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BI-IDEALS IN CLIFFORD ORDERED SEMIGROUP

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Abstract

In this paper we characterize both the Clifford and left Clifford ordered semigroups by their bi-ideals and quasi-ideals. Also we characterize principal bi-ideal generated by an ordered idempotent in a completely regular ordered semigroup.

Keywords: Clifford (completely regular) ordered semigroup, ordered idempotents, bi-ideals, quasi-ideals..

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1. Introduction

In 1952, R.A. Good and D.R. Hughes [2] introduced the notion of bi-ideals of a semigroup. A special case of (m, n)-ideal was introduced by S. Lajos [11]. Quasi ideals which was introduced by Otto Steinfeld [18] are generalizations of both left ideals and right ideals as well as a particular case of bi-ideals. The characterizations of regular and its subclasses like completely regular semigroups, Clifford semigroups by quasi-ideals and bi-ideals was beautifully presented by S. Lajos [13, 14].

Kehayopulu [7] introduced the notion of bi-ideals and quasi-ideals in an ordered semigroup and she developed the theory of ordered regular, completely regular and t-simple ordered semigroups by their bi-ideals and quasi-ideals. Later, these types of characterizations in ordered semigroups have been studied by Lee, Kang [16] and others.

This paper is inspired by the works of S. Lajos [13, 14, 15]. Clifford (left Clifford)ordered semigroup is a subclass of ordered regular semigroup. The Clifford (left Clifford) ordered semigroup has been introduced in [1]. The purpose of this paper is to characterize the Clifford and left Clifford ordered semigroups by bi-ideals and quasi-ideals.

Our paper is organized as follows. The basic definitions and properties of ordered semigroups are presented in Section 2. Section 3 is devoted to characterizing the Clifford and left Clifford ordered semigroups by bi-ideals and quasi-ideals. In Section 4 we discuss the principal bi-ideal generated by an ordered idempotent element in a completely regular ordered semigroup.

2. Preliminary

Here we introduce necessary notations and collect a few auxiliary results. In this paper $\mathbb{N}$ will provide the set of all natural numbers. An ordered semigroup is a partially ordered set S, and at the same time a semigroup $(S, \cdot)$ such that $(\forall a,\ b,\ x \in S)\ a \leq b \Rightarrow\ xa \leq xb$ and $ax \leq bx$. It is denoted by $(S, \cdot, \leq)$. For an ordered semigroup S and $H \subseteq S$, denote

$$(H]_S := \{t \in S : t \leq h, \quad \text{for some} \quad h \in H\}.$$

Also it is denoted by $(H]$ if there is no scope of confusion.

Let I be a nonempty subset of an ordered semigroup S. I is a left (right) ideal of S, if $SI \subseteq I$ $(IS \subseteq I)$ and $(I] = I$. I is an ideal of S if it is both a left and a right ideal of S. S is left (right) simple if it has no non-trivial proper left (right) ideal. Similarly we define simple ordered semigroups. S is called t-simple ordered semigroup if it is both left and right simple. A nonempty subset B of S is bi-ideal of S if $BSB \subseteq B$ and $(B] = B$ [7], a nonempty subset Q of S is called quasi-ideal of S if $QS \cap SQ \subseteq Q$ and $(Q] = Q$ [7]. The principal [10] left ideal, right ideal, ideal and bi-ideal [7] generated by $a \in S$ are denoted by $L(a)$, $R(a)$, $I(a)$ and $B(a)$ respectively. It is easy to check that

$$L(a) = (a \cup Sa],\ R(a) = (a \cup aS],\ I(a) = (a \cup Sa \cup aS \cup SaS] \text{ and } B(a) = (a \cup aSa],$$

and if moreover a is ordered regular then $L(a) = (Sa]$, $R(a) = (aS]$, $I(a) = (Sa \cup aS \cup SaS]$ and $B(a) = (aSa]$. Kehayopulu [10] defined Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$ and $\mathcal{H}$ on an ordered semigroup S as follows:

$$a\mathcal{L}b \ \text{ if } \ L(a) = L(b),$$

$$a\mathcal{R}b \ \text{ if } \ R(a) = R(b),$$

$$a\mathcal{J}b \ \text{ if } \ I(a) = I(b), \ \text{ and } \ \mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

These four relations are equivalence relations on S.

