Discussiones Mathematicae
General Algebra and Applications 33 (2013) 13–47
doi:10.7151/dmgaa.1194

THE EXOCENTER AND TYPE DECOMPOSITION OF A GENERALIZED PSEUDOEFFECT ALGEBRA

David J. Foulis

Department of Mathematics an Statistics
Univ. of Massachusetts, Amherst, MA, USA

e-mail: foulis@math.umass.edu

Silvia Pulmannová

Mathematical Institute, Slovak Academy of Sciences
Štefánikova 49, 814 73 Bratislava, Slovakia

e-mail: pulmann@mat.savba.sk

and

Elena Vinceková

Mathematical Institute, Slovak Academy of Sciences
Štefánikova 49, 814 73 Bratislava, Slovakia

e-mail: vincekova@mat.savba.sk

Abstract

We extend the notion of the exocenter of a generalized effect algebra (GEA) to a generalized pseudoeffect algebra (GPEA) and show that elements of the exocenter are in one-to-one correspondence with direct decompositions of the GPEA; thus the exocenter is a generalization of the center of a pseudoeffect algebra (PEA). The exocenter forms a boolean algebra and the central elements of the GPEA correspond to elements of a sublattice of the exocenter which forms a generalized boolean algebra. We extend the notion of central orthocompleteness to GPEA, prove that the exocenter of a centrally orthocomplete GPEA (COGPEA) is a complete boolean algebra and show that the sublattice corresponding to the center is a complete boolean subalgebra. We also show that in a COGPEA, every element admits an exocentral cover and that the family of all exocentral covers, the so-called exocentral cover system, has the properties of a hull system on a generalized effect algebra. We extend the notion of type determining (TD) sets, originally introduced for effect algebras and then extended to GEAs and PEAs, to GPEAs, and prove a type-decomposition theorem, analogous to the type decomposition of von Neumann algebras.

Keywords: pseudoeffect algebra, generalized pseudoeffect algebra, center, exocenter, central orthocompleteness, type determining set, type decomposition.

2010 Mathematics Subject Classification: Primary: 81P10, 08A55, Secondary: 03G12.

1. Introduction

Our purpose in this article is to define and study extensions to generalized pseudoeffect algebras of the notions of the center, central orthocompleteness, central cover, type determining sets and type decompositions for an effect algebra, resp. for a pseudoeffect algebra (see [10, 11, 13, 14, 16, 18]).

Effect algebras (EAs) [9] were originally introduced as a basis for the representation of quantum measurements [1], especially those that involve fuzziness or unsharpness. Special kinds of effect algebras include orthoalgebras, MV-algebras, Heyting MV-algebras, orthomodular posets, orthomodular lattices, and boolean algebras. An account of the axiomatic approach to quantum mechanics employing EAs can be found in [4].

Several authors have studied or employed algebraic structures that, roughly speaking, are EAs "without a largest element." These studies go back to M.H. Stone's work [34] on generalized boolean algebras; later M.F. Janowitz [21] extended Stone's work to generalized orthomodular lattices. More recent developments along these lines include [9, 20, 24, 25, 28, 30, 33, 35].

The notion of a (possibly) non-commutative effect algebra, called a pseudoeffect algebra, was introduced and studied in [5, 6, 3]. Whereas a prototypic example of an effect algebra is the order interval from 0 to a positive element in a partially ordered abelian group, an analogous interval in a partially ordered non-commutative group is a prototype of a pseudoeffect algebra. Pseudoeffect algebras "without a largest element", called generalized pseudoeffect algebras, also have been studied in the literature [7, 8, 31, 36].

The classic decomposition of a von Neumann algebra as a direct sum of subalgebras of types I, II and III [29], which plays an important role in the theory of von Neumann algebras, is reflected by a direct sum decomposition of the complete orthomodular lattice (OML) of its projections. The type-decomposition for a von Neumann algebra is dependent on the von Neumann-Murray dimension theory, and likewise the early type-decomposition theorems for OMLs were based on the dimension theories of L. Loomis [26] and of S. Maeda [27]. Decompositions of complete OMLs into direct summands with various special properties were obtained in [2, 23, 32] without explicitly employing lattice dimension theory. More recent and considerably more general results on type-decompositions based on dimension theory can be found in [17]. Dimension theory for effect algebras was developed in [12].

As a continuation of the aforementioned work, the theory of so called type determining sets was introduced and applied, first to obtain direct decompositions for centrally orthocomplete effect algebras [10, 11], and later for centrally orthocomplete pseudoeffect algebras [16]. While direct decompositions of effect algebras and pseudoeffect algebras are completely described by their central elements [3, 18], for the generalized structures without a top element, we need to replace the center by the so called exocenter, which is composed of special endomorphisms, resp. ideals [13, 22].

The present paper is organized as follows. In Section 2, we introduce basic definitions and facts concerning generalized pseudoeffect algebras (GPEAs). In Section 3 we introduce the notion of the exocenter of a GPEA and study its properties. Section 4 is devoted to central elements in a GPEA and relations between the center and the exocenter. The notion of central orthocompleteness is extended to GPEAs in Section 5 where it is shown that the center of a centrally orthocomplete GPEA (COGPEA) is a complete boolean algebra. In Section 6 we introduce the exocentral cover, which extends the notion of a central cover for an EA. In Section 7, we develop the theory of type determining sets for GPEAs and show some examples. Finally, in Section 8, we develop the theory of type decompositions of COGPEAs into direct summands of various types. We note that COGPEAs are, up to now, the most general algebraic structures for which the theory of type determining sets has been applied to obtain direct decompositions.

2. Generalized pseudoeffect algebras

We abbreviate 'if and only if' as 'iff' and the notation := means 'equals by definition'.

Definition 2.1. A *generalized pseudoeffect algebra* (GPEA) is a partial algebraic structure $(E, \oplus, 0)$, where $\oplus$ is a partial binary operation on E called the *orthosummation*, 0 is a constant in E called the *zero element*, and the following conditions hold for all $a, b, c \in E$:

- (GPEA1) (*associativity*) $(a \oplus b)$ and $(a \oplus b) \oplus c$ exist iff $b \oplus c$ and $a \oplus (b \oplus c)$ exist and in this case $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
- (GPEA2) (*conjugacy*) If $a \oplus b$ exists, then there are elements $d, e \in E$ such that $a \oplus b = d \oplus a = b \oplus e$.

(GPEA3) (*cancellation*) If $a \oplus b = a \oplus c$, or $b \oplus a = c \oplus a$, then $b = c$.

(GPEA4) (*positivity*) If $a \oplus b = 0$, then $a = b = 0$.

(GPEA5) (*zero element*) $a \oplus 0$ and $0 \oplus a$ always exist and are both equal to a.

As a consequence of (GPEA3), the elements d and e in (GPEA2) are uniquely determined by a and b. Following the usual convention, we often refer to a GPEA $(E, \oplus, 0)$ simply as E.

If E and F are GPEAs, then a mapping $\phi: E \to F$ is a *GPEA-morphism* iff, for all $a, b \in E$, if $a \oplus b$ exists in E, then $\phi(a) \oplus \phi(b)$ exists in F and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. If $\phi: E \to F$ is a bijective GPEA-morphism and $\phi^{-1}: F \to E$ is also a GPEA-morphism, then ϕ is a *GPEA-isomorphism*.

Standing Assumption:
In what follows, $(E, \oplus, 0)$ is a generalized pseudoeffect algebra. In general, lower case Latin letters $a, b, c, \ldots, x, y, z$, with or without subscripts, will denote elements of E. If we write an equation involving an orthosum, e.g. $x \oplus y = z$, we tacitly assume its existence.

Definition 2.2. The relation $\leq$ is defined on the GPEA E by

$$a \leq b \text{ iff } a \oplus x = b \text{ for some } x \in E$$

or equivalently (in view of (GPEA2)), by

$$a \leq b \text{ iff } y \oplus a = b \text{ for some } y \in E.$$

If $a \leq b$, then by (GPEA3) the elements x and y such that $a \oplus x = y \oplus a = b$ are uniquely determined by a and b, and we define the (left and right) differences

$$a/b := x \text{ and } b \backslash a := y.$$

In the event that $a \leq b$ and a/b coincides with $b \backslash a$, we also define

$$b \ominus a := a/b = b \backslash a.$$

We say that elements p and q in E are *orthogonal*, in symbols $p \perp q$, iff $p \oplus q$ and $q \oplus p$ both exist and are equal. The GPEA E is *commutative* iff $p \perp q$ holds whenever $p \oplus q$ is defined.

