# THE INERTIA OF UNICYCLIC GRAPHS AND BICYCLIC GRAPHS 

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#### Abstract

Let $G$ be a graph with $n$ vertices and $\nu(G)$ be the matching number of $G$. The inertia of a graph $G, \operatorname{In}(G)=\left(n_{+}, n_{-}, n_{0}\right)$ is an integer triple specifying the numbers of positive, negative and zero eigenvalues of the adjacency matrix $A(G)$, respectively. Let $\eta(G)=n_{0}$ denote the nullity of $G$ (the multiplicity of the eigenvalue zero of $G$ ). It is well known that if $G$ is a tree, then $\eta(G)=n-2 \nu(G)$. Guo et al. [Ji-Ming Guo, Weigen Yan and Yeong-Nan Yeh. On the nullity and the matching number of unicyclic graphs, Linear Algebra and its Applications, 431 (2009), 1293-1301.] proved if $G$ is a unicyclic graph, then $\eta(G)$ equals $n-2 \nu(G)-1, n-2 \nu(G)$ or $n-2 \nu(G)+2$. Barrett et al. determined the inertia sets for trees and graphs with cut vertices. In this paper, we give the nullity of bicyclic graphs $\mathcal{B}_{n}^{++}$. Furthermore, we determine the inertia set in unicyclic graphs and $\mathcal{B}_{n}^{++}$, respectively.


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## 1. Introduction

Let $G=(V(G), E(G))$ be a simple graph with vertex set $V(G)=\left\{v_{1}, \ldots, v_{n}\right\}$ and edge set $E(G)$. The inertia of a graph $G, \operatorname{In}(G)=\left(n_{+}, n_{-}, n_{0}\right)$ is an integer triple specifying the numbers of positive, negative and zero eigenvalues of the adjacency matrix $A(G)$, respectively. It is well known if $G$ is a bipartite graph, then $n_{+}=$ $n_{-}$. Barrett, Hall, and Loewy [1] determined the inertia sets for trees and graphs with cut vertices. The nullity of $G$, denoted by $\eta=\eta(G)=n_{0}$, is the multiplicity
of the number zero in the spectrum of $G$. Then $n_{+}+n_{-}=n-r(A(G))=$ $\eta$. The nullity of graphs is of interest in chemistry since the occurrence of a zero eigenvalue of a bipartite graph (corresponding to an alternant hydrocarbon) indicates the chemical instability of the molecule which such a graph represents. The question is of interest also for non-alternant hydrocarbons (non-bipartite graph), but a direct connection with the chemical stability in these cases is not so straightforward. The nullity has been determined for trees, unicyclic graphs and bicyclic graphs, respectively [4, 5, 6]. Recently, Gutman and Borovićanin give a survey on the nullity of graphs.

A unicyclic graph is a simple connected graph with equal numbers of vertices and edges. For the sake of a convenient description, let $\mathcal{U}_{n}$ be the set of unicyclic graphs with $n$ vertices. A bicyclic graph is a simple connected graph in which the number of edges equals the number of vertices plus one.

Let $C_{p}$ and $C_{q}$ be two vertex-disjoint cycles. Suppose that $v_{1} \in C_{p}, v_{l} \in C_{q}$. Joining $v_{1}$ and $v_{l}$ by a path $v_{1} v_{2} \cdots v_{l}$ of length $l-1$, where $l \geq 1$ and $l=1$ means identifying $v_{1}$ with $v_{l}$, resultant graph, denoted by $\infty(p, l, q)$, is called an $\infty$-graph. Let $P_{l+1}, P_{p+1}$ and $P_{q+1}$ be three vertex-disjoint paths, where $\min \{p, l, q\} \geq 1$ and at most one of them is 1 . Identifying the three initial vertices and terminal vertices of them, respectively, resultant graph, denoted by $\theta(p, l, q)$, is called a $\theta$-graph (see Figure 1).


