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ON k-RADICALS OF GREEN'S RELATIONS IN SEMIRINGS WITH A SEMILATTICE ADDITIVE REDUCT

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Abstract

We introduce the k-radicals of Green's relations in semirings with a semilattice additive reduct, introduce the notion of left k-regular (right k-regular) semirings and characterize these semirings by k-radicals of Green's relations. We also characterize the semirings which are distributive lattices of left k-simple subsemirings by k-radicals of Green's relations.

Keywords: k-radicals of Green's relations, left k-regular semiring, left k-simple semiring, distributive lattices of left k-simple semirings.

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1. Introduction

The notion of semirings was introduced by Vandiver [18] in 1934. Though they appeared in mathematics long before, for example, the semiring of all ideals of a

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ring, the semiring of all endomorphisms on a commutative semigroup etc., they found their full place in mathematics recently, for example, in idempotent analysis [13] which is applied in theoretical physics, optimization etc., various applications in theoretical computer science and algorithm theory [10, 12]. The underlying semirings applicable both in idempotent analysis and theoretical computer science is one in which the additive reduct is a semilattice, i.e., both idempotent and commutative.

While studying the structure of semigroups, semilattice decomposition of semigroups, an elegant technique, was first defined and studied by A.H. Clifford [9]. This motivates us to study on the structure of semirings whose additive reduct is a semilattice [3, 4, 5, 14]. M.K. Sen and A.K. Bhuniya [16] also investigated these semirings. Interestingly, the semiring of all finite subsets of a semigroup is a free model in the variety of such semirings and there are interesting connections between the subvarieties of semigroups and different subclasses of these semirings.

The preliminaries and prerequisites for this article are discussed in Section 2, where we also provide some important and useful examples of semirings with a semilattice additive reduct. In Section 3, we introduce the k-radicals of Green's relations, give the notion of left k-regular(right k-regular) semirings and present a necessary and sufficent condition for a semiring to be a left k-regular(right k-regular) semiring in terms of k-radicals of Green's relations. The semirings which are distributive lattices of left k-simple semirings via k-radicals of Green's relations have also been given in this section.

2. Preliminaries

A *semiring* $(S, +, \cdot)$ is an algebra with two binary operations $+$ and $\cdot$ such that both the *additive reduct* $(S, +)$ and the *multiplicative reduct* $(S, \cdot)$ are semigroups and such that the following distributive laws hold:

$$x(y+z) = xy + xz \quad \text{and} \quad (x+y)z = xz + yz, \quad \text{for all} \quad x, y, z \in S.$$

Thus the semirings can be viewed as a common generalization of both rings and distributive lattices. By $\mathbb{SL}^+$ we denote the variety of all semirings $(S, +, \cdot)$ such that $(S, +)$ is a semilattice, i.e. a commutative and idempotent semigroup. Throughout this paper, unless otherwise stated, S is always a semiring in $\mathbb{SL}^+$. The natural examples are: the semiring $(\mathbb{N}, +, \cdot)$, where $+$ and $\cdot$ are defined by: $a + b =$max$\{a, b\}$, $a \cdot b$ the usual multiplication of a and b, the Boolean semiring $\mathbb{B} = \{0,\ 1\}$, with $0, 1 \in \mathbb{Z}$ and $+$ and $\cdot$ defined as maximum and minimum respectively, the tropical semiring $\mathbb{N}_{min} = \mathbb{N} \cup \{\infty\}$, with $a + b = min\{a, b\}$ and $a.b = a + b$. Let τ be a topology on a set X. Then τ is a semiring, with $+$ and $\cdot$ defined as union and intersection respectively. Here we observe that the underlying semirings are at least as common as topological spaces. Let R be a commutative ring and let $Spec(R)$ denotes the set of all ideals of R. Then $Spec(R)$ becomes a semiring where $+$ and $\cdot$ are defined as $I+J = \{a+b \mid a \in I, b \in J\}$ and $I.J = \{\sum_{i=1}^{n} a_i b_i \mid a_i \in I, b_j \in J\}$. Maslov's dequantization semiring, applicable in physics, $\mathbb{R}_{max} = \mathbb{R} \cup \{-\infty\}$ is equipped with operations $\oplus = \max$ and $\odot = +$. Let F be a semigroup and $P_f(F)$ be the set of all finite subsets of F. Define addition and multiplication on $P_f(F)$ by:

$$U + V = U \cup V,\ U \cdot V = \{ab \mid a \in U, b \in V\},\ \text{for all } U, V \in P_f(F).$$

Then $(P_f(F), +, \cdot)$ is a semiring in $\mathbb{SL}^+$.