Evidently, if either a/b or $b \backslash a$ exists, then both exist and $a \leq b$; conversely, if $a \leq b$, then both a/b and $b \backslash a$ exist and $b = a \oplus (a/b) = (b \backslash a) \oplus a$. Also, if $b \ominus a$ exists, then $a, b \ominus a \leq b$, $a \perp (b \ominus a)$ and $a \oplus (b \ominus a) = (b \ominus a) \oplus a = b$. We note that a commutative GPEA is the same thing as a *generalized effect algebra* [33].

The GPEA E is partially ordered by $\leq$ and 0 is the smallest element in E. The cancellation laws in (GPA3) are easily extended to $\leq$ as follows:

$$\text{If } a \oplus b \leq a \oplus c, \text{or if } b \oplus a \leq c \oplus a, \text{ then } b \leq c.$$

An existing supremum (resp. infimum) in the partially ordered set (poset) E of elements a and b is denoted by $a \vee b$ (resp. by $a \wedge b$). We say that a and b are *disjoint* iff $a \wedge b = 0$. We note that a GPEA-morphism preserves inequalities and corresponding left and right differences.

An important example of a GPEA ([7], Example 2.3) is a subset of the positive cone in a partially ordered group (po-group). Let $(G, +, 0, \leq)$ be a po-group with $G^+ := \{g \in G : 0 \leq g\}$. Let G^0 be a nonempty subset of G^+ such that for all $a, b \in G^0$, if $b \leq a$ then $-a + b,\ b - a \in G^0$. Then $(G^0, \oplus, 0)$, where $\oplus$ is the group addition restricted to those pairs of elements whose sum is again in G^0, is a GPEA whose partial order coincides with the group partial order restricted to G^0.

Lemma 2.3. *Let $a, b, c, d \in E$ with $a \leq b$. Then:*

(i) *$b\backslash a,\ a/b \leq b$ and $(b\backslash a)/b = b\backslash(a/b) = a$.*

(ii) *$d \leq a/b \Leftrightarrow a \oplus d \leq b \Leftrightarrow d \leq b$ and $a \leq b\backslash d$.*

(iii) *If $b \oplus d$ exists, then $a/(b \oplus d) = (a/b) \oplus d$, $a \oplus d$ exists, and $a \oplus d \leq b \oplus d$. Also, if $d \oplus b$ exists, then $(d \oplus b)\backslash a = d \oplus (b\backslash a)$, $d \oplus a$ exists, and $d \oplus a \leq b \oplus a$.*

(iv) *If $a \leq b \leq c$, then $a/c = a/b \oplus b/c$ and $c\backslash a = c\backslash b \oplus b\backslash a$.*

Proof. (i) As $b = b\backslash a \oplus a$, we get $(b\backslash a)/b = a$, and $b = a \oplus a/b$ implies $b\backslash(a/b) = a$.

(ii) If $d \leq a/b$, then $\exists x \in E$ with $d \oplus x = a/b$, so $(a \oplus d) \oplus x = a \oplus (d \oplus x) = b$, and therefore $a \oplus d \leq b$. If $a \oplus d \leq b$, then $\exists y \in E$, $y \oplus (a \oplus d) = (y \oplus a) \oplus d = b$, whence $d \leq b$, $y \oplus a = b\backslash d$, and $a \leq b\backslash d$. Thus $d \leq a/b \Rightarrow a \oplus d \leq b \Rightarrow d \leq b$ and $a \leq b\backslash d$. Proofs of the converse implications are straightforward.

(iii) Assume that $b \oplus d$ exists. Then $a \leq b \leq b \oplus d$ and $a \oplus a/(b \oplus d) = b \oplus d = (a \oplus a/b) \oplus d = a \oplus ((a/b) \oplus d)$, whence $a/(b \oplus d) = (a/b) \oplus d$ by cancellation. Also, as $a \leq b$, we have $b\backslash a \oplus a = b$, whence $b \oplus d = (b\backslash a \oplus a) \oplus d = b\backslash a \oplus (a \oplus d)$, whence $a \oplus d$ exists and $a \oplus d \leq b \oplus d$. The remaining assertion is proved analogously.

(iv) As $a \leq b \leq c$, we have $a \oplus (a/b \oplus b/c) = (a \oplus a/b) \oplus b/c = b \oplus b/c = c = a \oplus a/c$, whence $a/b \oplus b/c = a/c$ by cancellation. The second equality is proved similarly. ■

Lemma 2.4. *Let $e \in E$, and let $(f_i)_{i \in I}$ be a family of elements of E such that the supremum $f := \bigvee_{i \in I} f_i$ exists in E. Suppose that $e \oplus f$ (resp. $f \oplus e$) exists. Then $e \oplus f_i$ (resp. $f_i \oplus e$) exists for all $i \in I$, the supremum $\bigvee_{i \in I}(e \oplus f_i)$ (resp. the supremum $\bigvee_{i \in I}(f_i \oplus e)$) exists in E, and $e \oplus f = \bigvee_{i \in I}(e \oplus f_i)$ (resp. $f \oplus e = \bigvee_{i \in I}(f_i \oplus e)$).*

Proof. We prove the lemma under the hypothesis that $e \oplus f$ exists. The proof under the alternative hypothesis is similar. For each $i \in I$, we have $f_i \leq f$, and therefore $e \oplus f_i$ exists and $e \oplus f_i \leq e \oplus f$ (Lemma 2.3 (iii)). Suppose that $e \oplus f_i \leq b \in E$ for all $i \in I$, i.e., there exists x_i with $b = (e \oplus f_i) \oplus x_i = e \oplus (f_i \oplus x_i)$. Then $e \leq b$ and $f_i \leq f_i \oplus x_i = e/b$ for all $i \in I$, whence $f \leq e/b$, and it follows from Lemma 2.3 (ii) that $e \oplus f \leq b$, proving that $e \oplus f = \bigvee_{i \in I}(e \oplus f_i)$. ∎

By (GPEA1), we may omit parentheses in expressions such as $a \oplus b \oplus c$. By recursion, the partial operation $\oplus$ can be extended to finite sequences $e_1, e_2, \ldots, e_n$ as follows: The orthosum $e_1 \oplus \cdots \oplus e_n$ exists iff the elements $f := e_1 \oplus e_2 \oplus \cdots \oplus e_{n-1}$ and $f \oplus e_n$ both exist, and then $e_1 \oplus \cdots \oplus e_n := f \oplus e_n$. In general, the orthosum may depend on the order of its orthosummands.

In a similar way, by recursion, we also define *orthogonality* and the corresponding *orthosum* for a finite sequence of elements in E, and it turns out that the orthosum does not depend on the order of the orthosummands. Therefore, in the obvious way, we define orthogonality and the corresponding orthosum for finite families in E. (We understand that the empty family in E is orthogonal and that its orthosum is 0.) The notion of orthogonality and the orthosum for arbitrary families is defined as follows: A family $(e_i)_{i \in I}$ in E is said to be *orthogonal* iff every finite subfamily $(e_i)_{i \in F}$ ($I \supseteq F$ is finite) is orthogonal in E. The family $(e_i)_{i \in I}$ is *orthosummable* with *orthosum* $\oplus_{i \in I} e_i$ iff it is orthogonal and the supremum $\bigvee_{F \subseteq I}(\oplus_{i \in F} e_i)$ over all finite subsets F of I exists in E, in which case $\oplus_{i \in I} e_i := \bigvee_{F \subseteq I}(\oplus_{i \in F} e_i)$.