Figure 1

Let $\mathcal{B}_{n}$ be the set of all bicyclic graphs of order $n$. $\mathcal{B}_{n}$ consists of three types of graphs: the first type denoted by $\mathcal{B}_{n}^{+}$is the set of those graphs each of which is an $\infty$-graph with trees attached when $l>1$; the second type denoted by $\mathcal{B}_{n}^{++}$is the set of those graphs each of which is an $\infty$-graph with trees attached when $l=1$; the third type denoted by $\theta_{n}$ is the set of those graphs each of which is an $\theta$-graph with trees attached.

In Section 3, we study the inertia in $\mathcal{U}_{n}$. In Section 4, we give the nullity and the inertia sets in $\mathcal{B}_{n}^{++}$, respectively.

## 2. Main Lemmas

A matching of $G$ is a collection of independent edges of $G$. A maximum matching is a matching with the maximum possible number of independent edges. The size of a maximum matching of $G$, i.e., the maximum number of independent edges of $G$, is denoted by $\nu=\nu(G)$.

Denote by $\varphi(x)=\varphi_{G}(x)$ the characteristic polynomial of $G$. Let

$$
\begin{equation*}
\varphi(x)=|x I-A|=x^{n}+a_{1} x^{n-1}+a_{2} x^{n-2}+\cdots+a_{n-1} x+a_{n} . \tag{1}
\end{equation*}
$$

Then [2]

$$
\begin{equation*}
a_{i}=\sum_{U}(-1)^{p(U)} 2^{c(U)} \quad(i=1,2, \ldots, n), \tag{2}
\end{equation*}
$$

where the sum is over all subgraphs $U$ of $G$ consisting of disjoint edges and cycles and having exactly $i$ vertices (called "basic figures"). If $U$ is such a subgraph, then $p(U)$ is the number of its components, of which $c(U)$ components are cycles.

Example 1. Let $G$ is a bipartite graph, then $G$ does not contain an odd cycle, so $a_{2 i+1}=0(i \geq 1)$.
Example 2. Considering equation (1) with equation (2), it is easy to obtain $a_{1}=0$ and $a_{2}=2 m(m$ is the number of edges of $G)$. In the following, we calculate $a_{3}$ and $a_{4}$. The subgraphs $U$ of $G$ having exactly 3 vertices consist of only the cycle $C_{3}$. Suppose that $n_{\Delta}$ is the number of the cycles $C_{3}$ in $G$, then $a_{3}=-2 n_{\Delta}$. Let $n_{\square}$ and $\nu_{2}(G)$ be the number of the cycles $C_{4}$, and two mutually disjoint edges in $G$, respectively, then $a_{4}=\nu_{2}(G)-2 n_{\square}$.

Next, we introduce the well-known Cauchy's interlacing theorem in matrix theory.
Lemma 3 [2]. Let $A$ be symmetric and $A^{\prime}$ be one of its principal submatrices. Let $\lambda_{1} \geq \cdots \geq \lambda_{n}$ and $\lambda_{1}^{\prime} \geq \cdots \geq \lambda_{m}^{\prime}$ be the eigenvalues of $A$ and $A^{\prime}$, respectively. Then the inequality $\lambda_{i} \geq \lambda_{i}^{\prime} \geq \lambda_{n-m+i}$ holds for all $i=1,2, \ldots, m$.
Applying the Cauchy's interlacing theorem to the adjacency matrix $A(G)$ of the graph $G$, we have the following corollary.

Corollary 4. Let $V_{0}$ be the $k$-subset of $G=(V, E)$ with $n$ vertices $(0 \leq k \leq n-1)$, and $G-V_{0}$ be the subgraph induced by removing the vertices $V_{0}$ and their incident edges. Then $\lambda_{i}(G) \geq \lambda_{i}\left(G-V_{0}\right) \geq \lambda_{i+k}(G) \quad(1 \leq i \leq n-k)$.

The next lemma is useful to the proof of our main results.
Lemma 5 [2]. For a graph $G$ containing a pendent vertex, if the induced subgraph $H$ of $G$ is obtained by deleting this vertex together with the vertex adjacent to it, then the relation $\eta(H)=\eta(G)$ holds.