Let S be a semiring and $\phi \neq A \subseteq S$. Then the k-closure of A is defined by $\overline{A} = \{x \in S \mid x + a_1 = a_2 \text{ for some } a_1, a_2 \in A\}$, and the k-radical of A by $\sqrt{A} = \{x \in S \mid (\exists\ n \in \mathbb{N})\ x^n \in \overline{A}\}$. Then $\overline{A} \subseteq \sqrt{A}$ by definition, and $A \subseteq \overline{A}$ since $(S, +)$ is a semilattice. Moreover, if $(A, +)$ is a subsemigroup of $(S, +)$ then $\overline{A} = \{x \in S \mid x + a = a \text{ for some } a \in A\}$. An ideal A of S is called a k-ideal of S if and only if $\overline{A} = A$. The principal k-ideal (resp. left k-ideal, right k-ideal) [3, 4, 15] generated by a are given by:

$$I_k(a) = \{x \in S \mid (\exists s \in S) x + a + sa + as + sas = a + sa + as + sas\},$$

$$L_k(a) = \{x \in S \mid (\exists s \in S) x + a + sa = a + sa\}$$

and

$$R_k(a) = \{x \in S \mid (\exists s \in S) x + a + as = a + as\}.$$

A semiring S is said to be left k-simple (k-simple) [5] if it has no non trivial proper left k-ideals (k-ideals).

The authors [5, 15] introduced the Green's relations $\overline{\mathcal{L}}$, $\overline{\mathcal{R}}$, $\overline{\mathcal{J}}$ and $\overline{\mathcal{H}}$ on a semiring S in $\mathbb{SL}^+$ and they are

$$\overline{\mathcal{L}} = \{(x, y) \in S \times S \mid L_k(x) = L_k(y)\},$$

$$\overline{\mathcal{J}} = \{(x, y) \in S \times S \mid I_k(x) = I_k(y)\},$$

$$\overline{\mathcal{R}} = \{(x, y) \in S \times S \mid R_k(x) = R_k(y)\}$$

and

$$\overline{\mathcal{H}} = \overline{\mathcal{L}} \cap \overline{\mathcal{R}}.$$

Since $S \in \mathbb{SL}^+$, both $\overline{\mathcal{L}}$ and $\overline{\mathcal{R}}$ are additive congruences on S and $\overline{\mathcal{L}}$ is a right congruence and $\overline{\mathcal{R}}$ is a left congruence on S.

We also recall that S is left k-simple (k-simple) if and only if $\overline{\mathcal{L}} = S \times S (\overline{\mathcal{J}} = S \times S)$.

A semiring S is k-regular if and only if for all $a \in S$ there exists $x \in S$ such that $a + axa = axa$. Let D be a distributive lattice. Then the semiring $S = M_2(D)$ of all 2×2 matrices oover D is a k-regular semiring. Also, if F be a semigroup then the semiring $P_f(F)$ is k-regular if and only if F is a regular semigroup.

For undefined concepts in semigroup theory we refer to [11], for undefined concepts in semiring theory see [10].

3. k-radicals of Green's relations

For a relation ρ on a semigroup S the radical of ρ, in notation $\sqrt{\rho}$, is a relation introduced by L.N. Shevrin [17] as follows:

$$(a, b) \in \sqrt{\rho} \Longleftrightarrow (a^m, b^n) \in \rho$$

for some $m, n \in \mathbb{N}$. Here we introduce k-radicals of the Green's relations $\overline{\mathcal{L}}, \overline{\mathcal{R}}, \overline{\mathcal{J}}, \overline{\mathcal{H}}$ in a semiring S in $\mathbb{SL}^+$ as follows:
$(a, b) \in \sqrt{\mathcal{L}} \Longleftrightarrow (a^m, b^n) \in \overline{\mathcal{L}} \Longleftrightarrow L_k(a^m) = L_k(b^n)$ for some $m, n \in \mathbb{N}$.