In [1] A.K. Bhuniya and author have explored a natural analogy between groups and t-simple and also observed that t-simple ordered semigroups are ordered regular. In the context of that article we renamed the t-simple ordered semigroups which carries a natural importance as shown in that article.

An ordered semigroup S is called a group like ordered semigroup or GLO-semigroup if for all $a, b \in S$ there are $x, y \in S$ such that $a \leq xb$ and $a \leq by$ [1]. An ordered regular semigroup S is called a left group like ordered semigroup or LGLO- semigroup if for all $a, b \in S$ there are $x \in S$ such that $a \leq xb$. Similarly we define right group like ordered semigroups or RGLO-semigroups [1]. S is called an ordered regular (completely regular ordered) semigroup [6] if for every $a \in S,\ a \in (aSa]$ $(a \in (a^2Sa^2])$. An element $b \in S$ is an inverse of a if $a \leq aba$ and $b \leq bab$. We denote set of all ordered regular, right ordered regular, completely ordered regular elements in ordered semigroup S have denoted by $Reg_{\leq}(S)$ and $Gr_{\leq}(S)$ respectively. The set of all inverses of an element $a \in S$ is denoted by $V_{\leq}(a)$.

By an ordered idempotent [1] in an ordered semigroup S, we shall mean an element $e \in S$ such that $e \leq e^2$. The set of all ordered idempotents in S will denoted by $E_{\leq}(S)$.

For example consider the ordered semigroup $(\mathbb{N},\ \cdot, \leq)$. Then $(\mathbb{N},\ \cdot)$ is not regular as a semigroup but it is ordered regular, as for example $2 \leq 2 \cdot 2 \cdot 2$.

Again 1 is the only idempotent in the semigroup $(\mathbb{N},\ \cdot)$ where as each natural number n is an ordered idempotent.

In an ordered semigroup S, every left (right) ideal a quasi-ideal and every quasi-ideal is a bi-ideal. Keeping in mind that every t-simple ordered semigroup is a group like ordered semigroup, we restate the result of Kehayopulu [9].

Theorem 1 [9]**.** *An ordered semigroup S is a group like ordered semigroup if and only if it has no proper bi-ideal.*

In [1], we have introduced the notion of Clifford and left Clifford ordered semigroups and characterized their complete semilattice decomposition. For the convenience of general reader, we collect few auxiliary result of [1].

Theorem 2 [1]**.** *An ordered semigroup S is completely regular if and only if for all $a \in S$ there exists $a' \in V_{\leq}(a)$ such that $aa' \leq a'ua$ and $a'a \leq ava'$ for some $u, v \in S$.*

Lemma 3 [1]**.** *Let S be a completely regular ordered semigroup. Then every $\mathcal{H}$-class is an ordered subsemigroup. Moreover if H is an $\mathcal{H}$-class then for every $a \in H$ there is $h \in H$ such that $a \leq aha,\ a \leq a^2h$, and $a \leq ha^2$.*

Proof. This is fairly straightforward. $h = a'$ as in Theorem 2 serves our purpose. ∎

Definition [1]**.** Let S be an ordered regular ordered semigroup. Then S is called Clifford ordered semigroup if for all $a \in S$ and $e \in E_{\leq}(S)$ there are $u, v \in S$ such that

$$ae \leq eua \text{ and } ea \leq ave.$$

Theorem 4 [1]**.** *Let S be an ordered regular semigroup. Then S is Clifford if and only if for all $a, b \in S$ there is $x \in S$ such that $ab \leq bxa$.*