Lemma 2.5. *Let $e, f \in E$. If $e \perp f$ and $e \vee f$ exists in E, then $e \wedge f$ exists in E, $(e \vee f) \perp (e \wedge f)$, and $e \oplus f = (e \vee f) \oplus (e \wedge f)$.*

Proof. As $e \perp f$, we have $e \oplus f = f \oplus e$. Evidently $e, f \leq e \oplus f$, so $e \leq e \vee f \leq e \oplus f$, and by Lemma 2.3 (iv), $e/(e \vee f) \oplus (e \vee f)/(e \oplus f) = e/(e \oplus f) = f$, whence $(e \vee f)/(e \oplus f) \leq f$. Likewise, $(e \vee f)/(e \oplus f) \leq e$. Suppose that $d \leq e, f$. By Lemma 2.3 (iii), $f \leq f \oplus (e \backslash d) = (f \oplus e) \backslash d = (e \oplus f) \backslash d$. Likewise, $e \leq (e \oplus f) \backslash d$, and we have $e \vee f \leq (e \oplus f) \backslash d$; hence by Lemma 2.3 (ii), $d \leq (e \vee f)/(e \oplus f)$. This proves that $(e \vee f)/(e \oplus f) = e \wedge f$, from which we obtain $e \oplus f = (e \vee f) \oplus (e \wedge f)$. Similarly, by considering $(e \oplus f) \backslash (e \vee f)$, which is again under e and f, and arguing that $(e \oplus f) \backslash (e \vee f) = e \wedge f$, we find that $e \oplus f = (e \wedge f) \oplus (e \vee f)$. ∎

Definition 2.6. A *pseudoeffect algebra* (PEA) is a partial algebraic structure $(E, \oplus, 0, 1)$, where $\oplus$ is a partial operation and 0 and 1 are constants, and the following hold:

(PEA1) $a \oplus b$ and $(a \oplus b) \oplus c$ exist iff $b \oplus c$ and $a \oplus (b \oplus c)$ exist, and in this case $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

(PEA2) There is exactly one $d \in E$ and exactly one $e \in E$ such that $a \oplus d = e \oplus a = 1$.

(PEA3) If $a \oplus b$ exists, there are elements $d, e \in E$ such that $a \oplus b = d \oplus a = b \oplus e$.

(PEA4) If $1 \oplus a$ or $a \oplus 1$ exists, then $a = 0$

The partial ordering for a PEA is defined in the same way as the partial ordering for a GPEA. It is easy to see, that a PEA is the same thing as a GPEA with a greatest element. We claim the following statement from ([7], Proposition 2.7):

Proposition 2.7. *Let $(E, \oplus, 0)$ be a GPEA and let $u \in E$. Then $(E[0, u], \oplus_u, 0, u)$ is a PEA, where $E[0, u] := \{a \in E : a \le u\}$ and where $a \oplus_u b$ is defined for $a, b \in E[0, u]$ iff $a \oplus b$ exists in E and $a \oplus b \le u$, in which case $a \oplus_u b := a \oplus b$.*

Definition 2.8. An *ideal* of the GPEA E is a nonempty subset $I \subseteq E$ such that:

(I1) If $a \in I$, $b \in E$, and $b \le a$, then $b \in I$.

(I2) If $a, b \in I$ and $a \oplus b$ exists, then $a \oplus b \in I$.

If I is an ideal in E, then I is said to be *normal* iff,

(N) whenever $a, x, y \in E$ and $a \oplus x = y \oplus a$, then $x \in I \Leftrightarrow y \in I$.

Definition 2.9. We say, that an ideal S in the GPEA E is *central*, or equivalently, that it is a *direct summand* of E, iff there is an ideal S' in E such that

(1) $a \in S, b \in S' \Rightarrow a \perp b$, and

(2) every $a \in E$ can be uniquely written a an orthosum $a = a_1 \oplus a_2$ with "coordinates" $a_1 \in S$ and $a_2 \in S'$.

We write $E = S \oplus S'$ iff (1) and (2) hold.

If $E = S \oplus S'$, then S' is also a central ideal (direct summand) in E, S' is uniquely determined by S (cf. the proof of [14, Lemma 4.3]), and all GPEA calculations on E can be conducted "coordinatewise" in the obvious sense. If $E = S \oplus S'$, we refer to S and S' as *complementary direct summands* of E.

Proposition 2.10. *Any central ideal (direct summand) of a GPEA E is normal.*

Proof. Let S be a central ideal of E with S' as its complementary direct summand, and assume that $a, x, y \in E$ with $a \oplus x = y \oplus a$. We can write a uniquely as $a = a_1 \oplus a_2$ with $a_1 \in S$ and $a_2 \in S'$. Then $a_1 \oplus a_2 \oplus x = y \oplus a_1 \oplus a_2$. Suppose that $x \in S$. Then, as $a_2 \in S'$, we have $x \perp a_2$, so $a_2 \oplus x = x \oplus a_2$, whence $a_1 \oplus x \oplus a_2 = y \oplus a_1 \oplus a_2$, and by cancellation $a_1 \oplus x = y \oplus a_1$. Therefore, $y \le a_1 \oplus x \in S$, and it follows that $y \in S$. By a similar argument, if $y \in S$, then $x \in S$. ∎

The notion that E is a direct sum $E = S \oplus S'$ of two central ideals is extended to finitely many direct summands $E = S_1 \oplus S_2 \oplus \cdots \oplus S_n$ in the obvious way, each S_i, $i = 1, 2, \ldots, n$, being a central ideal (direct summand) in E with complementary direct summand $(S_i)' = S_1 \oplus \cdots S_{i-1} \oplus S_{i+1} \cdots \oplus S_n$.

3. The exocenter of a GPEA

Definition 3.1. The exocenter of the GPEA E, denoted by $\Gamma_{\text{ex}}(E)$, is the set of all mappings $\pi : E \to E$ such that for all $e, f \in E$ the following hold:

- (EXC1) $\pi : E \to E$ is a PGEA-endomorphism of E, that is: if $e \oplus f$ exists, then $\pi e \oplus \pi f$ exists and $\pi(e \oplus f) = \pi e \oplus \pi f$.
- (EXC2) π is idempotent (i.e., $\pi(\pi e) = \pi e$).
- (EXC3) π is decreasing (i.e., $\pi e \leq e$).
- (EXC4) π satisfies the following orthogonality condition: if $\pi e = e$ and $\pi f = 0$, then $e \perp f$ (i.e., $e \oplus f = f \oplus e$).

If $\pi \in \Gamma_{\text{ex}}(E)$ and $e \in E$, then as $\pi e \leq e$ by (EXC3), we can (and do) define $\pi' e := (\pi e)/e$ for all $e \in E$.

Lemma 3.2. *If $\pi \in \Gamma_{\text{ex}}(E)$ and $e \in E$, then $\pi' e = (\pi e)/e = e \backslash (\pi e) = e \ominus \pi e$ and $\pi e \perp \pi' e$ with $\pi e \oplus \pi' e = \pi' e \oplus \pi e = e$.*

Proof. Let $\pi \in \Gamma_{\text{ex}}(E)$ and $e \in E$. As $\pi e \leq e$, both $\pi' e = (\pi e)/e$ and $e \backslash (\pi e)$ are defined, and with $x := (\pi e)/e$ and $y := e \backslash (\pi e)$, we have $\pi e \oplus x = e = y \oplus \pi e$. We apply the mapping π and obtain $\pi e \oplus \pi x = \pi e = \pi y \oplus \pi e$; hence $\pi x = \pi y = 0$ and by (EXC4), $\pi e \oplus x = x \oplus \pi e = e$ and also $\pi e \oplus y = y \oplus \pi e = e$. Therefore by cancellation, $\pi' e = (\pi e)/e = x = y = e \backslash (\pi e) = e \ominus \pi e$, and $\pi e \perp \pi' e$ with $\pi e \oplus \pi' e = \pi' e \oplus \pi e = e$. ∎

Theorem 3.3. *If $\pi \in \Gamma_{\text{ex}}(E)$, then for all $e, f \in E$ the following hold:*

- (i) $\pi(\pi' e) = \pi'(\pi e) = 0$.
- (ii) $\pi' \in \Gamma_{\text{ex}}(E)$ *and* $(\pi')' = \pi$.
- (iii) *If $e \leq \pi f$, then $e = \pi e$.*
- (iv) *If $e \leq f$, then $\pi e = e \wedge \pi f$.*
- (v) $\pi(E) := \{\pi e : e \in E\} = \{e \in E : e = \pi e\}$ *is an ideal in E.*
- (vi) *$\pi(E)$ is sup/inf-closed in E (i.e., $\pi(E)$ is closed under the formation of existing suprema and infima in E of nonempty families in $\pi(E)$).*

(vii) *If* $e \in \pi(E)$ *and* $f \in \pi'(E)$, *then* $e \perp f, e \oplus f = e \vee f$ *and* $e \wedge f = 0$.

