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# THE INERTIA OF UNICYCLIC GRAPHS AND BICYCLIC GRAPHS

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#### Abstract

Let G be a graph with n vertices and  $\nu(G)$  be the matching number of G. The inertia of a graph G,  $In(G) = (n_+, n_-, n_0)$  is an integer triple specifying the numbers of positive, negative and zero eigenvalues of the adjacency matrix A(G), respectively. Let  $\eta(G) = n_0$  denote the nullity of G (the multiplicity of the eigenvalue zero of G). It is well known that if G is a tree, then  $\eta(G) = n - 2\nu(G)$ . Guo et al. [Ji-Ming Guo, Weigen Yan and Yeong-Nan Yeh. On the nullity and the matching number of unicyclic graphs, Linear Algebra and its Applications, 431 (2009), 1293–1301.] proved if G is a unicyclic graph, then  $\eta(G)$  equals  $n - 2\nu(G) - 1, n - 2\nu(G)$  or  $n - 2\nu(G) + 2$ . Barrett et al. determined the inertia sets for trees and graphs with cut vertices. In this paper, we give the nullity of bicyclic graphs and  $\mathcal{B}_n^{++}$ , respectively.

**Keywords:** matching number, inertia, nullity, unicyclic graph, bicyclic graph.

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### 1. INTRODUCTION

Let G = (V(G), E(G)) be a simple graph with vertex set  $V(G) = \{v_1, \ldots, v_n\}$  and edge set E(G). The inertia of a graph G,  $In(G) = (n_+, n_-, n_0)$  is an integer triple specifying the numbers of positive, negative and zero eigenvalues of the adjacency matrix A(G), respectively. It is well known if G is a bipartite graph, then  $n_+ =$  $n_-$ . Barrett, Hall, and Loewy [1] determined the inertia sets for trees and graphs with cut vertices. The nullity of G, denoted by  $\eta = \eta(G) = n_0$ , is the multiplicity of the number zero in the spectrum of G. Then  $n_+ + n_- = n - r(A(G)) = \eta$ . The nullity of graphs is of interest in chemistry since the occurrence of a zero eigenvalue of a bipartite graph (corresponding to an alternant hydrocarbon) indicates the chemical instability of the molecule which such a graph represents. The question is of interest also for non-alternant hydrocarbons (non-bipartite graph), but a direct connection with the chemical stability in these cases is not so straightforward. The nullity has been determined for trees, unicyclic graphs and bicyclic graphs, respectively [4, 5, 6]. Recently, Gutman and Borovićanin give a survey on the nullity of graphs.

A unicyclic graph is a simple connected graph with equal numbers of vertices and edges. For the sake of a convenient description, let  $\mathcal{U}_n$  be the set of unicyclic graphs with *n* vertices. A bicyclic graph is a simple connected graph in which the number of edges equals the number of vertices plus one.

Let  $C_p$  and  $C_q$  be two vertex-disjoint cycles. Suppose that  $v_1 \in C_p$ ,  $v_l \in C_q$ . Joining  $v_1$  and  $v_l$  by a path  $v_1v_2\cdots v_l$  of length l-1, where  $l \ge 1$  and l = 1 means identifying  $v_1$  with  $v_l$ , resultant graph, denoted by  $\infty(p, l, q)$ , is called an  $\infty$ -graph. Let  $P_{l+1}, P_{p+1}$  and  $P_{q+1}$  be three vertex-disjoint paths, where min $\{p, l, q\} \ge 1$ and at most one of them is 1. Identifying the three initial vertices and terminal vertices of them, respectively, resultant graph, denoted by  $\theta(p, l, q)$ , is called a  $\theta$ -graph (see Figure 1).





Let  $\mathcal{B}_n$  be the set of all bicyclic graphs of order n.  $\mathcal{B}_n$  consists of three types of graphs: the first type denoted by  $\mathcal{B}_n^+$  is the set of those graphs each of which is an  $\infty$ -graph with trees attached when l > 1; the second type denoted by  $\mathcal{B}_n^{++}$  is the set of those graphs each of which is an  $\infty$ -graph with trees attached when l = 1; the third type denoted by  $\theta_n$  is the set of those graphs each of which is an  $\theta$ -graph with trees attached.

In Section 3, we study the inertia in  $\mathcal{U}_n$ . In Section 4, we give the nullity and the inertia sets in  $\mathcal{B}_n^{++}$ , respectively.

