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THE INERTIA OF UNICYCLIC GRAPHS AND BICYCLIC GRAPHS

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Abstract

Let G be a graph with n vertices and $\nu(G)$ be the matching number of G. The inertia of a graph G, $In(G) = (n_+, n_-, n_0)$ is an integer triple specifying the numbers of positive, negative and zero eigenvalues of the adjacency matrix $A(G)$, respectively. Let $\eta(G) = n_0$ denote the nullity of G (the multiplicity of the eigenvalue zero of G). It is well known that if G is a tree, then $\eta(G) = n - 2\nu(G)$. Guo *et al.* [Ji-Ming Guo, Weigen Yan and Yeong-Nan Yeh. On the nullity and the matching number of unicyclic graphs, Linear Algebra and its Applications, 431 (2009), 1293–1301.] proved if G is a unicyclic graph, then $\eta(G)$ equals $n - 2\nu(G) - 1, n - 2\nu(G)$ or $n - 2\nu(G) + 2$. Barrett *et al.* determined the inertia sets for trees and graphs with cut vertices. In this paper, we give the nullity of bicyclic graphs $\mathcal{B}_n^{++}$. Furthermore, we determine the inertia set in unicyclic graphs and $\mathcal{B}_n^{++}$, respectively.

Keywords: matching number, inertia, nullity, unicyclic graph, bicyclic graph.

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1. Introduction

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G)$. The inertia of a graph G, $In(G) = (n_+, n_-, n_0)$ is an integer triple specifying the numbers of positive, negative and zero eigenvalues of the adjacency matrix $A(G)$, respectively. It is well known if G is a bipartite graph, then $n_+ = n_-$. Barrett, Hall, and Loewy [1] determined the inertia sets for trees and graphs with cut vertices. The nullity of G, denoted by $\eta = \eta(G) = n_0$, is the multiplicity of the number zero in the spectrum of G. Then $n_+ + n_- = n - r(A(G)) = \eta$. The nullity of graphs is of interest in chemistry since the occurrence of a zero eigenvalue of a bipartite graph (corresponding to an alternant hydrocarbon) indicates the chemical instability of the molecule which such a graph represents. The question is of interest also for non-alternant hydrocarbons (non-bipartite graph), but a direct connection with the chemical stability in these cases is not so straightforward. The nullity has been determined for trees, unicyclic graphs and bicyclic graphs, respectively [4, 5, 6]. Recently, Gutman and Borovićanin give a survey on the nullity of graphs.

A unicyclic graph is a simple connected graph with equal numbers of vertices and edges. For the sake of a convenient description, let $\mathcal{U}_n$ be the set of unicyclic graphs with n vertices. A bicyclic graph is a simple connected graph in which the number of edges equals the number of vertices plus one.

Let C_p and C_q be two vertex-disjoint cycles. Suppose that $v_1 \in C_p, v_l \in C_q$. Joining v_1 and v_l by a path $v_1 v_2 \cdots v_l$ of length $l-1$, where $l \geq 1$ and $l = 1$ means identifying v_1 with v_l, resultant graph, denoted by $\infty(p, l, q)$, is called an ∞-graph. Let P_{l+1}, P_{p+1} and P_{q+1} be three vertex-disjoint paths, where $\min\{p, l, q\} \geq 1$ and at most one of them is 1. Identifying the three initial vertices and terminal vertices of them, respectively, resultant graph, denoted by $\theta(p, l, q)$, is called a θ-graph (see Figure 1).

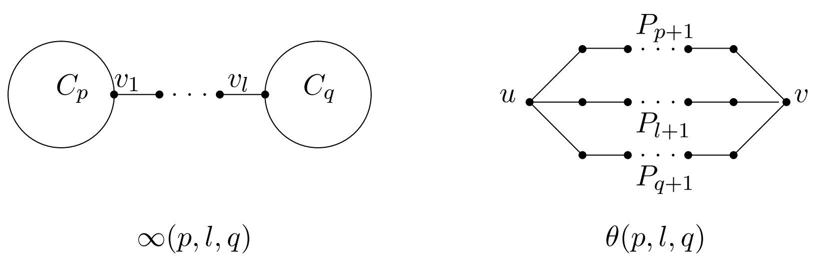

Figure 1

Let $\mathcal{B}_n$ be the set of all bicyclic graphs of order n. $\mathcal{B}_n$ consists of three types of graphs: the first type denoted by $\mathcal{B}_n^+$ is the set of those graphs each of which is an ∞-graph with trees attached when $l > 1$; the second type denoted by $\mathcal{B}_n^{++}$ is the set of those graphs each of which is an ∞-graph with trees attached when $l = 1$; the third type denoted by θ_n is the set of those graphs each of which is an θ-graph with trees attached.