Theorem 5 [1]**.** *Let S be an ordered regular semigroup. Then the following conditions are equivalent:*

1. *S is a Clifford ordered semigroup;*
2. *$\mathcal{L} = \mathcal{R}$;*
3. *$(aS] = (Sa]$ for all $a \in S$;*
4. *every left ideal and right ideal are (two sided) ideal and $A \cap B = (AB]$ for any two of them;*
5. *$L \cap R = (LR]$ for every left ideal L and every right ideal R of S.*

Definition [1]**.** An ordered regular ordered semigroup S is called left Clifford ordered semigroup if for all $a \in S$, $(aS] \subseteq (Sa]$.

Theorem 6 [1]**.** *Let S be an ordered regular semigroup. Then the following conditions are equivalent:*

1. *S is a left Clifford ordered semigroup;*
2. *for all $e \in E_{\leq}(S)$, $(eS] \subseteq (Se]$;*
3. *for all $a \in S$, and $e \in E_{\leq}(S)$ there is $x \in S$ such that $ea \leq xe$;*
4. *for all $a, b \in S$ there is $x \in S$ such that $ab \leq xa$;*
5. *$\mathcal{R} \subseteq \mathcal{L}$.*

Theorem 7 [1]**.** *The following conditions are equivalent on an ordered regular semigroup S:*

1. *S is a left Clifford ordered semigroup;*
2. *every left ideal of S is two sided and $A \cap B = (AB]$ for any two ideals A, B of S;*
3. *$L_1 \cap L_2 = (L_1L_2]$ for any two left ideals L_1, L_2 of S.*

We now list some properties on ideals, quasi-ideals and bi-ideals in an arbitrary ordered semigroup.

Theorem 8. *Let S be an ordered semigroup. Then the following statements hold in S:*

1. *The intersection of quasi-ideals in S is a quasi-ideal in S.*
2. *Let L be a left and R be a right ideal of S, then $L \cap R$ is a quasi-ideal in S.*
3. *Let T be a two sided ideal in S, then each quasi-ideal Q of T is a bi-ideal of S.*

Proof. These are straightforward. ∎

Theorem 9 [9]**.** *Let S be ordered regular semigroup, and B a bi-ideal of S. Then $B = (BSB]$.*

In view of the above theorem we show that every bi-ideal is a quasi-ideal in an ordered regular semigroup in the following lemma.

Lemma 10. *Let S be an ordered regular semigroup. Then the following statements hold in S:*

1. *Every bi-ideal of S is a quasi-ideal.*
2. *For every quasi-ideal Q of S, $Q = (QSQ]$.*

Proof. (1) Suppose that B is a bi-ideal of S. Let $x \in BS \cap SB$. Then $x \leq b_1 s_1$ and $x \leq s_2 b_2$ for some $s_1, s_2 \in S$ and $b_1, b_2 \in B$. Since S is ordered regular, there is $z \in S$ such that $x \leq xzx$. This implies that $x \leq b_1 s_1 z s_2 b_2$. Then by Theorem 9, it follows that $x \in (BSB] = B$. Thus $BS \cap SB \subseteq B$ and hence B is a quasi-ideal of S.

(2) This follows from the conditions (1). Since every quasi-ideal of S is a bi-ideal of S, by Theorem 9 the statement follows immediately. ∎

3. Bi-ideals in Clifford and left Clifford ordered semigroup

Kehayopulu [9] showed that an ordered semigroup S is an t-simple ordered semigroup if and only if it has no proper bi-ideal. We now characterize the Clifford ordered semigroups by their bi-ideals and quasi-ideals. For this we need some results which we prove first.

Theorem 11. *Let S be a Clifford ordered semigroup. Then every bi-ideal in S is an ideal.*

Proof. (1) Suppose that B is a bi-ideal of S. Choose $b \in B$, and $s \in S$. Since S is a Clifford ordered semigroup there is $x \in S$ such that $bs \leq sxb$, by Theorem 4. By the ordered regularity of S we have that $b \leq bzb$ for some $z \in S$. Then

$bs \leq bzbs \leq bzsxb$, that is, $bs \in (BSB] = (B] = B$. Similarly $sb \in B$. This shows that B is an ideal of S.