## 3. The inertia of unicyclic graphs

$n$ this section, we determine the inertia in $\mathcal{U}_{n}$. In order to prove our result, the following lemma is necessary.

Lemma 6 [5]. Suppose $G \in \mathcal{U}_{n}$ with the cycle $C_{l}$. Then
(1) $\eta(G)=n-2 \nu(G)-1$, if $\nu(G)=\frac{l-1}{2}+\nu\left(G-C_{l}\right)$;
(2) $\eta(G)=n-2 \nu(G)+2$, if $G$ satisfies: $\nu(G)=\frac{l}{2}+\nu\left(G-C_{l}\right), l \equiv 0(\bmod 4)$ and no maximum matching contains an edge incident to $C_{l}$;
(3) $\eta(G)=n-2 \nu(G)$, otherwise.

If $G \in \mathcal{U}_{n}$ is a bipartite graph, we know $n_{+}=n_{-}$and $n_{+}+n_{-}=n-\eta(G)$, then $\operatorname{In}(G)=(\nu(G)-1, \nu(G)-1, n-2 \nu(G)+2)$ or $\operatorname{In}(G)=(\nu(G), \nu(G), n-2 \nu(G))$, So we only consider those graphs $G \in \mathcal{U}_{n}$ which are non-bipartite.
Lemma 7. If $G \in \mathcal{U}_{n}$ is a non-bipartite graph, then $\operatorname{In}(G)=(\nu(G)+1, \nu(G)$, $n-2 \nu(G)-1), \operatorname{In}(G)=(\nu(G), \nu(G)+1, n-2 \nu(G)-1)$ or $\operatorname{In}(G)=(\nu(G), \nu(G)$, $n-2 \nu(G))$.
Proof. Since $G \in \mathcal{U}_{n}$ with the cycle $C_{l}$ is a non-bipartite graph, then $l$ is odd. Let $v_{i} \in V\left(C_{l}\right)$ and $d_{i} \geq 3$. Suppose that $T_{1}, \ldots, T_{d_{i}}$ are the components of $G-v_{i}$ where $d_{i}=d\left(v_{i}\right)$. Let $V\left(T_{j}\right)=n_{j}$ and $\nu_{j}=\nu\left(T_{j}\right)\left(j=1, \ldots, d_{i}\right)$, so we have $\sum_{j=1}^{d_{i}} n_{j}=n-1$ and $\sum_{j=1}^{d_{i}} \nu_{j}=\nu(G)$ or $\nu(G)-1$. And $\operatorname{In}\left(T_{j}\right)=\left(\nu_{j}, \nu_{j}, n_{j}-2 \nu_{j}\right)$. We discuss two cases in the following.
(1) $\nu(G)=\frac{l-1}{2}+\nu\left(G-C_{l}\right)$, then $\eta(G)=n-2 \nu(G)-1$ and $\sum_{j=1}^{d_{i}} \nu_{j}=$ $\nu(G)$. We know $\eta\left(G-v_{i}\right)=\sum_{j=1}^{d_{i}} \eta\left(T_{j}\right)=n-1-2 \sum_{j=1}^{d_{i}} \nu_{j}=n-$ $2 \nu(G)-1$. Let $\lambda_{1}^{\prime}, \ldots, \lambda_{\nu(G)}^{\prime}, \underbrace{\lambda_{\nu(G)+1}^{\prime}, \ldots, \lambda_{n-1-\nu(G)}^{\prime}}_{n-2 \nu(G)-1}, \lambda_{n-\nu(G)}^{\prime}, \ldots, \lambda_{n-1}^{\prime}$ be the eigenvalues of $G-v_{i}$ according to nondecreasing order. By Corollary 4, we have $\lambda_{n-\nu(G)+1}(G) \leq \lambda_{n-\nu(G)}^{\prime}<0$ and $\lambda_{\nu(G)}(G) \geq \lambda_{\nu(G)}^{\prime}>0$. So $\operatorname{In}(G)=(\nu(G)+1, \nu(G), n-2 \nu(G)-1)$ or $\operatorname{In}(G)=(\nu(G), \nu(G)+1$, $n-2 \nu(G)-1)$.
(2) $\nu(G) \neq \frac{l-1}{2}+\nu\left(G-C_{l}\right)$, then $\eta(G)=n-2 \nu(G)$ and $\sum_{j=1}^{d_{i}} \nu_{j}=\nu(G)-1$. We know $\eta\left(G-v_{i}\right)=\sum_{j=1}^{d_{i}} \eta\left(T_{j}\right)=n-1-2 \sum_{j=1}^{d_{i}} \nu_{j}=n-2 \nu(G)+1$. Let $\lambda_{1}^{\prime}, \ldots, \lambda_{\nu(G)}^{\prime}, \underbrace{\lambda_{\nu(G)+1}^{\prime}, \ldots, \lambda_{n-\nu(G)+1}^{\prime}}_{n-2 \nu(G)+1}, \lambda_{n-\nu(G)+2}^{\prime}, \ldots, \lambda_{n-1}^{\prime}$ be the eigenvalues of $G-v_{i}$ according to nondecreasing order. By Corollary 4, we have $\lambda_{n-\nu(G)+2}(G) \leq \lambda_{n-\nu(G)+1}^{\prime}<0$ and $\lambda_{\nu(G)}(G) \geq \lambda_{\nu(G)}^{\prime}>0$. And $\eta(G)=n-2 \nu(G)$, so $\operatorname{In}(G)=(\nu(G), \nu(G), n-2 \nu(G))$.