Similarly we define $\sqrt{\mathcal{R}}, \sqrt{\mathcal{J}}$ and $\sqrt{\mathcal{H}}$.

Bhuniya and Jana [2] introduced intra k-regular semirings and characterized these semirings as the semirings for which every k-ideal is semiprime. A semiring S is said to be *intra-k-regular* if for all $a \in S$ we have $a + sa^2s = sa^2s$ for some $s \in S$. On these semirings we find the least distributive lattice congruence which we present in the following lemma.

Lemma 1. *Let S be an intra-k-regular semiring. Then one has*

1. *for every $a, b \in S$ and $m \in \mathbb{N}, a\overline{\mathcal{J}}a^m$ and $ab\overline{\mathcal{J}}ba$.*
2. *$\overline{\mathcal{J}}$ is the least distributive lattice congruence on S.*

Proof. 1. Since $\overline{\mathcal{J}}$ is reflexive, $a\overline{\mathcal{J}}a^m$ holds for $m = 1$. Assume that the statement is true for some $r \in \mathbb{N}$. Then $I_k(a) = I_k(a^r)$ so that there exists $s \in S$ such that $a + a^r + sa^r + a^rs + sa^rs = a^r + sa^r + a^rs + sa^rs$. Also we have $a^r + ua^{4r}u = ua^{4r}u$, for some $u \in S$. Then one gets $a + ua^{4r}u + sua^{4r}u + ua^{4r}us + sua^{4r}us = ua^{4r}u + sua^{4r}u + ua^{4r}us + sua^{4r}us$ which can be written as $a + ua^{2r+1}a^{2r-1}u + sua^{2r+1}a^{2r-1}u + ua^{2r+1}a^{2r-1}us + sua^{2r+1}a^{2r-1}us = ua^{2r+1}a^{2r-1}u + sua^{2r+1}a^{2r-1}u + ua^{2r+1}a^{2r-1}us + sua^{2r+1}a^{2r-1}us$ so that $a \in I_k(a^{r+1})$. Thus $I_k(a) \subseteq I_k(a^{r+1})$. The opposite inclusion is clear and hence $a\overline{\mathcal{J}}a^{r+1}$. So by induction one gets $a\overline{\mathcal{J}}a^m$ for every $m \in \mathbb{N}$. Since S is intra-k-regular, for $a, b \in S$ one has $ab \in \overline{SbaS} \subseteq I_k(ba)$ so that one gets $I_k(ab) \subseteq I_k(ba)$. Interchanging the role of a and b we get the opposite inclusion, and so $I_k(ab) = I_k(ba)$. Thus $ab\overline{\mathcal{J}}ba$.

2. By (1) we have $I_k(ab) = I_k(ba)$ and $a \in I_k(a^2)$ for all $a, b \in S$, and then by Theorem 4.2 [5], one gets (2). ■

Here we characterize the intra-k-regular semirings by k-radicals of Green's relations.

Theorem 2. *In a semiring S the following conditions are equivalent:*

1. *S is an intra-k-regular semiring;*
2. $\sqrt{\overline{\mathcal{J}}} = \overline{\mathcal{J}}$*;*
3. $\sqrt{\overline{\mathcal{L}}} \subseteq \overline{\mathcal{J}}$*;*
4. $\sqrt{\overline{\mathcal{H}}} \subseteq \overline{\mathcal{J}}$*.*

Proof. (1) $\Rightarrow$ (2): For $(a,b) \in \sqrt{\overline{\mathcal{J}}}$ there are $m,n \in \mathbb{N}$ such that $(a^m, b^n) \in \overline{\mathcal{J}}$. Then by Lemma 1, we have $a\overline{\mathcal{J}}a^m\overline{\mathcal{J}}b^n\overline{\mathcal{J}}b$ so that $a\overline{\mathcal{J}}b$ that yields $\sqrt{\overline{\mathcal{J}}} \subseteq \overline{\mathcal{J}}$. The other inclusion is clear. Thus $\sqrt{\overline{\mathcal{J}}} = \overline{\mathcal{J}}$.

