ideal of M and d be a nonzero derivation of M. If $d(U) \subset U$ and $d^2(U) \subset Z$, then $U \subset Z$.

Proof. For all $x \in M$, $u \in U$ and $\alpha \in \Gamma$ we have

(4)
$$d^2([x,u]_\alpha) \in Z.$$

Replacing x by $x\beta d^2(v)$, where $v \in U$ and $\beta \in \Gamma$, and using $d^2(U) \subset Z$, we get

(5)
$$2d([x,u]_{\alpha})\beta d^{3}(v) + [x,u]_{\alpha}\beta d^{4}(v) \subset Z, for all u, v \in U, x \in Mand\alpha, \beta \in \Gamma.$$

Replacing x by $x\gamma d^2(w)$ in relation (5), where $w \in U$ and $\gamma \in \Gamma$, and using $d^2(U) \subset Z$ and M is 2-torsion free, then the relation (5) reduced to

$$[x, u]_{\alpha} \gamma d^3(w) \beta d^3(v) \in \mathbb{Z}$$
, for all $v, u, w \in U, x \in M$ and $\alpha, \beta, \gamma \in \Gamma$.

Since $d^2(U) \subset Z$ and $d(U) \subset U$, then $d^3(U) \subset Z$ and thus by Lemma 2(ii) we have $d^3(U) = 0$ or $[x, u]_{\alpha} \subset Z$. Therefore if $d^3(U) = 0$, hence by Lemma 10 yields $U \subset Z$. If $[M, U]_{\alpha} \subset Z$, then by Lemma 4 we get $U \subset Z$.

References

- W. E. Barness, On the Γ-Rings of Nobusawa, Pacific J. Math. 18 (1966) 411–422. doi:10.2140/pjm.1966.18.411
- [2] J. Bergan, I. N. Herstein, and W. Kerr, *Lie Ideals and Derivations of Prime Rings*, J. Algebra **71** (1981) 259–267. doi:10.1016/0021-8693(81)90120-4
- [3] A. K. Halder and A. C. Paul, Jordan Left Derivations on Lie Ideals of Prime Γ-Rings, Punjab Univ. J. of Math. (2011) 1–7.
- [4] P.H.Lee and T.K.Lee, *Lie Ideals of Prime Rings with Derivations*, Bull. Inst. Math. Acad. Scin. **11** (1983) 75–80.
- [5] N. Nobusawa, On a Generalzetion of the Ring Theory, Osaka J. Math. 1 (1964) 81–89.
- [6] M. Soyturk, The Commutativity in Prime Gamma Rings with Derivation, Tr. J. Math. 18 (1994) 149–155.

Received 24 October 2012 Revised 3 January 2013