In Section 3, we study the inertia in $\mathcal{U}_n$. In Section 4, we give the nullity and the inertia sets in $\mathcal{B}_n^{++}$, respectively.

2. Main Lemmas

A matching of G is a collection of independent edges of G. A maximum matching is a matching with the maximum possible number of independent edges. The size of a maximum matching of G, i.e., the maximum number of independent edges of G, is denoted by $\nu = \nu(G)$.

Denote by $\varphi(x) = \varphi_G(x)$ the characteristic polynomial of G. Let

$$\varphi(x) = |xI - A| = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n. \tag{1}$$

Then [2]

$$a_i = \sum_U (-1)^{p(U)} 2^{c(U)} \quad (i = 1, 2, \ldots, n), \tag{2}$$

where the sum is over all subgraphs U of G consisting of disjoint edges and cycles and having exactly i vertices (called "basic figures"). If U is such a subgraph, then $p(U)$ is the number of its components, of which $c(U)$ components are cycles.

Example 1. Let G is a bipartite graph, then G does not contain an odd cycle, so $a_{2i+1} = 0$ $(i \geq 1)$.

Example 2. Considering equation (1) with equation (2), it is easy to obtain $a_1 = 0$ and $a_2 = 2m$ (m is the number of edges of G). In the following, we calculate a_3 and a_4. The subgraphs U of G having exactly 3 vertices consist of only the cycle C_3. Suppose that n_Δ is the number of the cycles C_3 in G, then $a_3 = -2n_\Delta$. Let $n_\Box$ and $\nu_2(G)$be the number of the cycles C_4, and two mutually disjoint edges in G, respectively, then $a_4 = \nu_2(G) - 2n_\Box$.

Next, we introduce the well-known Cauchy's interlacing theorem in matrix theory.

Lemma 3 [2]**.** *Let A be symmetric and A' be one of its principal submatrices. Let $\lambda_1 \geq \cdots \geq \lambda_n$ and $\lambda'_1 \geq \cdots \geq \lambda'_m$ be the eigenvalues of A and A', respectively. Then the inequality $\lambda_i \geq \lambda'_i \geq \lambda_{n-m+i}$ holds for all $i = 1, 2, \ldots, m$.*

Applying the Cauchy's interlacing theorem to the adjacency matrix $A(G)$ of the graph G, we have the following corollary.

Corollary 4. *Let V_0 be the k-subset of $G = (V, E)$ with n vertices ($0 \leq k \leq n-1$), and $G - V_0$ be the subgraph induced by removing the vertices V_0 and their incident edges. Then $\lambda_i(G) \geq \lambda_i(G - V_0) \geq \lambda_{i+k}(G) \quad (1 \leq i \leq n-k)$.*

The next lemma is useful to the proof of our main results.

Lemma 5 [2]**.** *For a graph G containing a pendent vertex, if the induced subgraph H of G is obtained by deleting this vertex together with the vertex adjacent to it, then the relation $\eta(H) = \eta(G)$ holds.*

3. The inertia of unicyclic graphs

n this section, we determine the inertia in $\mathcal{U}_n$. In order to prove our result, the following lemma is necessary.

Lemma 6 [5]**.** *Suppose $G \in \mathcal{U}_n$ with the cycle C_l. Then*

(1) $\eta(G) = n - 2\nu(G) - 1$, *if* $\nu(G) = \frac{l-1}{2} + \nu(G - C_l)$;

(2) $\eta(G) = n - 2\nu(G) + 2$, *if G satisfies:* $\nu(G) = \frac{l}{2} + \nu(G - C_l)$, $l \equiv 0$ (*mod* 4) *and no maximum matching contains an edge incident to* C_l;

(3) $\eta(G) = n - 2\nu(G)$, *otherwise.*

If $G \in \mathcal{U}_n$ is a bipartite graph, we know $n_+ = n_-$ and $n_+ + n_- = n - \eta(G)$, then $In(G) = (\nu(G) - 1, \nu(G) - 1, n - 2\nu(G) + 2)$ or $In(G) = (\nu(G), \nu(G), n - 2\nu(G))$, So we only consider those graphs $G \in \mathcal{U}_n$ which are non-bipartite.