(2) $\Rightarrow$ (1): For $a \in S$, we have $a\sqrt{\overline{\mathcal{J}}}a^4$ so that $a\overline{\mathcal{J}}a^4$. Then there exists $s \in S$ such that $a + a^4 + a^4s + sa^4 + sa^4s = a^4 + a^4s + sa^4 + sa^4s$, which implies $a + ua^2u = ua^2u$, where $u = a + a^2 + s + a^2s$. Thus S is intra-k-regular.

(2) $\Rightarrow$ (3): Let $(a,b) \in \sqrt{\overline{\mathcal{L}}}$. Then there exist $m,n \in \mathbb{N}$ such that $L_k(a^m) = L_k(b^n)$, and then $a^m \in L_k(b^n) \subseteq I_k(b^n)$ and $b^n \in L_k(a^m) \subseteq I_k(a^m)$ implies that $I_k(a^m) = I_k(b^n)$. Thus $(a,b) \in \sqrt{\overline{\mathcal{J}}} = \overline{\mathcal{J}}$.

(3) $\Rightarrow$ (1): For $a \in S$, we have $a\sqrt{\overline{\mathcal{L}}}a^4$ so that $a\overline{\mathcal{J}}a^4$, and then as above S is intra-k-regular.

The proof that (4) is equivalent to the other conditions follows from the corresponding proof of (3) and its dual. ■

Several characterizations of the regular semigroups by the Green's relations were given by A.H. Cliffford and G.B. Preston [8]. Later, S. Bogdanović and M. Ćirić [6] characterized the regularity of the semigroups, using the radicals of the Green's relations, and also characterized left regular semigroups while investigating new decompositions of left regular semigroups [7]. This motivates us to find a class of semirings which are distributive lattices of left k-simple subsemirings, to introduce left k-regular semirings and show that every left k-regular semiring is a distributive lattice of left k-simple subsemirings. Several other characterizations were presented in [5].

Definition. A semiring S is said to be left k-regular if for every $a \in S, a \in \overline{Sa^2}$.

Similarly we define right k-regular semirings. Since $(S,+)$ is a semilattice, it is easy to observe that a semiring S is left k-regular if and only if for each $a \in S$ there exists $s \in S$ such that $a + sa^2 = sa^2$.

Example 3. Let $\mathbb{Q}^+$ be the set of all positive rationals, and consider the semiring $(\mathbb{Q}^+ \cup \{0\}, \max, \cdot)$. Set $S = \{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a,b \in \mathbb{Q}^+\}$. Then $(S,+,\cdot)$ is a left k-regular semiring over $\mathbb{Q}^+ \cup \{0\}$, where $+$ and $\cdot$ are usual addition and multiplication of matrices.

The class of left k-regular (right k-regular) semirings contains $P_f(F)$ when F is a left regular (right regular) semigroup [8]. In the following proposition we show that the left k-regularity is an extension of the left regularity of semigroups to the semirings in $\mathbb{SL}^+$.

Proposition 4. *Let F be a semigroup. Then the semiring $P_f(F)$ is left k-regular if and only if F is left regular.*

Proof. Let $P_f(F)$ be a left k-regular semiring and $a \in F$. Then $A = \{a\} \in P_f(F)$ and so there exists $X \in P_f(F)$ such that $A + XA^2 = XA^2$ so that $A \subseteq XA^2$. So there is an $x \in X$ such that $a = xa^2$, whence F is left regular.

Conversely suppose that F is left regular and $A \in P_f(F)$. Then for each $a \in A$, there exists $x_a \in F$ such that $a = x_a a^2$. We set $X = \{x_a : \ a \in A\}$. Then $X \in P_f(F)$ and $A \subseteq XA^2$ implies $A + XA^2 = XA^2$ which shows that $P_f(F)$ is left k-regular. ■

A similar proposition holds with respect to right k-regularity.