(2) Consider a bi-ideal B of S. Let $b \in B$. Since S is ordered regular, there is $z \in S$ such that $b \leq bzb$ and so for any $s \in S$, $bs \leq bzbs$. As S is left Clifford by Theorem 6, it follows that $bs \leq bztb$ for some $t \in S$. Thus $bs \in (BSB] = B$ and hence B is an ideal of S. ■

Lemma 12. *In a Clifford ordered semigroup S, the following statements hold:*

1. *For a bi-ideal B of S, $B = (BS] = (SB]$.*
2. *For an ideal I and a quasi-ideal Q of S, $(IQ] = (IQI]$.*
3. *For an ideal I and a bi-ideal B of S, $(IB] = (IBI]$.*

Proof. (1) Since B is a bi-ideal of S and S is ordered regular semigroup, by Theorem 9, $B = (BSB]$. Also as B is an ideal of S, $(BS] \subseteq (B] = B$. Again $B = (BSB] \subseteq (BS]$. Thus $B = (BS]$. Similarly $B = (SB]$.

(2) Let I be an ideal and Q be a quasi-ideal of S. Since S is a Clifford ordered semigroup, Q is an ideal of S, by Theorem 11. So $(IQI] \subseteq (IQ]$. Next for any $z \in (IQ]$, $z \leq sa$ for $s \in I$ and $a \in Q$. From the ordered regularity of S, there is $x \in S$ such that $s \leq sxs$. Since S is a Clifford ordered semigroup, there is $t \in S$ such that $sa \leq ats$. Consequently, we have $s \leq sa \leq sxsa \leq sxats$. Since I is an ideal of S, sx, $tx \in I$. Hence $(IQI] = (IQ]$.

(3) This is obvious. ■

In the following theorem the Clifford ordered semigroups are characterized by their left ideals, quasi- ideals and bi- ideals.

Theorem 13. *In an ordered regular semigroup S, the following conditionsare equivalent:*

1. *S is a Clifford ordered semigroup;*
2. *$Q_1 \cap Q_2 = (Q_1Q_2]$ for every pair of quasi-ideals Q_1 and Q_2 of S;*
3. *$B_1 \cap B_2 = (B_1B_2]$ for every pair of bi-ideals B_1 and B_2 of S;*
4. *$Q \cap B = (QB]$ for every quasi ideal Q and every bi-ideal B of S;*
5. *$L \cap Q = (LQ]$ for every left ideal L and every quasi-ideal Q of S;*
6. *$L \cap B = (LB]$ for every left ideal L and every bi-ideal B of S.*

Proof. (1) ⇒ (2): Suppose that S is a Clifford ordered semigroup. Let Q_1 and Q_2 are two quasi ideals of S. Consider $x \in (Q_1Q_2]$. Then $x \leq ab$ for some $a \in Q_1$ and $b \in Q_2$. Since S is ordered regular there is $y \in S$ such that $x \leq abyab$. Also since S is a Clifford semigroup, by Theorem 4, $ab \leq bza$ for some $z \in S$, from which it follows that $x \leq ab \leq abybza$, that is, $x \in (Q_1SQ_2]$. Therefore by Lemma 12, $x \in Q_1$. Similarly $x \in Q_2$. Thus $x \in Q_1 \cap Q_2$ and hence $(Q_1Q_2] \subseteq Q_1 \cap Q_2$. For the converse part, let $s \in Q_1 \cap Q_2$. Since S is ordered regular, $s \leq ststs$ for some $t \in S$. Now as S is Clifford ordered semigroup, by Theorem 4, $ts \leq sut$ for some $u \in S$. This implies that $s \leq s(sut)ts$. It is known that every quasi-ideal is a bi-ideal. Thus $(sut)ts \in Q_2SQ_2 \subseteq Q_2$. Also $s \in Q_1$. Thus $s \in (Q_1Q_2]$ and so $Q_1 \cap Q_2 \subseteq (Q_1Q_2]$. Hence $Q_1 \cap Q_2 = (Q_1Q_2]$.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4): By Lemma 10, a simple argument shows that statements hold.