### 2. Main Lemmas

A matching of G is a collection of independent edges of G. A maximum matching is a matching with the maximum possible number of independent edges. The size of a maximum matching of G, i.e., the maximum number of independent edges of G, is denoted by  $\nu = \nu(G)$ .

Denote by  $\varphi(x) = \varphi_G(x)$  the characteristic polynomial of G. Let

(1) 
$$\varphi(x) = |xI - A| = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n.$$

Then [2]

(2) 
$$a_i = \sum_U (-1)^{p(U)} 2^{c(U)} \quad (i = 1, 2, \dots, n),$$

where the sum is over all subgraphs U of G consisting of disjoint edges and cycles and having exactly *i* vertices (called "basic figures"). If U is such a subgraph, then p(U) is the number of its components, of which c(U) components are cycles.

**Example 1.** Let G is a bipartite graph, then G does not contain an odd cycle, so  $a_{2i+1} = 0$   $(i \ge 1)$ .

**Example 2.** Considering equation (1) with equation (2), it is easy to obtain  $a_1 = 0$  and  $a_2 = 2m$  (*m* is the number of edges of *G*). In the following, we calculate  $a_3$  and  $a_4$ . The subgraphs *U* of *G* having exactly 3 vertices consist of only the cycle  $C_3$ . Suppose that  $n_{\Delta}$  is the number of the cycles  $C_3$  in *G*, then  $a_3 = -2n_{\Delta}$ . Let  $n_{\Box}$  and  $\nu_2(G)$  be the number of the cycles  $C_4$ , and two mutually disjoint edges in *G*, respectively, then  $a_4 = \nu_2(G) - 2n_{\Box}$ .

Next, we introduce the well-known Cauchy's interlacing theorem in matrix theory.

**Lemma 3** [2]. Let A be symmetric and A' be one of its principal submatrices. Let  $\lambda_1 \geq \cdots \geq \lambda_n$  and  $\lambda'_1 \geq \cdots \geq \lambda'_m$  be the eigenvalues of A and A', respectively. Then the inequality  $\lambda_i \geq \lambda'_i \geq \lambda_{n-m+i}$  holds for all  $i = 1, 2, \ldots, m$ .

Applying the Cauchy's interlacing theorem to the adjacency matrix A(G) of the graph G, we have the following corollary.

**Corollary 4.** Let  $V_0$  be the k-subset of G = (V, E) with n vertices  $(0 \le k \le n-1)$ , and  $G - V_0$  be the subgraph induced by removing the vertices  $V_0$  and their incident edges. Then  $\lambda_i(G) \ge \lambda_i(G - V_0) \ge \lambda_{i+k}(G)$   $(1 \le i \le n-k)$ .

The next lemma is useful to the proof of our main results.

**Lemma 5** [2]. For a graph G containing a pendent vertex, if the induced subgraph H of G is obtained by deleting this vertex together with the vertex adjacent to it, then the relation  $\eta(H) = \eta(G)$  holds.

### 3. The inertia of unicyclic graphs

n this section, we determine the inertia in  $\mathcal{U}_n$ . In order to prove our result, the following lemma is necessary.

**Lemma 6** [5]. Suppose  $G \in \mathcal{U}_n$  with the cycle  $C_l$ . Then

- (1)  $\eta(G) = n 2\nu(G) 1$ , if  $\nu(G) = \frac{l-1}{2} + \nu(G C_l)$ ;
- (2)  $\eta(G) = n 2\nu(G) + 2$ , if G satisfies:  $\nu(G) = \frac{l}{2} + \nu(G C_l)$ ,  $l \equiv 0 \pmod{4}$ and no maximum matching contains an edge incident to  $C_l$ ;
- (3)  $\eta(G) = n 2\nu(G)$ , otherwise.

If  $G \in \mathcal{U}_n$  is a bipartite graph, we know  $n_+ = n_-$  and  $n_+ + n_- = n - \eta(G)$ , then  $In(G) = (\nu(G) - 1, \nu(G) - 1, n - 2\nu(G) + 2)$  or  $In(G) = (\nu(G), \nu(G), n - 2\nu(G))$ , So we only consider those graphs  $G \in \mathcal{U}_n$  which are non-bipartite.