Lemma 7. *If $G \in \mathcal{U}_n$ is a non-bipartite graph, then $In(G) = (\nu(G) + 1, \nu(G), n - 2\nu(G) - 1), In(G) = (\nu(G), \nu(G) + 1, n - 2\nu(G) - 1)$ or $In(G) = (\nu(G), \nu(G), n - 2\nu(G))$.*

Proof. Since $G \in \mathcal{U}_n$ with the cycle C_l is a non-bipartite graph, then l is odd. Let $v_i \in V(C_l)$ and $d_i \geq 3$. Suppose that $T_1, \ldots, T_{d_i}$ are the components of $G - v_i$ where $d_i = d(v_i)$. Let $V(T_j) = n_j$ and $\nu_j = \nu(T_j)$ $(j = 1, \ldots, d_i)$, so we have $\sum_{j=1}^{d_i} n_j = n - 1$ and $\sum_{j=1}^{d_i} \nu_j = \nu(G)$ or $\nu(G) - 1$. And $In(T_j) = (\nu_j, \nu_j, n_j - 2\nu_j)$. We discuss two cases in the following.

(1) $\nu(G) = \frac{l-1}{2} + \nu(G - C_l)$, then $\eta(G) = n - 2\nu(G) - 1$ and $\sum_{j=1}^{d_i} \nu_j = \nu(G)$. We know $\eta(G - v_i) = \sum_{j=1}^{d_i} \eta(T_j) = n - 1 - 2\sum_{j=1}^{d_i} \nu_j = n - 2\nu(G) - 1$. Let $\lambda'_1, \ldots, \lambda'_{\nu(G)}, \underbrace{\lambda'_{\nu(G)+1}, \ldots, \lambda'_{n-1-\nu(G)}}_{n-2\nu(G)-1}, \lambda'_{n-\nu(G)}, \ldots, \lambda'_{n-1}$ be the eigenvalues of $G - v_i$ according to nondecreasing order. By Corollary 4, we have $\lambda_{n-\nu(G)+1}(G) \leq \lambda'_{n-\nu(G)} < 0$ and $\lambda_{\nu(G)}(G) \geq \lambda'_{\nu(G)} > 0$. So $In(G) = (\nu(G) + 1, \nu(G), n - 2\nu(G) - 1)$ or $In(G) = (\nu(G), \nu(G) + 1, n - 2\nu(G) - 1)$.

(2) $\nu(G) \neq \frac{l-1}{2} + \nu(G - C_l)$, then $\eta(G) = n - 2\nu(G)$ and $\sum_{j=1}^{d_i} \nu_j = \nu(G) - 1$. We know $\eta(G - v_i) = \sum_{j=1}^{d_i} \eta(T_j) = n - 1 - 2\sum_{j=1}^{d_i} \nu_j = n - 2\nu(G) + 1$. Let $\lambda'_1, \ldots, \lambda'_{\nu(G)}, \underbrace{\lambda'_{\nu(G)+1}, \ldots, \lambda'_{n-\nu(G)+1}}_{n-2\nu(G)+1}, \lambda'_{n-\nu(G)+2}, \ldots, \lambda'_{n-1}$ be the eigenvalues of $G - v_i$ according to nondecreasing order. By Corollary 4, we have $\lambda_{n-\nu(G)+2}(G) \leq \lambda'_{n-\nu(G)+1} < 0$ and $\lambda_{\nu(G)}(G) \geq \lambda'_{\nu(G)} > 0$. And $\eta(G) = n - 2\nu(G)$, so $In(G) = (\nu(G), \nu(G), n - 2\nu(G))$. ■

Basing on the above detailed account, we obtain the next theorem.

Theorem 8. *If $G \in \mathcal{U}_n$, then $In(G) = (\nu(G)-1, \nu(G)-1, n-2\nu(G)+2)$, $(\nu(G), \nu(G), n-2\nu(G))$, $(\nu(G)+1, \nu(G), n-2\nu(G)-1)$ or $(\nu(G), \nu(G)+1, n-2\nu(G)-1)$.*

4. THE INERTIA OF BICYCLIC GRAPHS

In this section, we only consider $\mathcal{B}_n^{++}$. For $G \in \mathcal{B}_n^{++}$, we give the nullity of G and determine the inertia of G according to $\nu(G)$, respectively.