Theorem 5. *A semiring S is left k-regular if and only if $\sqrt{\mathcal{L}} = \overline{\mathcal{L}}$.*

Proof. Let S be left k-regular and $(a, b) \in \sqrt{\mathcal{L}}$. Then $(a^m, b^n) \in \overline{\mathcal{L}}$ for some $m, n \in \mathbb{N}$. Now there is $s \in S$ such that $a + sa^2 = sa^2$, which yields $a \in L_k(a^2)$, and so $L_k(a) \subseteq L_k(a^2)$. Also we have $L_k(a^2) \subseteq L_k(a)$, which implies that $L_k(a) = L_k(a^2)$. Thus $a\overline{\mathcal{L}}a^2$. Again $a^2 \in \overline{Sa^4} \subseteq \overline{Sa^3}$. Also $a^3 \in \overline{Sa^2}$ is clear, and so $a^2\overline{\mathcal{L}}a^3$. Similarly we get $a^k\overline{\mathcal{L}}a^{k+1}$ for $k \geq 3$. This implies $a\overline{\mathcal{L}}a^m\overline{\mathcal{L}}b^n\overline{\mathcal{L}}b$. Thus $a\overline{\mathcal{L}}b$. Also the opposite inclusion is clear and thus $\sqrt{\mathcal{L}} = \overline{\mathcal{L}}$.

Conversely, suppose that $\sqrt{\mathcal{L}} = \overline{\mathcal{L}}$. Then for $a \in S$ one has $a\sqrt{\mathcal{L}}a^4 \Rightarrow a\overline{\mathcal{L}}a^4$ which implies $a \in \overline{Sa^2}$ and thus S is left k-regular. ■

Similarly we have the dual of the above:

Theorem 6. *A semiring S is right k-regular if and only if $\sqrt{\mathcal{R}} = \overline{\mathcal{R}}$.*

Theorem 7. *A k-regular semiring S is left k-regular if it is a distributive lattice of left k-simple semirings.*

Proof. Let $a \in S$, then there exists $x \in S$ such that $a + axa = axa$. Also there is a $u \in S$ such that $ax + a + ua = a + ua$, since S is a distributive lattice of left k-simple subsemirings [5]. Then we have $a + (ax + a + ua)a = (ax + a + ua)a$ which implies $a + (a + ua)a = (a + ua)a$. Adding axa^2 to both sides, we get $a + (axa + ua)a = (axa + ua)a$, i.e., $a + va^2 = va^2$, where $v = ax + u$. Thus S is left k-regular. ■

Theorem 8. *Every left k-regular semiring S is a distributive lattice of left k-simple subsemirings.*

Proof. Let $a, x \in S$ such that $x^2 \in L_k(a)$ Then there exists $s \in S$ such that $x^2 + a + sa = a + sa$. Also there is a $u \in S$ such that $x + ux^2 = ux^2$. Then one gets $x + u(x^2 + a + sa) = u(x^2 + a + sa)$ implying $x + u(a + sa) = u(a + sa)$, i.e., $x + (u + us)\,a = (u + us)a$ so that $x \in \overline{Sa} \subseteq L_k(a)$. Thus $L_k(a)$ is semiprime. Then by Theorem 3.2 [5], S is a distributive lattice of left k-simple subsemirings. ∎

Now we give a necessary and sufficient condition for a semiring S to be a distributive lattice of left k-simple semirings via k-radicals of Green's relations, and during the proof we observe that a semiring S is a distributive lattice of left k-Archimedean semirings if $\sqrt{\mathcal{L}} = \overline{\mathcal{J}}$. A semiring S is said to be a left k-Archimedean semiring if $S = \sqrt{Sa}$, for every $a \in S$ [4]. A semiring S is said to be a distributive lattice of left k-Archimedean semirings if there exists a congruence ρ on S such that the quotient set S/ρ is a distributive lattice and each ρ-class is a left k-Archimedean semiring [4].