**Lemma 7.** If  $G \in U_n$  is a non-bipartite graph, then  $In(G) = (\nu(G) + 1, \nu(G), n - 2\nu(G) - 1), In(G) = (\nu(G), \nu(G) + 1, n - 2\nu(G) - 1)$  or  $In(G) = (\nu(G), \nu(G), n - 2\nu(G)).$ 

**Proof.** Since  $G \in \mathcal{U}_n$  with the cycle  $C_l$  is a non-bipartite graph, then l is odd. Let  $v_i \in V(C_l)$  and  $d_i \geq 3$ . Suppose that  $T_1, \ldots, T_{d_i}$  are the components of  $G - v_i$ where  $d_i = d(v_i)$ . Let  $V(T_j) = n_j$  and  $\nu_j = \nu(T_j)$   $(j = 1, \ldots, d_i)$ , so we have  $\sum_{j=1}^{d_i} n_j = n-1$  and  $\sum_{j=1}^{d_i} \nu_j = \nu(G)$  or  $\nu(G)-1$ . And  $In(T_j) = (\nu_j, \nu_j, n_j - 2\nu_j)$ . We discuss two cases in the following.

(1)  $\nu(G) = \frac{l-1}{2} + \nu(G - C_l)$ , then  $\eta(G) = n - 2\nu(G) - 1$  and  $\sum_{j=1}^{d_i} \nu_j = \nu(G)$ . We know  $\eta(G - v_i) = \sum_{j=1}^{d_i} \eta(T_j) = n - 1 - 2\sum_{j=1}^{d_i} \nu_j = n - 2\nu(G) - 1$ . Let  $\lambda'_1, \dots, \lambda'_{\nu(G)}, \underbrace{\lambda'_{\nu(G)+1}, \dots, \lambda'_{n-1-\nu(G)}}_{n-2\nu(G)-1}, \lambda'_{n-\nu(G)}, \dots, \lambda'_{n-1}$  be

 $n-2\nu(G)-1$ the eigenvalues of  $G - v_i$  according to nondecreasing order. By Corollary 4, we have  $\lambda_{n-\nu(G)+1}(G) \leq \lambda'_{n-\nu(G)} < 0$  and  $\lambda_{\nu(G)}(G) \geq \lambda'_{\nu(G)} > 0$ . So  $In(G) = (\nu(G) + 1, \nu(G), n - 2\nu(G) - 1)$  or  $In(G) = (\nu(G), \nu(G) + 1, n - 2\nu(G) - 1)$ .

(2)  $\nu(G) \neq \frac{l-1}{2} + \nu(G - C_l)$ , then  $\eta(G) = n - 2\nu(G)$  and  $\sum_{j=1}^{d_i} \nu_j = \nu(G) - 1$ . We know  $\eta(G - v_i) = \sum_{j=1}^{d_i} \eta(T_j) = n - 1 - 2\sum_{j=1}^{d_i} \nu_j = n - 2\nu(G) + 1$ . Let  $\lambda'_1, \dots, \lambda'_{\nu(G)}, \underbrace{\lambda'_{\nu(G)+1}, \dots, \lambda'_{n-\nu(G)+1}}_{n-2\nu(G)+1}, \lambda'_{n-\nu(G)+2}, \dots, \lambda'_{n-1}$  be the eigen-

values of  $G - v_i$  according to nondecreasing order. By Corollary 4, we have  $\lambda_{n-\nu(G)+2}(G) \leq \lambda'_{n-\nu(G)+1} < 0$  and  $\lambda_{\nu(G)}(G) \geq \lambda'_{\nu(G)} > 0$ . And  $\eta(G) = n - 2\nu(G)$ , so  $In(G) = (\nu(G), \nu(G), n - 2\nu(G))$ .

Basing on the above detailed account, we obtain the next theorem.

**Theorem 8.** If  $G \in U_n$ , then  $In(G) = (\nu(G) - 1, \nu(G) - 1, n - 2\nu(G) + 2)$ ,  $(\nu(G), \nu(G), n - 2\nu(G))$ ,  $(\nu(G) + 1, \nu(G), n - 2\nu(G) - 1)$  or  $(\nu(G), \nu(G) + 1, n - 2\nu(G) - 1)$ .

### 4. The inertia of bicyclic graphs

In this section, we only consider  $\mathcal{B}_n^{++}$ . For  $G \in \mathcal{B}_n^{++}$ , we give the nullity of G and determine the inertia of G according to  $\nu(G)$ , respectively.