Basing on the above detailed account, we obtain the next theorem.
Theorem 8. If $G \in \mathcal{U}_{n}$, then $\operatorname{In}(G)=(\nu(G)-1, \nu(G)-1, n-2 \nu(G)+2)$, $(\nu(G), \nu(G), n-2 \nu(G)),(\nu(G)+1, \nu(G), n-2 \nu(G)-1)$ or $(\nu(G), \nu(G)+1$, $n-2 \nu(G)-1)$.

## 4. The inertia of bicyclic graphs

In this section, we only consider $\mathcal{B}_{n}^{++}$. For $G \in \mathcal{B}_{n}^{++}$, we give the nullity of $G$ and determine the inertia of $G$ according to $\nu(G)$, respectively.

Lemma 9. The graph $\infty(p, 1, q)$ is defined as above, then
(1) $\eta(\infty(4 s, 1,4 t+2))=1(s, t \geq 1)$;
(2) $\eta(\infty(4 s, 1,4 t))=3(s, t \geq 1)$.

Proof. Let $\varphi_{1}(x)=|x I-A|=x^{n}+a_{1} x^{n-1}+a_{2} x^{n-2}+\cdots+a_{4 s+4 t} x+a_{4 s+4 t+1}$ and $\varphi_{2}(x)=|x I-B|=x^{n}+b_{1} x^{n-1}+b_{2} x^{n-2}+\cdots+b_{4 s+4 t-2} x+b_{4 s+4 t-1}$ be the polynomials of $\infty(4 s, 1,4 t+2)$ and $\infty(4 s, 1,4 t)$, respectively. Since $\infty(4 s, 1,4 t+2)$ and $\infty(4 s, 1,4 t)$ are bipartite graph, so by the equation (2), we have $a_{2 i+1}=0$ and $b_{2 i+1}=0$ for $i \geq 1$. First of all, we consider $a_{4 s+4 t}$ using the equation (2), then $a_{4 s+4 t}=2 m_{1}(-1)^{2 t+1}+2 m_{2}(-1)^{2 s}+\left(2 m_{1}+2 m_{2}\right) \neq 0$, where $m_{1}$ is the number of methods picking up $2 t$ disjoint edges from $P_{4 t+1}$ and $m_{2}$ is the number of methods picking up $2 s-1$ disjoint edges from $P_{4 s-1}$. So $\eta(\infty(4 s, 1,4 t+2))=1$.