Theorem 9. *A semiring S is a distributive lattice of left k-simple semirings if and only if $\sqrt{\mathcal{L}} = \overline{\mathcal{J}}$.*

Proof. Let ξ be a congruence on S such that $\mathcal{D} = S/\xi$ is a distributive lattice and each ξ-class is a left k-simple subsemiring. Let $(a, b) \in \sqrt{\mathcal{L}}$. Then $L_k(a^m) = L_k(b^n)$ for some $m, n \in \mathbb{N}$ which implies $a \in L_k(a^m) = L_k(b^n) \subseteq L_k(b) \subseteq I_k(b)$, since $L_k(a)$ is semiprime by Theorem 3.2 [5]. Similarly, $b \in I_k(a)$. Thus $(a, b) \in \overline{\mathcal{J}}$. Now let $(a, b) \in \overline{\mathcal{J}}$. Then there exists $s \in S$ such that $a + b + sb + bs + sbs = b + sb + bs + sbs$. By Theorem 3.2 [5], $bs \in L_k(b), sbs \in L_k(sb) \subseteq L_k(b)$ and so one gets $a \in L_k(b)$. Similarly $b \in L_k(a)$ implies $(a, b) \in \sqrt{\mathcal{L}}$. Thus $\sqrt{\mathcal{L}} = \overline{\mathcal{J}}$.

Conversely, suppose that $\sqrt{\mathcal{L}} = \overline{\mathcal{J}}$. By Theorem 2, S is intra k-regular and then by Lemma 1, $ab\overline{\mathcal{J}}ba$ for all $a, b \in S$. So $ab\sqrt{\mathcal{L}}ba$ for all $a, b \in S$, i.e., $(ab)^m\overline{\mathcal{L}}(ba)^n$ for some $m, n \in \mathbb{N}$ and for all $a, b \in S$, i.e., there exists $m \in \mathbb{N}$ such that $(ab)^m \in L_k((ba)^n) \subseteq L_k(ba) \subseteq \overline{Sa}$ for all $a, b \in S$. Then by Theorem 3.3 [4], S is a distributive lattice $\mathcal{D} = S/\rho$ of left k-Archimedean semirings $S_\alpha, \alpha \in \mathcal{D}$. Let $a \in S_\alpha$ for some $\alpha \in \mathcal{D}$. Now there exists $s \in S$ such that $a + sa^4s = sa^4s$. Then $a + as, a + sa \in S_\alpha$ and so $a + (a + sa)a^2(a + as) = (a + sa)a^2(a + as)$, i.e., $a + xa^2x = xa^2x$, where $x = a + sa + as \in S_\alpha$. This implies $a + x^r a(ax)^r = x^r a(ax)^r$ for every $r \in \mathbb{N}$. Now since S_α is left k-Archimedean and $a^2, ax \in S_\alpha$, we have $(ax)^n \in \overline{S_\alpha a^2}$ for some $n \in \mathbb{N}$. Then $a + x^n a(ax)^n = x^r a(ax)^n$ implies $a \in \overline{S_\alpha a^2}$. This immediately implies that $a \in \overline{S_\alpha a^r}$ for every $r \in \mathbb{N}$. Again, for $a, b \in S_\alpha$ there are $u \in S_\alpha$ and $r \in \mathbb{N}$ such that $a^r + ub = ub$. Then $a \in \overline{S_\alpha a^r}$ implies $a \in \overline{S_\alpha b}$, i.e., S_α is left k-simple. ∎

Completely k-regular semirings were introduced in [1], and several equivalent conditions have been given for a semiring to be completely k-regular. One of them is: S is k-regular and $a \in \overline{a^2S} \cap \overline{Sa^2}$ for all $a \in S$. Here, finally we characterize the k-regular semirings which are completely k-regular via k-radicals of Green's relations.

Theorem 10. *The following conditions are equivalent on a k-regular semiring S:*

1. *S is a completely k-regular semiring;*
2. *$\sqrt{\mathcal{L}} = \overline{\mathcal{L}}$ and $\sqrt{\mathcal{R}} = \overline{\mathcal{R}}$;*
3. *$\sqrt{\mathcal{H}} = \overline{\mathcal{H}}$.*

Proof. (1) $\Rightarrow$ (2): Let $a \in S$. Then there exists $x, u, v \in S$ such that $a + axa = axa$, $ax + xua = xua$ and $xa + avx = avx$ so that $a \in \overline{Sa^2} \cap \overline{a^2S}$. Then by Theorem 5 and its dual, we get $\sqrt{\mathcal{L}} = \overline{\mathcal{L}}$ and $\sqrt{\mathcal{R}} = \overline{\mathcal{R}}$.