(viii) *For each element* $e \in E$ *there are uniquely determined elements* $e_1 \in \pi(E), e_2 \in \pi'(E)$ *such that* $e = e_1 \oplus e_2$; *in fact,* $e_1 = \pi e$ *and* $e_2 = \pi' e$.

(ix) *If* $e = e_1 \oplus e_2, f = f_1 \oplus f_2$, *where* $e_1, f_1 \in \pi(E)$, $e_2, f_2 \in \pi'(E)$, *then* $e \oplus f$ *exists iff both* $e_1 \oplus f_1$ *and* $e_2 \oplus f_2$ *exist.*

(x) $\pi'(E) = \{f \in E : f \wedge e = 0, \forall e \in \pi(E)\}$.

Proof. (i) $\pi(\pi' e) = \pi(e \backslash \pi e) = \pi e \backslash \pi\pi e = \pi e \backslash \pi e = 0$ and $\pi'(\pi e) = \pi e \backslash \pi\pi e = 0$ too.

(ii) By Lemma 2.3 (i), $(\pi')' e = e \backslash \pi' e = e \backslash (\pi e / e) = \pi e$. To prove that π' is a GPEA-endomorphism of E, suppose that $e \oplus f$ exists. Then by (EXC1) $\pi'(e \oplus f) = (e \oplus f) \backslash (\pi e \oplus \pi f)$, whence $\pi'(e \oplus f) \oplus \pi e \oplus \pi f = e \oplus f$ and so by Lemma 2.3 (iii), $\pi'(e \oplus f) \oplus \pi e = (e \oplus f) \backslash \pi f = e \oplus (f \backslash \pi f) = e \oplus \pi' f$. As $\pi\pi e = \pi e$ and by (i), $\pi\pi'(e \oplus f) = 0$, we have $e \perp \pi' f$ by (EXC4), whence $\pi e \oplus \pi'(e \oplus f) = \pi'(e \oplus f) \oplus \pi e = e \oplus \pi' f$, i.e., $\pi'(e \oplus f) = \pi e / (e \oplus \pi' f)$, and a second application of Lemma 2.3 (iii) yields $\pi'(e \oplus f) = (\pi e / e) \oplus \pi' f = \pi' e \oplus \pi' f$. Thus, π' satisfies (EXC1). Moreover, by (i), $\pi'(\pi' e) = \pi'(e \backslash \pi e) = \pi' e \backslash \pi' \pi e = \pi' e$, whence π' satisfies (EXC2). Obviously, (EXC3) holds for π'. Finally to prove that π' satisfies (EXC4), suppose that $\pi' e = e$ and $\pi' f = 0$. Then $\pi e = 0$ because $\pi' e = e \backslash \pi e = e$ and $\pi f = f$ because $\pi' f = f \backslash \pi f = 0$. Therefore, since π satisfies (EXC4), we have $e \perp f$, and π' also satisfies (EXC4). Therefore, $\pi' \in \Gamma_{\mathrm{ex}}(E)$.

(iii) If $e \leq \pi f$, then $e \backslash \pi e = \pi' e \leq \pi' \pi f = 0$, whence $e = \pi e$.

(iv) Suppose that $e \leq f$. Then $\pi e \leq e$ and $\pi e \leq \pi f$. Suppose that $d \leq e, \pi f$. Since $d \leq \pi f$, (iii) implies that $d = \pi d \leq \pi e$, so $\pi e = e \wedge \pi f$.

(v) If $e = \pi e$, then $e \in \pi(E)$. Vice versa, if $e \in \pi(E)$, then $e = \pi f$ for some $f \in E$, so $\pi e = \pi\pi f = \pi f = e$, and we have $\pi(E) = \{e \in E : e = \pi e\}$.

(vi) Assume that $(e_i)_{i \in I} \subseteq \pi(E)$ and $e = \bigvee_{i \in I} e_i$ exists in E. As $e_i \leq e$, we have $e_i = \pi e_i \leq \pi e$ for all $i \in I$, whence $e \leq \pi e$. But also $\pi e \leq e$ and thus $\pi e = e \in \pi(E)$. Since $\pi(E)$ is an ideal, it is automatically closed under the formation of existing infima in E of nonempty families in $\pi(E)$.

(vii) Let $e \in \pi(E)$ and $f \in \pi'(E)$. Then $e = \pi e$, and $\pi f = \pi\pi' f = 0$, whence by (EXC4) $e \perp f$. Clearly, $e, f \leq e \oplus f$. If now $e, f \leq d \in E$, then $e = \pi e \leq \pi d$ and $f = \pi' f \leq \pi' d$, thus $e \oplus f \leq \pi d \oplus \pi' d = d$, whence $e \oplus f = e \vee f$. Finally, by Lemma 2.5, $e \wedge f = 0$.

(viii) Obviously, $e = \pi e \oplus \pi' e$, $\pi e \in \pi(E)$ and $\pi' e \in \pi'(E)$. Suppose $e = e_1 \oplus e_2$ with $e_1 \in \pi(E)$, $e_2 \in \pi'(E)$. Then $e_1 = \pi e_1$, $e_2 = \pi' e_2$, $\pi e = \pi e_1 \oplus \pi e_2 = e_1$, and $\pi' e = \pi' e_1 \oplus \pi' e_2 = e_2$.

(ix) Suppose $e = e_1 \oplus e_2$ and $f = f_1 \oplus f_2$, with $e_1, f_1 \in \pi(E)$, $e_2, f_2 \in \pi'(E)$. If $e_1 \oplus f_1$ and $e_2 \oplus f_2$ both exist, then $e_1 \oplus f_1 \in \pi(E)$ and $e_2 \oplus f_2 \in \pi'(E)$ by (v). Then by (vii) $(e_1 \oplus f_1) \perp (e_2 \oplus f_2)$, so $(e_1 \oplus f_1) \oplus (e_2 \oplus f_2)$ exists and equals $e_1 \oplus (f_1 \oplus e_2) \oplus f_2 = e_1 \oplus (e_2 \oplus f_1) \oplus f_2 = (e_1 \oplus e_2) \oplus (f_1 \oplus f_2) = e \oplus f$. If, on the other hand, $e \oplus f$ exists, then $e_1 \oplus e_2 \oplus f_1 \oplus f_2$ exists and equals $e_1 \oplus f_1 \oplus e_2 \oplus f_2$, which implies that $e_1 \oplus f_1$ and $e_2 \oplus f_2$ both exist.

(x) Assume that $f \wedge e = 0$ for all $e \in \pi(E)$. As $f = f_1 \oplus f_2$ with $f_1 \in \pi(E)$, $f_2 \in \pi'(E)$, we have $f_1 = f \wedge f_1 = 0$, whence $f = f_2 \in \pi'(E)$. The converse follows from (vii). ■

Lemma 3.4. *Let* $\xi, \pi \in \Gamma_{\mathrm{ex}}(E)$. *Then*:

(i) $\xi \circ \pi = \pi \circ \xi \in \Gamma_{\mathrm{ex}}(E)$.

(ii) $\xi = \xi \circ \pi \Leftrightarrow \xi e \leq \pi e, \forall\, e \in E \Leftrightarrow \xi(E) \subseteq \pi(E)$.

Proof. (i) Since $\xi(\pi e) \leq \pi e$, part (iii) of Theorem 3.3 yields $\xi(\pi e) = \pi(\xi(\pi e))$. Also, since $\pi e \leq e$ and both, π and ξ are order-preserving mappings, it follows that $\xi(\pi e) = \pi(\xi(\pi e)) \leq \pi(\xi e)$. By symmetry $\pi(\xi e) \leq \xi(\pi e)$, which gives $\xi \circ \pi = \pi \circ \xi$.