(4) $\Rightarrow$ (5): It follows from the fact that every left ideal is a quasi-ideal.

(5) $\Rightarrow$ (6): This follows from the Lemma 10.

(6) $\Rightarrow$ (1): Let L and R be left and right ideal of S respectively. Now every right ideal is a quasi-ideal and hence a bi-ideal. Thus R is a bi-ideal. So by the given condition it follows that $L \cap R = (LR]$. Accordingly, S is a Clifford ordered semigroup, by Theorem 5. ∎

In the following theorem Clifford ordered semigroup is characterized by their ideals, quasi-ideals and bi-ideals.

Theorem 14. *In an ordered regular semigroup S, the following conditions are equivalent:*

1. *S is a Clifford ordered semigroup;*
2. *$I \cap Q = (IQI]$ for every ideal I and every quasi ideal Q of S;*
3. *$I \cap B = (IBI]$ for every ideal I and every bi-ideal B of S.*

Proof. (1) $\Rightarrow$ (2): Let I be an ideal and Q a quasi ideal of S. Since S is an Clifford ordered semigroup, Q is also an ideal of S, by Theorem 11. Then $I \cap Q = (IQ]$, by Theorem 5. Hence by Lemma 12, $I \cap Q = (IQ] = (IQI]$.

(2) $\Rightarrow$ (3) This follows trivially.

(3) $\Rightarrow$ (1): Consider a bi ideal B of S. Using the given condition we have $B = S \cap B = (SBS]$. Now $BS = (SBS]S \subseteq (S(BSB]S]S \subseteq ((SBSBS]]S \subseteq (SBSBS]S \subseteq (SBS]S \subseteq (SBS] = B$. Thus $BS \subseteq B$, similarly $SB \subseteq B$. This conclude that any bi ideal in an ordered semigroup that satisfies the given condition is also a two sided ideal.

Let B_1 and B_2 be two bi-ideals of S. Treating B_1 as an ideal of S, we have $B_1 \cap B_2 = (B_1B_2B_1]$. On the other hand, treating both B_1 *and* B_2 as two sided ideals of S, it follows that $(B_1B_2B_1] \subseteq (B_1B_2]$. Hence $B_1B_2 = (B_1B_2B_1] = B_1 \cap B_2$. Therefore S is an Clifford ordered semigroup, by Theorem 13. ∎

We now characterize the left Clifford ordered semigroups by their bi-ideals and quasi-ideals.

Theorem 15. *Let S be a left Clifford ordered semigroup. Then every bi-ideal in S is a right ideal.*

Proof. Consider a bi-ideal B of S. Let $b \in B$. Since S is ordered regular, there is $z \in S$ such that $b \leq bzb$ and so for any $s \in S$, $bs \leq bzbs$. Also, as S is left Clifford ordered semigroup, by Theorem 6, $bs \leq bztb$ for some $t \in S$. Thus $bs \in (BSB] = B$ and hence B is a right ideal of S. ■

Theorem 16. *Let S be an ordered regular semigroup. Then the following conditions are equivalent:*

1. *S is a left Clifford ordered semigroup;*
2. *$Q \cap L = (QL]$ for every quasi-ideal Q and every left ideal L of S;*
3. *$B \cap L = (BL]$ for every bi-ideal B and every left ideal L of S.*

Proof. (1) $\Rightarrow$ (2): Suppose that S is a left Clifford ordered semigroup. Let $x \in Q \cap L$. Since S is an ordered regular semigroup, there is $y \in S$ such that $x \leq xyx$. Then from the fact that $x \in Q$ and $yx \in L$ it is evident that $x \in (QL]$.