Lemma 9. *The graph $\infty(p,1,q)$ is defined as above, then*

(1) $\eta(\infty(4s,1,4t+2)) = 1$ $(s,t \geq 1)$;

(2) $\eta(\infty(4s,1,4t)) = 3$ $(s,t \geq 1)$.

Proof. Let $\varphi_1(x) = |xI - A| = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{4s+4t}x + a_{4s+4t+1}$ and $\varphi_2(x) = |xI - B| = x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_{4s+4t-2}x + b_{4s+4t-1}$ be the polynomials of $\infty(4s,1,4t{+}2)$ and $\infty(4s,1,4t)$, respectively. Since $\infty(4s,1,4t{+}2)$ and $\infty(4s,1,4t)$ are bipartite graph, so by the equation (2), we have $a_{2i+1} = 0$ and $b_{2i+1} = 0$ for $i \geq 1$. First of all, we consider a_{4s+4t} using the equation (2), then $a_{4s+4t} = 2m_1(-1)^{2t+1} + 2m_2(-1)^{2s} + (2m_1 + 2m_2) \neq 0$, where m_1 is the number of methods picking up $2t$ disjoint edges from P_{4t+1} and m_2 is the number of methods picking up $2s-1$ disjoint edges from P_{4s-1}. So $\eta(\infty(4s,1,4t{+}2)) = 1$.

Next, we prove $b_{4s+4t-2} = 0$ and $b_{4s+4t-4} \neq 0$. Using the similar method as above, we have $b_{4s+4t-2} = 2m_1(-1)^{2t} + 2m_2(-1)^{2s} - (2m_1 + 2m_2) = 0$, where m_1 is the number of methods picking up $2t-1$ disjoint edges from P_{4t-1} and m_2 is the number of methods picking up $2s-1$ disjoint edges from P_{4s-1}. And $b_{4s+4t-4} \geq m_3 > 0$ where m_3 is the number of methods picking up $2t-1$ disjoint edges from P_{4t} and picking up $2s-1$ disjoint edges from P_{4s-1}. So we complete the proof. ■

Using the similar method of proof in Lemma 9 and the equation (2), we obtain the following lemma.

Lemma 10. *The graph $\infty(p,1,q)$ is defined as above, then*

(1) $\eta(\infty(2s+1,1,4t)) = \eta(\infty(4s+1,1,4t+3)) = 1$;

(2) $\eta(\infty(2s+1,1,4t+2)) = \eta(\infty(4s+1,1,4t+1)) = 0$.

Lemma 11 [3]**.** *If a bipartite graph G with $n \geq 1$ vertices does not contain any cycle of length $4s$ $(s \geq 1)$, then $\eta(G) = n - 2\nu(G)$.*

In accordance with Lemma 11, it is easy to know for $G \in \mathcal{B}_n^{++}$ is a bipartite graph with not containing cycle C_{4s} $(s \geq 1)$, then $\eta(G) = n - 2\nu(G)$, so $In(G) = (\nu(G), \nu(G), n - 2\nu(G))$. Hence in the following, we discuss the case $G \in \mathcal{B}_n^{++}$ is a bipartite graph with containing cycles C_{4s} $(s \geq 1)$.

Lemma 12. *If $G \in \mathcal{B}_n^{++}$ is a bipartite graph with containing cycle C_{4s} $(s \geq 1)$, then $\eta(G) = n - 2\nu(G)$ or $\eta(G) = n - 2\nu(G) + 2$.*

Proof. Putting to use the Lemma 5 a times, we can obtain the following cases:

(1) T_i $(1 \leq i \leq s)$ are the components where T_i $(1 \leq i \leq s)$ are trees with n_i vertices. Then $\eta(G) = \sum_{i=1}^{s} \eta(T_i) = \sum_{i=1}^{s}(n_i - 2\nu(T_i)) = n - a - 2(\nu(G) - a) = n - 2\nu(G)$.

(2) U_0, T_i $(1 \leq i \leq s)$ are the components where T_i $(1 \leq i \leq s)$ are trees with n_i vertices and U_0 is a unicyclic graph with n_0 vertices. By Lemma 6, we know $\eta(U_0) = n_0 - 2\nu(U_0)$ or $n_0 - 2\nu(U_0) + 2$, so $\eta(G) = \eta(U_0) + \sum_{i=1}^{s} \eta(T_i) = n - 2\nu(G)$ or $n - 2\nu(G) + 2$.