Obviously, $\xi \circ \pi$ is a GPEA-endomorphism. Furthermore, $(\xi \circ \pi) \circ (\xi \circ \pi) = \xi \circ \pi \circ \pi \circ \xi = \xi \circ \pi \circ \xi = \xi \circ \xi \circ \pi = \xi \circ \pi$, whence $\xi \circ \pi$ is idempotent. Moreover, $(\xi \circ \pi)e = \xi(\pi e) \leq \pi e \leq e$, so (EXC3) holds. Finally, suppose that $e, f \in E$ with $e = \xi(\pi e)$ and $\xi(\pi f) = 0$. Then $e = \pi(\xi e)$, so $e = \xi e = \pi e$. We put $d := \pi f$, so that $\xi d = 0$, $d \leq f$, $d = \pi d = \pi f$, and $\pi' f = (\pi f)/f = f \backslash \pi f = d/f = f \backslash d$. Therefore, $\pi' f \oplus d = d \oplus \pi' f = f$. As $e = \xi e$ and $\xi d = 0$, (EXC4) implies that $e \perp d$, i.e., $e \oplus d = d \oplus e$. Also, $\pi(e \oplus d) = \pi e \oplus \pi d = e \oplus d$, and it follows from $\pi(\pi' f) = 0$ and (EXC4) that $(e \oplus d) \perp \pi' f$. Consequently,

$$e \oplus f = e \oplus d \oplus \pi' f = \pi' f \oplus e \oplus d = \pi' f \oplus d \oplus e = f \oplus e,$$

proving that $\xi \circ \pi$ satisfies (EXC4).

(ii) If $\xi = \xi \circ \pi$, then $\xi e = \xi(\pi e) \leq \pi e$ for all $e \in E$. Conversely, if $\xi e \leq \pi e$ for all $e \in E$, then $\xi e = \xi(\xi e) \leq \xi(\pi e)$. Also, as $\xi(\pi e) \leq \xi e$ always holds, $\xi e = \xi(\pi e)$ for all $e \in E$, which means that $\xi = \xi \circ \pi$. Now if $\xi e \leq \pi e$ for all $e \in E$, then if $e \in \xi(E)$, we get $e = \xi e \leq \pi e$, whence $\pi e = e \in \pi(E)$. Conversely, if $\xi(E) \subseteq \pi(E)$, then every $\xi e \in \pi(E)$, thus by (i), $\xi e = \pi(\xi e) = \xi(\pi e) \leq \pi e$. ■

Theorem 3.5. *Let* $\pi, \xi \in \Gamma_{\mathrm{ex}}(E)$ *and let* $e \in E$. *Then* $\Gamma_{\mathrm{ex}}(E)$ *is partially ordered by* $\xi \leq \pi \Leftrightarrow \xi = \xi \circ \pi \Leftrightarrow \xi e \leq \pi e, \forall e \in E \Leftrightarrow \xi(E) \subseteq \pi(E)$, *with* 0 (*the zero mapping*) *as the smallest element and* 1 (*the identity mapping*) *as the largest element. Moreover,* $\Gamma_{\mathrm{ex}}(E)$ *is a boolean algebra with* $\pi \mapsto \pi'$ *as the boolean complementation, with* $\pi \wedge \xi = \pi \circ \xi = \xi \circ \pi$, *and with* $\pi \vee \xi = (\pi' \circ \xi')'$.

Proof. Let $\pi, \xi \in \Gamma_{\rm ex}(E)$. By Lemma 3.4, $\leq$ is a partial order on $\Gamma_{\rm ex}(E)$ and $0 \leq \pi \leq 1$ holds for every $\pi \in \Gamma_{\rm ex}(E)$. Clearly, $\pi \circ \xi$ is the infimum $\pi \wedge \xi$ of π and ξ in $\Gamma_{\rm ex}(E)$. We also have $\pi \wedge \xi = 0$ iff $\pi(\xi e) = 0$ for every $e \in E$, which is equivalent to $\pi(e \backslash \xi e) = \pi e$, $\forall\, e \in E$. But this means that $\pi(\xi' e) = \pi e$, $\forall\, e \in E$, that is $\pi \circ \xi' = \pi$, which holds iff $\pi \leq \xi'$. So by [19, Theorem 4, p. 49], $\Gamma_{\rm ex}(E)$ is a boolean algebra, π' is the complement of π in $\Gamma_{\rm ex}(E)$, and $\pi \vee \xi = (\pi' \circ \xi')'$. ■

Lemma 3.6. *Let $\pi, \xi \in \Gamma_{\rm ex}(E)$ with $\pi \wedge \xi = 0$ and let $e, f \in E$. Then:*

(i) *If $e \in \pi(E), f \in \xi(E)$, then $e \perp f$ and $e \oplus f \in (\pi \vee \xi)(E)$, $e \oplus f = e \vee f$ and $e \wedge f = 0$.*

(ii) *$\pi e \perp \xi e, (\pi \vee \xi)e = \pi e \vee \xi e = \pi e \oplus \xi e$ and $\pi e \wedge \xi e = 0$.*

Proof. (i) By the hypotheses $e = \pi e$ and $f = \xi f$. As $\pi f = \pi(\xi f) = 0$ (by Theorem 3.5), we get $\pi' f = f \backslash \pi f = f$. Therefore, $f \in \pi'(E)$, and by Theorem 3.3 (vii), $e \perp f$, $e \oplus f = e \vee f$ and $e \wedge f = 0$. Also $e = \pi e \leq (\pi \vee \xi)e \leq e$, whence $(\pi \vee \xi)e = e$. Likewise, $(\pi \vee \xi)f = f$, whence $e \oplus f = (\pi \vee \xi)(e \oplus f) \in (\pi \vee \xi)(E)$.

(ii) We need only replace e by πe and f by ξe in (i) to obtain $\pi e \perp \xi e$, $\pi e \oplus \xi e = \pi e \vee \xi e$ and $\pi e \wedge \xi e = 0$. As $\pi \wedge \xi = 0$ in the boolean algebra $\Gamma_{\rm ex}(E)$, we have $\pi \leq \xi'$, whence $\pi e = (\pi \wedge \xi')e = (\pi \circ \xi')e = \pi(\xi' e)$. Thus, combining the equalities $\xi e \oplus \xi' e = e$ and $\pi e \oplus (\pi' \circ \xi')e = \pi(\xi' e) \oplus \pi'(\xi' e) = \xi' e$, we obtain $\xi e \oplus \pi e \oplus (\pi' \circ \xi')e = e$. Therefore, as $(\pi' \circ \xi')' e \oplus (\pi' \circ \xi')e = e$, we infer by cancellation that $(\pi \vee \xi)e = (\pi' \circ \xi')' e = \pi e \oplus \xi e = \pi e \vee \xi e$. ■

Theorem 3.7. *Let $\pi_1, \pi_2, \ldots, \pi_n$ be pairwise disjoint elements of the boolean algebra $\Gamma_{\rm ex}(E)$ and let $e \in E, e_i \in \pi_i(E)$ for $i = 1, 2, \ldots, n$. Then:*

(i) *$(e_i)_{i=1,2,\ldots,n}$ is an orthogonal sequence in E and $\oplus_{i=1}^n e_i = \bigvee_{i=1}^n e_i$.*

(ii) *$(\pi_i e)_{i=1}^n$ is an orthogonal sequence in E and $(\pi_1 \vee \pi_2 \vee \ldots \vee \pi_n)e = \oplus_{i=1}^n \pi_i e = \bigvee_{i=1}^n \pi_i e$.*

Proof. For $n = 1$ the assertions hold trivially, and the results for $n = 2$ are consequences of Lemma 3.6. The results for an arbitrary $n \in \mathbb{N}$ then follow from a straightforward induction argument. ■

Theorem 3.8. *Let $\pi_1, \pi_2, \ldots, \pi_n \in \Gamma_{\rm ex}(E)$, $e \in E$. Then:*

(i) *$(\pi_1 \wedge \pi_2 \wedge \ldots \wedge \pi_n)e = \pi_1 e \wedge \pi_2 e \wedge \ldots \wedge \pi_n e$.*

(ii) *$(\pi_1 \vee \pi_2 \vee \ldots \vee \pi_n)e = \pi_1 e \vee \pi_2 e \vee \ldots \vee \pi_n e$.*

Proof. We will prove the assertions for $n = 2$ and the general cases will then follow by induction.