(2) $\Rightarrow$ (3): The proof is clear.

(3) $\Rightarrow$ (1): Let $a \in S$. Then there is $u \in S$ such that $a + aua = aua$, since S is k-regular. Now $(a, a^2) \in \sqrt{\mathcal{H}} = \overline{\mathcal{H}} = \overline{\mathcal{L}} \cap \overline{\mathcal{R}}$, and this implies $a \in \overline{Sa^2} \cap \overline{a^2S}$. Thus by Theorem 3.1.2 [1] S is a completely k-regular semiring. ∎

References

[1] A.K. Bhuniya, *On the additively regular semirings*, Ph.D Thesis, University of Calcutta, 2009.

[2] A.K. Bhuniya and K. Jana, *Bi-ideals in k-regular and intra k-regular semirings*, Discuss. Math. General Algebra and Applications **31** (2011) 5–25. ISSN 2084-0373

[3] A.K. Bhuniya and T.K. Mondal, *Distributive lattice decompositions of semirings with a semilattice additive reduct*, Semigroup Forum **80** (2010) 293–301. doi:10.1007/s00233-009-9205-6

[4] A.K. Bhuniya and T.K. Mondal, *Semirings which are distributive lattices of left k-Archimedean semirings*, Semigroup Forum (to appear). ISSN 1432-2137

[5] A.K. Bhuniya and T.K. Mondal, *Semirings which are distributive lattices of k-simple semirings*, Southeast Asian Bulletin of Mathematics **36** (2012) 309–318. ISSN 0129-2021

[6] S. Bogdanović and M. Ćirić, *A note on radicals of Green's relations*, PU.M.A **7** (1996) 215–219. ISSN 1218-4586

[7] S. Bogdanović and M. Ćirić, *A note on left regular semigroups* Publ. Math. Debrecen **48** (3–4) (1996) 285–291. ISSN 0033-3883

[8] A.H. Clifford and G.B. Preston, The algebraic theory of semigroups I, Amer. Math. Soc., 1977. ISBN 0-8218-0271-2

[9] A.H. Clifford, *Semigroups admitting relative inverses*, Ann. Math. **42** (1941) 1037–1049. ISSN 1939-0980

[10] U. Hebisch and H.J. Weinert, Semirings. Algebraic Theory and Applications in Computer Science (Singapore, World Scientific, 1998). ISBN 981-02-3601-8

[11] J.M. Howie, Fundamentals of semigroup theory (Clarendon, Oxford, 1995). Reprint in 2003. ISBN 0-19-851194-9

[12] G.L. Litvinov and V.P. Maslov, *The Correspondence principle for idempotent calculus and some computer applications*, in: Idempotency, J. Gunawardena (Editor), (Cambridge, Cambridge Univ. Press, 1998) 420–443.

[13] G.L. Litvinov, V.P. Maslov and A.N. Sobolevskii, Idempotent Mathematics and Interval Analysis, Preprint ESI 632, The Erwin Schrödinger International Institute for Mathematical Physics, Vienna, 1998; e-print: http://www.esi.ac.at and http://arXiv.org,math.Sc/9911126.

[14] T.K. Mondal, *Distributive lattices of t-k-Archimedean semirings*, Discuss. Math. General Algebra and Applications **31** (2011), 147–158. doi:10.7151/dmgaa.1179

[15] T.K. Mondal and A.K. Bhuniya, *Semirings which are distributive lattices of t-k-simple semirings*, submitted.

[16] M.K. Sen and A.K. Bhuniya, *On semirings whose additive reduct is a semilattice*, Semigroup forum **82** (2011) 131–140. doi:10.1007/s00233-010-9271-9

[17] L.N. Shevrin, *Theory of epigroups* I, Mat. Sbornic **185** (8) (1994) 129–160. ISSN 1468-4802

[18] H.S. Vandiver, *Note on a simple type of algebra in which the cancellation law of addition does not hold*, Bull. Amer. Math. Soc. **40** (1934) 914–920. ISSN 1088-9485

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