On the other hand, since L is a left ideal of S, it is obvious that $(QL] \subseteq (L] = L$. To show that $(QL] \subseteq Q$, let us consider $s \in (QL]$. Then for some $a \in Q$ and $b \in L$, $s \leq ab$. This together with the ordered regularity of S yields that $s \leq azab$ for some $z \in S$. Thus $s \in (aSab]$. Since S is left Clifford, $s \in (aSa]$, by Theorem 6. Consequently, $a \in (QSQ]$. Also by Lemma 10, $(QSQ] = Q$. This shows that $s \in Q$ and so $s \in L \cap Q$. Hence by the given condition we have $Q \cap L = (QL]$.

(2) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (1): Consider two left ideals L_1 and L_2 of S. Since every left ideal is itself a bi-ideal of S, using the given condition we have $L_1 \cap L_2 = (L_1L_2]$. So by Theorem 6, S is a left Clifford ordered semigroup. ■

In the previous theorem we have characterized a left Clifford ordered semigroup by their left ideals. An alternative characterization of these ordered semigroups by their left ideals, ideals, bi-ideals and quasi-ideals are given in the following results.

Lemma 17. *In a left Clifford ordered semigroup S, the following statements hold:*

1. *For a quasi-ideal Q and a left ideal L of S, $(QLQ] = (QL]$.*
2. *For a bi-ideal B and a left ideal L of S, $(BLB] = (BL]$..*

Proof. (1) Let $a_1, a_2 \in Q$ and $b \in L$. Since S is left Clifford ordered semigroup there is some $x \in S$ such that $a_1ba_2 \leq a_1xb$, by Theorem 6. Since $xb \in L$, $a_1ba_2 \in (QL]$. Therefore $(QLQ] \subseteq (QL]$.

To show the converse part, let us assume that $y \in (QL]$. By Theorem 16, $y \in Q \cap L$. Since S is ordered regular, there is $z \in S$ such that $y \leq yzyzy$. Since

S is left Clifford ordered semigroup, there is $t \in S$ such that $yz \leq ty$. Hence $y \leq yztyy$. Also, Q is bi-ideal of S so by Theorem 15, Q is a right ideal of S. So $yzt \in Q$. Thus $y \in (QLQ]$ and therefore $(QLQ] = (QL]$.

(2) This is obvious. ■

As an immediate consequence of the previous theorem we have the following theorem.

Theorem 18. *Let S be an ordered regular semigroup. Then the following conditions are equivalent:*

1. *S is a left Clifford ordered semigroup;*
2. *$Q \cap L = (QLQ]$ for every quasi-ideal Q and every left ideal L of S;*
3. *$B \cap L = (BLB]$ for every bi-ideal B and every left ideal L of S.*

Proof. (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3): By Lemma 17 and Theorem 16, a simple argument shows the statements hold.

(3) $\Rightarrow$ (1): Let L be a left ideal of S. Then $L = L \cap S = (SLS]$, which implies that $LS = (SLS]S \subseteq (SLS] = L$. Therefore L is also a right ideal of S.

Let L_1, L_2 be two left ideals of S. Then L_1 is quasi-ideal of S. So by using the given condition we have $L_1 \cap L_2 = (L_1L_2L_1]$. Since L_2 is a right ideal of S, $L_1 \cap L_2 \subseteq (L_1L_2]$. Also, it is obvious that $(L_1L_2] \subseteq L_1 \cap L_2$. Hence $(L_1L_2] = L_1 \cap L_2$. Therefore S is a left Clifford ordered semigroup, by Theorem 7. ■

The following theorem is an application of the previous theorem. It requires only routine verification and so its proof is omitted.