(i) Obviously, $(\pi \wedge \xi)e \leq \pi e, \xi e$. Suppose now that $f \leq \pi e, \xi e$. Then $f = \pi f = \xi f$ by Theorem 3.3 (iii) and therefore $f = (\pi \circ \xi) f \leq (\pi \circ \xi)\xi e = (\pi \circ \xi \circ \xi)e = (\pi \circ \xi)e = (\pi \wedge \xi)e$.

(ii) Working in the boolean algebra $\Gamma_{\rm ex}(E)$, we can write $\pi \vee \xi$ as a pairwise disjoint supremum:

$$\pi \vee \xi = (\pi \wedge \xi) \vee (\pi \wedge \xi') \vee (\pi' \wedge \xi).$$

Then we use Theorem 3.7 to get

$$(\pi \vee \xi)e = (\pi \wedge \xi)e \vee (\pi \wedge \xi')e \vee (\pi' \wedge \xi)e$$

where $\pi e = (\pi \wedge \xi)e \vee (\pi \wedge \xi')e$ and $\xi e = (\pi \wedge \xi)e \vee (\pi' \wedge \xi)e$. Therefore $(\pi \vee \xi)e = \pi e \vee \xi e$. ■

As is easily confirmed, a cartesian product of GPEAs, with the obvious pointwise operations and relations, is again a GPEA.

Theorem 3.9. *Let $\pi_1, \pi_2, \ldots, \pi_n$ be pairwise disjoint elements of $\Gamma_{\rm ex}(E)$ such that $\pi_1 \vee \pi_2 \vee \ldots \vee \pi_n = 1$ and let X be the cartesian product of $\pi_i(E)$ for $i = 1, 2 \ldots, n$. Then for $(e_1, e_2, \ldots, e_n) \in X$, the sequence $(e_i)_{i=1}^n$ is orthogonal in E and $\oplus_{i=1}^n e_i = \bigvee_{i=1}^n e_i$. Moreover, $\Phi : X \to E$ defined by $\Phi(e_1, e_2, \ldots, e_n) := e_1 \oplus e_2 \oplus \ldots \oplus e_n$, is a GPEA-isomorphism and for every $e \in E$, $\Phi^{-1}e = (\pi_1 e, \pi_2 e, \ldots, \pi_n e) \in X$.*

Proof. The first part has already been proved in Theorem 3.7. To prove that Φ is a GPEA-morphism, let $(e_1, e_2, \ldots, e_n), (f_1, f_2, \ldots, f_n) \in X$ and let $e_i \oplus f_i$ exist for all $i = 1, 2, \ldots, n$. Then $(e_1 \oplus f_1, e_2 \oplus f_2, \ldots, e_n \oplus f_n) \in X$ and so $(e_i \oplus f_i)_{i=1}^n$ is an orthogonal sequence. Using Theorem 3.3 (ix) and induction, we get $\oplus_{i=1}^n (e_i \oplus f_i) = \oplus_{i=1}^{n-1}(e_i \oplus f_i) \oplus e_n \oplus f_n = (\oplus_{i=1}^{n-1} e_i) \oplus (\oplus_{i=1}^{n-1} f_i) \oplus e_n \oplus f_n$. But, since f_i for $i = 1, 2 \ldots, n-1$ are all orthogonal to e_n, we have $(\oplus_{i=1}^{n-1} e_i) \oplus (\oplus_{i=1}^{n-1} f_i) \oplus e_n \oplus f_n = (\oplus_{i=1}^{n-1} e_i) \oplus e_n \oplus (\oplus_{i=1}^{n-1} f_i) \oplus f_n = (\oplus_{i=1}^n e_i) \oplus (\oplus_{i=1}^n f_i)$, whence $\Phi : X \to E$ is a GPEA-morphism. Define $\Psi : E \to X$ by $\Psi(e) := (\pi_1 e, \pi_2 e, \ldots, \pi_n e)$ for all $e \in E$. Then Ψ is also a GPEA-morphism and by Theorem 3.7 (ii), $\Phi \circ \Psi$ is the identity on E. Now consider $\pi_i e_j$ for $i, j = 1, 2, \ldots, n$. We have $\pi_i e_j = \pi_i(\pi_j e_j) = (\pi_i \wedge \pi_j)e_j$. Thus $\pi_i e_j = 0$ for $i \neq j$ and $\pi_i e_j = e_j$ for $i = j$ and so $\Psi \circ \Phi$ is the identity on X. Consequently $\Psi = \Phi^{-1}$ and Φ is a GPEA-isomorphism. ■

According to the previous theorem, we may consider E as a direct sum $E = \pi_1(E) \oplus \pi_2(E) \oplus \ldots \oplus \pi_n(E)$ whenever π_i are pairwise disjoint elements of $\Gamma_{\rm ex}(E)$ and $\bigvee_{i=1}^n \pi_i = 1$. In particular, $E = \pi(E) \oplus \pi'(E)$ for every π in the boolean algebra $\Gamma_{\rm ex}(E)$.

Theorem 3.10. *If $S \subseteq E$, then the following statements are equivalent:*

(i) *S is a central ideal (direct summand) of E.*

(ii) *There exists $\pi \in \Gamma_{\rm ex}(E)$ such that $S = \pi(E)$.*

Proof. Assume that $E = S \oplus S'$. We define for $e \in E$: $\pi e := s$ where $e = s \oplus t$, $s \in S$, $t \in S'$. Then $\pi \in \Gamma_{\rm ex}(E)$. Indeed, (EXC1) and (EXC2) hold trivially and since $s \leq s \oplus t$, (EXC3) also holds. If $e, f \in E$ are such that $\pi e = e$ and $\pi f = 0$, then $e \in S$ and $f \in S'$, thus $e \perp f$ and so (EXC4) holds too. If, on the other hand, $\pi \in \Gamma_{\rm ex}(E)$ and $S = \pi(E)$, then $E = S \oplus \pi'(E)$ and so S is a central ideal. ■

Corollary 3.11. *If $\pi \in \Gamma_{\rm ex}(E)$, then $\pi(E)$ is a normal ideal in E.*

Proof. By Theorem 3.10, $\pi(E)$ is a central ideal in E, and by Proposition 2.10, every central ideal in E is normal. ■

Corollary 3.12. *Let us partially order the set C of all central ideals (direct summands) of E by inclusion. Then there is an order isomorphism between $\Gamma_{\rm ex}(E)$ and C given by: $\pi \leftrightarrow S$ iff $\pi(E) = S$. Moreover, if $\pi(E) = S$, then $\pi'(E)$ is the direct summand S' of E that is complementary to S.*

Theorem 3.13. *Let $\pi \in \Gamma_{\rm ex}(E)$ and let $(e_i)_{i\in I}$ be a family of elements in E. Then:*

(i) *If $\bigvee_{i\in I} e_i$ exists in E, then so does $\bigvee_{i\in I} \pi e_i$ and $\pi(\bigvee_{i\in I} e_i) = \bigvee_{i\in I} \pi e_i$.*

(ii) *If $I \neq \emptyset$ and $\bigwedge_{i\in I} e_i$ exists in E, then so does $\bigwedge_{i\in I} \pi e_i$ and $\pi(\bigwedge_{i\in I} e_i) = \bigwedge_{i\in I} \pi e_i$.*

(iii) *If $(e_i)_{i\in I}$ is orthosummable, then so is $(\pi e_i)_{i\in I}$ and $\pi(\oplus_{i\in I} e_i) = \oplus_{i\in I} \pi e_i$.*

Proof. (i) Put $e := \bigvee_{i\in I} e_i$. As $e_i \leq e$, we also have $\pi e_i \leq \pi e$ for all $i \in I$. Now suppose that $\pi e_i \leq f$ for all $i \in I$. Then $\forall i \in I$: $\pi e_i = \pi(\pi e_i) \leq \pi f$. But we also have $\pi' e_i \leq \pi' e$ for all $i \in I$. So by (vii) and (viii) in Theorem 3.3, $e_i = \pi e_i \oplus \pi' e_i = \pi e_i \vee \pi' e_i \leq \pi f \vee \pi' e = \pi f \oplus \pi' e$ for all $i \in I$. Thus $e \leq \pi f \oplus \pi' e$ so $\pi e \leq \pi f \oplus \pi(\pi' e) = \pi f \leq f$. Hence $\pi e = \bigvee_{i\in I} \pi e_i$.