**Lemma 9.** The graph  $\infty(p, 1, q)$  is defined as above, then

- (1)  $\eta(\infty(4s, 1, 4t+2)) = 1 \ (s, t \ge 1);$
- (2)  $\eta(\infty(4s, 1, 4t)) = 3 \ (s, t \ge 1).$

**Proof.** Let  $\varphi_1(x) = |xI - A| = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{4s+4t} + a_{4s+4t+1}$ and  $\varphi_2(x) = |xI - B| = x^n + b_1 x^{n-1} + b_2 x^{n-2} + \dots + b_{4s+4t-2} x + b_{4s+4t-1}$  be the polynomials of  $\infty(4s, 1, 4t+2)$  and  $\infty(4s, 1, 4t)$ , respectively. Since  $\infty(4s, 1, 4t+2)$ and  $\infty(4s, 1, 4t)$  are bipartite graph, so by the equation (2), we have  $a_{2i+1} = 0$ and  $b_{2i+1} = 0$  for  $i \ge 1$ . First of all, we consider  $a_{4s+4t}$  using the equation (2), then  $a_{4s+4t} = 2m_1(-1)^{2t+1} + 2m_2(-1)^{2s} + (2m_1 + 2m_2) \ne 0$ , where  $m_1$  is the number of methods picking up 2t disjoint edges from  $P_{4t+1}$  and  $m_2$  is the number of methods picking up 2s-1 disjoint edges from  $P_{4s-1}$ . So  $\eta(\infty(4s, 1, 4t+2)) = 1$ .

Next, we prove  $b_{4s+4t-2} = 0$  and  $b_{4s+4t-4} \neq 0$ . Using the similar method as above, we have  $b_{4s+4t-2} = 2m_1(-1)^{2t} + 2m_2(-1)^{2s} - (2m_1 + 2m_2) = 0$ , where  $m_1$  is the number of methods picking up 2t - 1 disjoint edges from  $P_{4t-1}$  and  $m_2$  is the number of methods picking up 2s - 1 disjoint edges from  $P_{4s-1}$ . And  $b_{4s+4t-4} \geq m_3 > 0$  where  $m_3$  is the number of methods picking up 2t - 1 disjoint edges from  $P_{4t}$  and picking up 2s - 1 disjoint edges from  $P_{4s-1}$ . So we complete the proof.

Using the similar method of proof in Lemma 9 and the equation (2), we obtain the following lemma.

**Lemma 10.** The graph  $\infty(p, 1, q)$  is defined as above, then

- (1)  $\eta(\infty(2s+1,1,4t)) = \eta(\infty(4s+1,1,4t+3)) = 1;$
- (2)  $\eta(\infty(2s+1,1,4t+2)) = \eta(\infty(4s+1,1,4t+1)) = 0.$

**Lemma 11** [3]. If a bipartite graph G with  $n \ge 1$  vertices does not contain any cycle of length 4s  $(s \ge 1)$ , then  $\eta(G) = n - 2\nu(G)$ .

In accordance with Lemma 11, it is easy to know for  $G \in \mathcal{B}_n^{++}$  is a bipartite graph with not containing cycle  $C_{4s}$   $(s \ge 1)$ , then  $\eta(G) = n - 2\nu(G)$ , so  $In(G) = (\nu(G), \nu(G), n - 2\nu(G))$ . Hence in the following, we discuss the case  $G \in \mathcal{B}_n^{++}$  is a bipartite graph with containing cycles  $C_{4s}$   $(s \ge 1)$ .

**Lemma 12.** If  $G \in \mathcal{B}_n^{++}$  is a bipartite graph with containing cycle  $C_{4s}$   $(s \ge 1)$ , then  $\eta(G) = n - 2\nu(G)$  or  $\eta(G) = n - 2\nu(G) + 2$ .

**Proof.** Putting to use the Lemma 5 a times, we can obtain the following cases:

- (1)  $T_i$   $(1 \le i \le s)$  are the components where  $T_i$   $(1 \le i \le s)$  are trees with  $n_i$  vertices. Then  $\eta(G) = \sum_{i=1}^s \eta(T_i) = \sum_{i=1}^s (n_i 2\nu(T_i)) = n a 2(\nu(G) a) = n 2\nu(G)$ .
- (2)  $U_0, T_i \ (1 \le i \le s)$  are the components where  $T_i \ (1 \le i \le s)$  are trees with  $n_i$  vertices and  $U_0$  is a unicyclic graph with  $n_0$  vertices. By Lemma 6, we know  $\eta(U_0) = n_0 2\nu(U_0)$  or  $n_0 2\nu(U_0) + 2$ , so  $\eta(G) = \eta(U_0) + \sum_{i=1}^s \eta(T_i) = n 2\nu(G)$  or  $n 2\nu(G) + 2$ .
- (3)  $\infty(p, 1, q), T_i \ (1 \le i \le s)$  are the components where  $T_i \ (1 \le i \le s)$  are trees with  $n_i$  vertices and  $\infty(p, 1, q)$  is a bicyclic graph with  $n_0$  vertices. By Lemma 9, we have  $\eta(\infty(4s, 1, 4t + 2)) = 1$  or  $\eta(\infty(4s, 1, 4t)) = 3$ . Then  $\eta(G) = \eta(\infty(p, 1, q)) + \sum_{i=1}^{s} \eta(T_i) = n - 2\nu(G)$  or  $n - 2\nu(G) + 2$ .