Theorem 19. *Let S be a ordered regular semigroup. Then the following conditions are equivalent:*

1. *S is a left Clifford ordered semigroup;*
2. *$Q \cap I = (QI]$ for every quasi-ideal Q and every ideal I of S;*
3. *$B \cap I = (BI]$ for every bi-ideal B and every ideal I of S.*

4. BI-IDEALS IN AN ORDERED SEMIGROUP HAVING ORDERED IDEMPOTENT ELEMENTS:

In [1], A.K. Bhuniya and author have introduced the notion of ordered idempotents and studied the ordered semigroups in which every element is an ordered idempotent. In this section we characterize completely regular ordered semigroups, Clifford ordered semigroups by their bi-ideals generated by ordered idempotents.

Let a be an ordered regular element, then there is $x \in S$ such that $a \leq axa$. This yields that $ax \leq (ax)^2$ and $xa \leq (xa)^2$. Then $xa, ax \in E \leq (S)$, that is, $E_{\leq}(S) \neq \phi$. We shall now focus our attention primarily on some properties of left(right) ideals generated by ordered idempotent elements in an ordered regular semigroup.

Lemma 20. *Let L be a left and R be a right ideal of an ordered regular semigroup S. Then for $e, f \in E_{\leq}(S)$:*

1. $L \cap (eS] = (eL]$.
2. $R \cap (Se] = (Re]$.
3. $(Sf] \cap (eS] = (eSf]$.

Proof. (1) We have $(eL] \subseteq L$, since L is a left ideal of S. Also $(eL] \subseteq (eS]$. Thus $(eL] \subseteq L \cap (eS]$.

Conversely, suppose that $y \in L \cap (eS]$. Then there is $s \in S$ such that $y \leq es$. Since S is ordered regular, there is $z \in S$ such that $y \leq yzy \leq ezy$. Since L is left ideal of S, $szy \in L$ and hence $L \cap (eS] \subseteq (eL]$. Thus $(eL] = L \cap (eS]$.

(2) This is similar.

(3) We have $(eSf] \subseteq (Sf] \cap (eS]$. On the other hand, consider $z \in (Sf] \cap (eS]$. Then there are $s, t \in S$ such that $x \leq sf$ and $x \leq et$. By the ordered regularity of S, there is $w \in S$ such that $x \leq xwx \leq etwsf$. Thus $(Sf] \cap (eS] = (eSf]$. ■

As an immediate consequence of the above lemma we have the following theorem.

Theorem 21. *Let S be an ordered semigroup and L, R the left and right ideals of S respectively. Then for $e, f \in E_{\leq}(S)$ the subsets $(eL]$, $(Re]$ and $(eSf]$ are quasi-ideals of S.*

In the next theorem we shall characterize a completely regular ordered semigroup in terms of bi-ideal.

Theorem 22. *An ordered semigroup S is a completely regular ordered semigroup if and only if for every $a \in S$ there is some $e \in E_{\leq}(S)$ such that $a \leq ae$, $a \leq ea$ and $B(a) = B(e)$.*

Proof. First suppose that S is completely regular ordered semigroup. Let $a \in S$. Then by Theorem 2, there is $a' \in V_{\leq}(a)$ such that $a \leq aa'a$, $aa' \leq a'ua$ *and* $a'a \leq ava'$ for some $u, v \in S$. Take $e = aa'$. Then $e \in E_{\leq}(S)$ and $B(e) = (aa'Saa'] \subseteq (aa'Sa'ua] = B(a)$. Again from Lemma 3, for such $a' \in V_{\leq}(a)$ we have $a \leq aa'a$, $a \leq a^2a'$ and $a \leq a'a^2$. Then $B(a) = (aSa] \subseteq (aa'aSa^2a'] \subseteq (aa'Saa'] = B(e)$. Also $a \leq (aa')a$ and $a \leq a^2a' \leq a(aa') \leq ae$, by Lemma 3.