(ii) Put $e := \bigwedge_{i\in I} e_i$. As $e \leq e_i$, we have $\pi e \leq \pi e_i$ for all $i \in I$. Suppose $f \in E$ with $f \leq \pi e_i$ for all $i \in I$. As $I \neq \emptyset$, Theorem 3.3 (iii) implies that $f = \pi f$. Because $\pi e_i \leq e_i$, we have $f \leq e_i$ for all $i \in I$. Therefore $f \leq e$ and $\pi f = f \leq \pi e$.

(iii) For any finite subset F of I, as π is a GPEA-endomorphism, $\pi(\oplus_{i\in F} e_i) = \oplus_{i\in F} \pi e_i$. As $\oplus_{i\in I} \pi e_i = \bigvee_F \oplus_{i\in F} \pi e_i = \bigvee_F \pi(\oplus_{i\in F} e_i) = \pi \bigvee_F (\oplus_{i\in F} e_i) = \pi(\bigvee_{i\in I} e_i)$, the desired result follows from (i). ■

4. The center of a GPEA

Definition 4.1. An element $c \in E$ is *central* iff for every $a, b \in E$, the following hold:

(C1) There exist $a_1, a_2 \in E$ such that $a_1 \leq c$, $a_2 \oplus c$ exists and $a = a_1 \oplus a_2$.

(C2) If $a \leq c$ and if $b \oplus c$ exists, then $a \perp b$.

(C3) If $a, b \leq c$ and $a \oplus b$ exists, then $a \oplus b \leq c$.

(C4) If $a \oplus c$, $b \oplus c$ and $a \oplus b$ exist, then $a \oplus b \oplus c$ exists.

We denote the set of all central elements of the GPEA E by $\Gamma(E)$.

Lemma 4.2. *Let $a, x, y \in E$ and let $c \in \Gamma(E)$. Then*:

(i) *The elements a_1 and a_2 in* (*C1*) *of Definition* 4.1 *are unique and $a_1 \perp a_2$.*

(ii) *$\forall\, a \in E$, $a \oplus c$ exists iff $a \perp c$ iff $c \oplus a$ exists.*

(iii) *If $x \oplus y$ exists in E and at least one of the elements x, y is central, then $x \perp y$.*

Proof. (i) Suppose that $a_1, a_2, b_1, b_2 \in E$ with $a = a_1 \oplus a_2 = b_1 \oplus b_2$, where $a_1, b_1 \leq c$ and both $a_2 \oplus c$ and $b_2 \oplus c$ exist. Then by (C2), we have $a_1 \perp a_2$ and $b_1 \perp b_2$, whence $a = a_1 \oplus a_2 = a_2 \oplus a_1 = b_1 \oplus b_2 = b_2 \oplus b_1$. As $a_1 \leq c$, there exists $d \in E$ such that $a_1 \oplus d = c$ and we have $a_2 \oplus c = a_2 \oplus a_1 \oplus d = b_2 \oplus b_1 \oplus d$. Since $b_1, d \leq c$, (C3) implies that $b_1 \oplus d \leq c$, whence $a_2 \oplus c \leq b_2 \oplus c$, and it follows by cancellation that $a_2 \leq b_2$. By symmetry, $b_2 \leq a_2$, so $a_2 = b_2$, and therefore $a_1 = b_1$ by cancellation.

(ii) If $a \oplus c$ exists, then as $c \leq c$, we have $a \perp c$ by (C2). As $a \perp c$, then $c \oplus a$ exists. Finally, suppose that $c \oplus a$ exists. Then by (C1), there exist $d_1, d_2 \in E$ with $c \oplus a = d_1 \oplus d_2$, where $d_1 \leq c$ and $d_2 \oplus c$ exists. As $c \leq c$ and $d_2 \oplus c$ exists, (C2) implies that $c \perp d_2$. Also, by part (i), $d_1 \perp d_2$, and since $d_1 \leq c$, we have $c \oplus a = d_1 \oplus d_2 = d_2 \oplus d_1 \leq d_2 \oplus c = c \oplus d_2$, whence $a \leq d_2$ by cancellation. Thus, $a \leq d_2$ and $d_2 \perp c$, so $a \oplus c$ exists by Lemma 2.3 (iii). Part (iii) follows immediately from (ii). ■

Theorem 4.3. *If $c \in E$, then the following are equivalent:*

(i) *c is central, i.e., $c \in \Gamma(E)$.*

(ii) *$E[0, c]$ is a central ideal* (*direct summand*) *of E.*

(iii) *E decomposes as a direct sum $E = E[0, c] \oplus \{f \in E : f \perp c\}$.*

Proof. (i) $\Rightarrow$ (ii): If c is central, then by (C3), $E[0,c]$ is an ideal. We prove that it is moreover a central ideal; that is, there exists another ideal, namely $E[0,c]' := \{e \in E : e \perp c\}$, such that $E = E[0,c] \oplus E[0,c]'$. By Definition 4.1 and Lemma 4.2 (ii), for every $e \in E$ there exist $e_1, e_2 \in E$ such that $e = e_1 \oplus e_2$, where $e_1 \in E[0,c]$ and $e_2 \in E[0,c]'$. It will be sufficient to show that $E[0,c]'$ is an ideal in E. If $d \leq e$ and $e \in E[0,c]'$, then by Lemma 2.3 (iii), $d \oplus c$ exists; whence, as $c \in \Gamma(E)$, we have $d \in E[0,c]'$. Finally, suppose that $e, f \in E[0,c]'$ and $e \oplus f$ exists. Then by (C4), $e \oplus f \oplus c$ exists, and again, as $c \in \Gamma(E)$, it follows that $e \oplus f \in E[0,c]'$.

(ii) $\Rightarrow$ (iii): If $E[0,c]$ is a central ideal in E, then there is an ideal $E[0,c]'$ such that $E = E[0,c] \oplus E[0,c]'$. Evidently, if $f \in E[0,c]'$, then $f \perp c$. Conversely, if $f \in E$ with $f \perp c$, then $f = s \oplus t$, where $s \leq c$ and $t \in E[0,c]'$. As $s \leq f$ and $f \perp c$, we get $s \perp c$ and since $E[0,c]$ is an ideal, $s \oplus c \leq c$, which entails $s = 0$. Thus $f = t \in E[0,c]'$ and $E[0,c]' = \{f \in E : f \perp c\}$.

(iii) $\Rightarrow$ (i): Let $E = E[0,c] \oplus \{f \in E : f \perp c\}$. We prove (C1)–(C4). (C1) follows directly from the fact, that every $e \in E$ can be written as $e = e_1 \oplus e_2$, where $e_1 \in E[0,c]$ and $e_2 \in \{f \in E : f \perp c\}$. To prove (C2), suppose that $a \leq c$ and $b \oplus c$ exists. Then, we can write $b \oplus c = e_1 \oplus f_1$ where $e_1 \in E[0,c]$, $f_1 \in \{f \in E : f \perp c\}$, and $e_1 \perp f_1$. Therefore, $b \oplus c = f_1 \oplus e_1 \leq f_1 \oplus c$, so $b \leq f_1$ by cancellation; hence, since $\{f \in E : f \perp c\}$ is an ideal, it follows that $b \in \{f \in E : f \perp c\}$. Now we have $a \in E[0,c]$ and $b \in \{f \in E : f \perp c\}$, whence $a \perp b$, proving (C2). Because $E[0,c]$ is an ideal, (C3) follows immediately. For (C4), suppose $a \oplus c$, $b \oplus c$ and $a \oplus b$ all exist. As a consequence of (C2) and the fact that $c \leq c$, we have $a \perp c$ and $b \perp c$, i.e., $a, b \in \{f \in E : f \perp c\}$. Again, since $\{f \in E : f \perp c\}$ is an ideal, we infer that $a \oplus b \perp c$, so $a \oplus b \oplus c$ exists, proving (C4). ■

Definition 4.4. If $c \in \Gamma(E)$, then by Theorems 4.3 and 3.10, there exists uniquely determined mapping in $\Gamma_{\text{ex}}(E)$, henceforth denoted by π_c, such that $\pi_c(E) = E[0,c]$.