(3) $\infty(p, 1, q), T_i$ $(1 \leq i \leq s)$ are the components where T_i $(1 \leq i \leq s)$ are trees with n_i vertices and $\infty(p, 1, q)$ is a bicyclic graph with n_0 vertices. By Lemma 9, we have $\eta(\infty(4s, 1, 4t + 2)) = 1$ or $\eta(\infty(4s, 1, 4t)) = 3$. Then $\eta(G) = \eta(\infty(p, 1, q)) + \sum_{i=1}^{s} \eta(T_i) = n - 2\nu(G)$ or $n - 2\nu(G) + 2$. ■

Combining Lemmas 10 and 12, we obtain the following theorem.

Theorem 13. *If $G \in \mathcal{B}_n^{++}$ is a bipartite graph, then $\eta(G) = n - 2\nu(G)$ or $\eta(G) = n - 2\nu(G) + 2$.*

Lemma 14. *If $G \in \mathcal{B}_n^{++}$ is a non-bipartite graph, then $\eta(G) = n - 2\nu(G) - 1,\ n - 2\nu(G),\ n - 2\nu(G) + 1$ or $\eta(G) = n - 2\nu(G) + 2$.*

Proof. Putting to use the Lemma 5 b times, we can obtain the following cases:

(1) T_i $(1 \leq i \leq s)$ are the components where T_i $(1 \leq i \leq s)$ are trees with n_i vertices. Then $\eta(G) = \sum_{i=1}^{s} \eta(T_i) = n - 2\nu(G)$.

(2) U_0, T_i $(1 \leq i \leq s)$ are the components where T_i $(1 \leq i \leq s)$ are trees with n_i vertices and U_0 is a unicyclic graph with n_0 vertices. By Lemma 6, we know $\eta(U_0) = n_0 - 2\nu(U_0), n_0 - 2\nu(U_0) + 2$ or $n_0 - 2\nu(U_0) - 1$, so $\eta(G) = \eta(U_0) + \sum_{i=1}^{s} \eta(T_i) = n - 2\nu(G),\ n - 2\nu(G) + 2$ or $n - 2\nu(G) - 1$.

(3) $\infty(p,1,q), T_i$ $(1 \le i \le s)$ are the components where T_i $(1 \le i \le s)$ are trees with n_i vertices and $\infty(p,1,q)$ is a bicyclic graph with n_0 vertices. By Lemma 10, we have $\eta(\infty(2t+1,1,4s)) = 1$, $\eta(\infty(2t+1,1,4s+2)) = 0$, $\eta(\infty(4s+1,1,4t+1)) = 0$ or $\eta(\infty(4s+1,1,4t+3)) = 1$. Then $\eta(G) = \eta(\infty(p,1,q)) + \sum_{i=1}^{s} \eta(T_i) = n - 2\nu(G) + 1,\ n - 2\nu(G)$ or $n - 2\nu(G) - 1$. ■

Using the similar method of Lemma 7 and Lemma 14, we have the next lemma.

Lemma 15. *If $G \in \mathcal{B}_n^{++}$ is a non-bipartite graph, then* $In(G) = (\nu(G), \nu(G)+1, n-2\nu(G)-1),\ (\nu(G)+1, \nu(G), n-2\nu(G)-1),\ (\nu(G), \nu(G), n-2\nu(G)), (\nu(G), \nu(G)-1, n-2\nu(G)+1),\ (\nu(G)+1, \nu(G)-2, n-2\nu(G)+1),\ (\nu(G), \nu(G)-2, n-2\nu(G)+2)$.

So we obtain our main result.

Theorem 16. *If $G \in \mathcal{B}_n^{++}$, then* $In(G) = (\nu(G), \nu(G)+1, n-2\nu(G)-1),\ (\nu(G)+1, \nu(G), n-2\nu(G)-1),\ (\nu(G), \nu(G), n-2\nu(G)), (\nu(G), \nu(G)-1, n-2\nu(G)+1),$ $(\nu(G)+1,\ \nu(G)-2, n-2\nu(G)+1),\ (\nu(G), \nu(G)-2, n-2\nu(G)+2)$.

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