Next, we prove $b_{4 s+4 t-2}=0$ and $b_{4 s+4 t-4} \neq 0$. Using the similar method as above, we have $b_{4 s+4 t-2}=2 m_{1}(-1)^{2 t}+2 m_{2}(-1)^{2 s}-\left(2 m_{1}+2 m_{2}\right)=0$, where $m_{1}$ is the number of methods picking up $2 t-1$ disjoint edges from $P_{4 t-1}$ and $m_{2}$ is the number of methods picking up $2 s-1$ disjoint edges from $P_{4 s-1}$. And $b_{4 s+4 t-4} \geq m_{3}>0$ where $m_{3}$ is the number of methods picking up $2 t-1$ disjoint edges from $P_{4 t}$ and picking up $2 s-1$ disjoint edges from $P_{4 s-1}$. So we complete the proof.

Using the similar method of proof in Lemma 9 and the equation (2), we obtain the following lemma.

Lemma 10. The graph $\infty(p, 1, q)$ is defined as above, then
(1) $\eta(\infty(2 s+1,1,4 t))=\eta(\infty(4 s+1,1,4 t+3))=1$;
(2) $\eta(\infty(2 s+1,1,4 t+2))=\eta(\infty(4 s+1,1,4 t+1))=0$.

Lemma 11 [3]. If a bipartite graph $G$ with $n \geq 1$ vertices does not contain any cycle of length $4 s(s \geq 1)$, then $\eta(G)=n-2 \nu(G)$.

In accordance with Lemma 11, it is easy to know for $G \in \mathcal{B}_{n}^{++}$is a bipartite graph with not containing cycle $C_{4 s}(s \geq 1)$, then $\eta(G)=n-2 \nu(G)$, so $\operatorname{In}(G)=$ ( $\nu(G), \nu(G), n-2 \nu(G))$. Hence in the following, we discuss the case $G \in \mathcal{B}_{n}^{++}$is a bipartite graph with containing cycles $C_{4 s}(s \geq 1)$.

Lemma 12. If $G \in \mathcal{B}_{n}^{++}$is a bipartite graph with containing cycle $C_{4 s}(s \geq 1)$, then $\eta(G)=n-2 \nu(G)$ or $\eta(G)=n-2 \nu(G)+2$.

Proof. Putting to use the Lemma 5 a times, we can obtain the following cases:
(1) $T_{i}(1 \leq i \leq s)$ are the components where $T_{i}(1 \leq i \leq s)$ are trees with $n_{i}$ vertices. Then $\eta(G)=\sum_{i=1}^{s} \eta\left(T_{i}\right)=\sum_{i=1}^{s}\left(n_{i}-2 \nu\left(T_{i}\right)\right)=n-a-2(\nu(G)-$ $a)=n-2 \nu(G)$.
(2) $U_{0}, T_{i}(1 \leq i \leq s)$ are the components where $T_{i}(1 \leq i \leq s)$ are trees with $n_{i}$ vertices and $U_{0}$ is a unicyclic graph with $n_{0}$ vertices. By Lemma 6, we know $\eta\left(U_{0}\right)=n_{0}-2 \nu\left(U_{0}\right)$ or $n_{0}-2 \nu\left(U_{0}\right)+2$, so $\eta(G)=\eta\left(U_{0}\right)+\sum_{i=1}^{s} \eta\left(T_{i}\right)=$ $n-2 \nu(G)$ or $n-2 \nu(G)+2$.
(3) $\infty(p, 1, q), T_{i}(1 \leq i \leq s)$ are the components where $T_{i}(1 \leq i \leq s)$ are trees with $n_{i}$ vertices and $\infty(p, 1, q)$ is a bicyclic graph with $n_{0}$ vertices. By Lemma 9 , we have $\eta(\infty(4 s, 1,4 t+2))=1$ or $\eta(\infty(4 s, 1,4 t))=3$. Then $\eta(G)=\eta(\infty(p, 1, q))+\sum_{i=1}^{s} \eta\left(T_{i}\right)=n-2 \nu(G)$ or $n-2 \nu(G)+2$.