Conversely, suppose that the conditions hold in S. Then for an arbitrary $a \in S$ there is $e \in E_{\leq}(S)$ be such that $a \leq ae$, $a \leq ea$ *and* $B(a) = B(e)$. Then $B(a) = B(e) = (eSe]$. This implies that $a \leq ese$. Also for $e \in B(a) = (a \cup aSa]$, $e \leq a$ *or* $e \leq ata$.

If $e \leq a$. Then $a \leq ese \leq e^2se^2 \leq a^2sa^2$.

If $e \leq ata$. Then $a \leq ae$ and $a \leq ea$ implies that $a \leq a^2ta$ and $a \leq ata^2$, and finally $a \leq a^2tata^2$. Thus in either case S is completely regular ordered semigroup. ■

Every Clifford ordered semigroup is a completely regular ordered semigroup. So from the above theorem we have the following theorem. It requires only routine verification and so its proof is omitted.

Theorem 23. *Let S be an Clifford ordered semigroup. Then for every $a \in S$, there is an ordered idempotent e in S satisfying $a \leq ae$ and $a \leq ea$, and $B(a) = B(e)$.*

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References

[1] A.K. Bhuniya and K. Hansda, *Complete semilattice of ordered semigroups*, communicated.

[2] R.A. Good and D.R. Hughes, *Associated groups for a semigroup*, Bull. Amer. Math. Soc. **58** (1952) 624–625.

[3] U. Hebisch and H.J. Weinert, Semirings: Algebraic Theory and Applications in Computer Science, World Scientific, Singapore, 1998.

[4] J.M. Howie, Fundamentals of Semigroup Theory, Clarendon Press, Oxford, 1995.

[5] A. Iampan, *On bi-ideals of semigroups*, Lobachevskii J. Math. **29** (2) (2008) 68–72. doi:10.1134/S1995080208020042

[6] N. Kehayopulu, *Note on Green's relation in ordered semigroup*, Math. Japonica **36** (1991) 211–214.

[7] N. Kehayopulu, *On completely regular poe-semigroups*, Math. Japonica **37** (1992) 123–130.

[8] N. Kehayopulu, *On regular duo ordered semigroups*, Math. Japonica **37** (1992) 535–540.

[9] N. Kehayopulu, J.S. Ponizovskii and M. Tsingelis, *Bi-ideals in ordered semigroups and ordered groups*, J. Math. Sciences **112** (4) (2002). ISSN: 1573–8795

[10] N. Kehayopulu, *Ideals and Green's relations in ordered semigroups*, International Journal of Mathematics and Mathematical Sciences (2006), 1–8, Article ID 61286. doi:10.1155/IJMMS/2006/61286

[11] S. Lajos, *On (m, n)-ideals of semigroups*, Abstract of Second Hunger. Math. Congress I (1960) 42–44.

[12] S. Lajos, *On the bi-ideals in Semigroups*, Proc. Japan Acad. **45** (8) (1969) 710–712. doi:10.3792/pja/1195520625

[13] S. Lajos, *Notes on semilattices of groups*, Proc. Japan Acad. **46** (2) (1970) 151–152. doi:10.3792/pja/1195520460

[14] S. Lajos, *A characterization of Cliffordian semigroups*, Proc. Japan Acad. **52** (9) (1976) 496–497. doi:10.3792/pja/1195518214

[15] S. Lajos, *Some characterization of regular duo semigroup*, Proc. Japan Acad. **46** (10) (1970) 1099–1101. doi:10.3792/pja/1195526501

[16] S. Lee and S. Kang, *Characterization of regular po- semigroup*, Commu. Korean Math. Soc. **14** (4) (1999) 687–691.

[17] M.M. Miccoli, *Bi-ideals in regular semigroups and orthogroups*, Acta. Math. Hung. **47** (1-2) (1986) 3–6. doi:10.1007/BF01949118

[18] O. Steinfeld, *Quasi-ideals in rings and regular semigroups*, Akademiai Ki- ado, Budapest (1958). doi:10.1007/BF02572529

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