Combining Lemmas 10 and 12, we obtain the following theorem.

**Theorem 13.** If  $G \in \mathcal{B}_n^{++}$  is a bipartite graph, then  $\eta(G) = n - 2\nu(G)$  or  $\eta(G) = n - 2\nu(G) + 2$ .

**Lemma 14.** If  $G \in \mathcal{B}_n^{++}$  is a non-bipartite graph, then  $\eta(G) = n - 2\nu(G) - 1$ ,  $n - 2\nu(G)$ ,  $n - 2\nu(G) + 1$  or  $\eta(G) = n - 2\nu(G) + 2$ .

**Proof.** Putting to use the Lemma 5 b times, we can obtain the following cases:

- (1)  $T_i$   $(1 \le i \le s)$  are the components where  $T_i$   $(1 \le i \le s)$  are trees with  $n_i$  vertices. Then  $\eta(G) = \sum_{i=1}^s \eta(T_i) = n 2\nu(G)$ .
- (2)  $U_0, T_i$   $(1 \le i \le s)$  are the components where  $T_i$   $(1 \le i \le s)$  are trees with  $n_i$  vertices and  $U_0$  is a unicyclic graph with  $n_0$  vertices. By Lemma 6, we know  $\eta(U_0) = n_0 - 2\nu(U_0), n_0 - 2\nu(U_0) + 2$  or  $n_0 - 2\nu(U_0) - 1$ , so  $\eta(G) = \eta(U_0) + \sum_{i=1}^s \eta(T_i) = n - 2\nu(G), n - 2\nu(G) + 2$  or  $n - 2\nu(G) - 1$ .

(3)  $\infty(p,1,q), T_i$   $(1 \le i \le s)$  are the components where  $T_i$   $(1 \le i \le s)$  are trees with  $n_i$  vertices and  $\infty(p,1,q)$  is a bicyclic graph with  $n_0$  vertices. By Lemma 10, we have  $\eta(\infty(2t+1,1,4s)) = 1, \ \eta(\infty(2t+1,1,4s+2)) = 0, \ \eta(\infty(4s+1,1,4t+1)) = 0 \text{ or } \eta(\infty(4s+1,1,4t+3)) = 1.$  Then  $\eta(G) = \eta(\infty(p,1,q)) + \sum_{i=1}^{s} \eta(T_i) = n - 2\nu(G) + 1, \ n - 2\nu(G) \text{ or } n - 2\nu(G) - 1.$ 

Using the similar method of Lemma 7 and Lemma 14, we have the next lemma.

**Lemma 15.** If  $G \in \mathcal{B}_n^{++}$  is a non-bipartite graph, then  $In(G) = (\nu(G), \nu(G) + 1, n - 2\nu(G) - 1), (\nu(G) + 1, \nu(G), n - 2\nu(G) - 1), (\nu(G), n - 2\nu(G)), (\nu(G), \nu(G) - 1, n - 2\nu(G) + 1), (\nu(G) + 1, \nu(G) - 2, n - 2\nu(G) + 1), (\nu(G), \nu(G) - 2, n - 2\nu(G) + 2).$ 

So we obtain our main result.

**Theorem 16.** If  $G \in \mathcal{B}_n^{++}$ , then  $In(G) = (\nu(G), \nu(G)+1, n-2\nu(G)-1)$ ,  $(\nu(G)+1, \nu(G), n-2\nu(G)-1)$ ,  $(\nu(G), \nu(G), n-2\nu(G))$ ,  $(\nu(G), \nu(G)-1, n-2\nu(G)+1)$ ,  $(\nu(G)+1, \nu(G)-2, n-2\nu(G)+1)$ ,  $(\nu(G), \nu(G)-2, n-2\nu(G)+2)$ .

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