Combining Lemmas 10 and 12, we obtain the following theorem.
Theorem 13. If $G \in \mathcal{B}_{n}^{++}$is a bipartite graph, then $\eta(G)=n-2 \nu(G)$ or $\eta(G)=n-2 \nu(G)+2$.

Lemma 14. If $G \in \mathcal{B}_{n}^{++}$is a non-bipartite graph, then $\eta(G)=n-2 \nu(G)-$ $1, n-2 \nu(G), n-2 \nu(G)+1$ or $\eta(G)=n-2 \nu(G)+2$.

Proof. Putting to use the Lemma $5 b$ times, we can obtain the following cases:
(1) $T_{i}(1 \leq i \leq s)$ are the components where $T_{i}(1 \leq i \leq s)$ are trees with $n_{i}$ vertices. Then $\eta(G)=\sum_{i=1}^{s} \eta\left(T_{i}\right)=n-2 \nu(G)$.
(2) $U_{0}, T_{i}(1 \leq i \leq s)$ are the components where $T_{i}(1 \leq i \leq s)$ are trees with $n_{i}$ vertices and $U_{0}$ is a unicyclic graph with $n_{0}$ vertices. By Lemma 6 , we know $\eta\left(U_{0}\right)=n_{0}-2 \nu\left(U_{0}\right), n_{0}-2 \nu\left(U_{0}\right)+2$ or $n_{0}-2 \nu\left(U_{0}\right)-1$, so $\eta(G)=\eta\left(U_{0}\right)+\sum_{i=1}^{s} \eta\left(T_{i}\right)=n-2 \nu(G), n-2 \nu(G)+2$ or $n-2 \nu(G)-1$.
(3) $\infty(p, 1, q), T_{i}(1 \leq i \leq s)$ are the components where $T_{i}(1 \leq i \leq s)$ are trees with $n_{i}$ vertices and $\infty(p, 1, q)$ is a bicyclic graph with $n_{0}$ vertices. By Lemma 10 , we have $\eta(\infty(2 t+1,1,4 s))=1, \eta(\infty(2 t+1,1,4 s+2))=0$, $\eta(\infty(4 s+1,1,4 t+1))=0$ or $\eta(\infty(4 s+1,1,4 t+3))=1$. Then $\eta(G)=$ $\eta(\infty(p, 1, q))+\sum_{i=1}^{s} \eta\left(T_{i}\right)=n-2 \nu(G)+1, n-2 \nu(G)$ or $n-2 \nu(G)-1$.

Using the similar method of Lemma 7 and Lemma 14, we have the next lemma.
Lemma 15. If $G \in \mathcal{B}_{n}^{++}$is a non-bipartite graph, then $\operatorname{In}(G)=(\nu(G), \nu(G)+$ $1, n-2 \nu(G)-1),(\nu(G)+1, \nu(G), n-2 \nu(G)-1),(\nu(G), \nu(G), n-2 \nu(G)),(\nu(G)$, $\nu(G)-1, n-2 \nu(G)+1),(\nu(G)+1, \nu(G)-2, n-2 \nu(G)+1),(\nu(G), \nu(G)-2$, $n-2 \nu(G)+2)$.

So we obtain our main result.
Theorem 16. If $G \in \mathcal{B}_{n}^{++}$, then $\operatorname{In}(G)=(\nu(G), \nu(G)+1, n-2 \nu(G)-1),(\nu(G)+$ $1, \nu(G), n-2 \nu(G)-1),(\nu(G), \nu(G), n-2 \nu(G)),(\nu(G), \nu(G)-1, n-2 \nu(G)+1)$, $(\nu(G)+1, \nu(G)-2, n-2 \nu(G)+1),(\nu(G), \nu(G)-2, n-2 \nu(G)+2)$.

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