Corollary 4.5. *Let $\pi \in \Gamma_{\text{ex}}(E)$. Then the following statements are equivalent:*

(i) *There exists a largest element $c \in \pi(E)$.*

(ii) $\pi(E) = E[0,c]$.

(iii) $c \in \Gamma(E)$, $\pi = \pi_c$, *and* $\pi'(E) = \{f \in E : f \perp c\}$.

Proof. Since $\pi(E)$ is an ideal in E, $\pi(E) = E[0,c]$ iff c is the largest element in $\pi(E)$. The rest follows by Theorem 3.10, Theorem 4.3, and Definition 4.4. ■

If $c \in \Gamma(E)$ and $d \in E$ with $c \leq d$, then there exists $x := d \backslash c \in E$ with $x \oplus c = d$, and since $c \in \Gamma(E)$, it follows from Lemma 4.2 (iii) that $x \perp c$, whence $c \oplus x = d$

also holds, i.e., $x = c/d$. Consequently, $d \ominus c = d\backslash c = c/d$ exists (Definition 2.2). In particular, $d \ominus c$ is defined for $c, d \in \Gamma(E)$ iff $c \leq d$, and if $c \leq d$, then by part (x) of the next theorem, $d \ominus c \in \Gamma(E)$ and we have $d = c \oplus (d \ominus c) = (d \ominus c) \oplus c$.

We omit the proofs of the following two theorems as they can be obtained by easy modifications of the proofs of [13, Lemma 4.5, Theorem 4.6].

Theorem 4.6. *Let $c, d \in \Gamma(E)$, $e \in E$. Then*:

(i) $\pi_c e = e \wedge c$.

(ii) $\pi_c d = \pi_d c = c \wedge d$.

(iii) $e \wedge c = 0 \Leftrightarrow e \in (\pi_c)'(E) \Leftrightarrow e \perp c$.

(iv) $c \wedge d \in \Gamma(E)$ *and* $\pi_{c\wedge d} = \pi_c \wedge \pi_d$.

(v) $c \wedge d = 0 \Leftrightarrow \pi_c \wedge \pi_d = 0 \Leftrightarrow c \perp d$.

(vi) *If $c \perp d$, then $c \oplus d = c \vee d \in \Gamma(E)$ and $\pi_{c\oplus d} = \pi_{c\vee d} = \pi_c \vee \pi_d$.*

(vii) *π_c is the smallest $\pi \in \Gamma_{\rm ex}(E)$ such that $\pi c = c$.*

(viii) *If $\pi \in \Gamma_{\rm ex}(E)$ and $h \in E$, then $h \in \Gamma(E)$ iff $\pi e = e \wedge h$ for all $e \in E$, and in this case, $\pi = \pi_h$.*

(ix) $c \leq d \Leftrightarrow \pi_c \leq \pi_d$.

(x) *If $c \leq d$, then $d \ominus c$ exists, $d \ominus c \in \Gamma(E)$ and $\pi_{d\ominus c} = \pi_d \wedge (\pi_c)'$.*

(xi) *$c \vee d$ exists in E, $c \vee d \in \Gamma(E)$ and $\pi_{c\vee d} = \pi_c \vee \pi_d$.*

Theorem 4.7.

(i) *$\{\pi_c : c \in \Gamma(E)\}$ is a sublattice of the boolean algebra $\Gamma_{\rm ex}(E)$, and as such, it is a generalized boolean algebra.*

(ii) *$\Gamma(E)$ is a commutative lattice-ordered sub-GPEA (hence sub-GEA) of E.*

(iii) *The mapping $c \mapsto \pi_c$ from $\Gamma(E)$ onto $\{\pi_c : c \in \Gamma(E)\}$ is a lattice isomorphism.*

(iv) *$\Gamma(E)$ is a generalized boolean algebra, i.e., a distributive and relatively complemented lattice with smallest element 0.*

(v) *E is a PEA iff $\{\pi_c : c \in \Gamma(E)\} = \Gamma_{\rm ex}(E)$.*

If ϕ is a mapping defined on E and $S \subseteq E$, then $\phi|_S$ denotes the restriction of ϕ to S. The proofs of parts (i)–(iv) of the next theorem are easy modifications of the proofs of [14, Theorem 4.13, (i)–(iv)]; part (v) follows as in the proof of [14, Lemma 4.5 (iii)]; and with the aid of part (v), part (vi) follows as in the proof of [14, Theorem 4.13 (v)].

Theorem 4.8. *Let* $\xi, \pi \in \Gamma_{\text{ex}}(E)$. *Then:*

(i) $\xi|_{\pi(E)} \in \Gamma_{\text{ex}}(\pi(E))$.

(ii) *If* $\tau \in \Gamma_{\text{ex}}(\pi(E))$, *then* $\tau \circ \pi \in \Gamma_{\text{ex}}(E)$.

(iii) $\xi \mapsto \xi|_{\pi(E)}$ *is a surjective boolean homomorphism of* $\Gamma_{\text{ex}}(E)$ *onto* $\Gamma_{\text{ex}}(\pi(E))$.

(iv) *If* $p \in \pi(E)$, *then* $\pi(E)[0,p]$ *and* $E[0,p]$ *coincide both as sets and as pseudoeffect algebras.*

(v) *If* $p \in E$, *then* $\pi(E[0,p]) = E[0,\pi p] = \pi(E)[0,\pi p]$.

(vi) $\Gamma(\pi(E)) = \Gamma(E) \cap \pi(E)$.

Lemma 4.9. *If* $\pi \in \Gamma_{\text{ex}}(E)$ *and* $k \in E$, *then* $\pi|_{E[0,k]} \in \Gamma_{\text{ex}}(E[0,k])$.

Proof. We prove that $\pi|_{E[0,k]}$ satisfies (EXC1)–(EXC4) for the PEA $E[0,k]$. Let $a, b \in E[0,k]$. We have $\pi|_{E[0,k]}a = \pi a \leq a \leq k$, so $\pi|_{E[0,k]}\colon E[0,k] \to E[0,k]$. To prove (EXC1), suppose that $a \oplus_k b = a \oplus b \leq k$. Then $\pi|_{E[0,k]}(a \oplus_k b) = \pi(a \oplus b) = \pi(a) \oplus \pi(b) \leq a \oplus b \leq k$, so $\pi|_{E[0,k]}$ is a GPEA-endomorphism of $E[0,k]$. Conditions (EXC2) and (EXC3) hold trivially. To prove (EXC4), suppose that $\pi|_{E[0,k]}a = \pi a = a$ and $\pi|_{E[0,k]}b = \pi b = 0$. Then $a \perp b$, so $a \oplus b = b \oplus a$. Also $\pi' b = b$, and by Lemma 3.6 (i) with $\xi := \pi'$, $a \oplus b = b \oplus a = a \vee b \leq k$. Therefore, $a \oplus_k b = a \oplus b = b \oplus a = b \oplus_k a$, i.e., a is orthogonal to b in $E[0,k]$, proving (EXC4). ■

5. Central orthocompleteness

Definition 5.1. We say that elements $e, f \in E$ are Γ_{ex}-*orthogonal* iff there are $\pi, \xi \in \Gamma_{\text{ex}}(E)$ such that $\pi \wedge \xi = 0$, $\pi e = e$ and $\xi f = f$. More generally, an arbitrary family $(e_i)_{i \in I}$ in E is Γ_{ex}-orthogonal iff there is a pairwise disjoint family $(\pi_i)_{i \in I}$ in $\Gamma_{\text{ex}}(E)$ such that $\pi_i e_i = e_i$ for all $i \in I$.

As is easily seen, elements $e, f \in E$ are Γ_{ex}-orthogonal iff there is a direct sum decomposition $E = S \oplus S'$ such that $e \in S$ and $f \in S'$.

Lemma 5.2.

(i) *A finite family* $(e_i)_{i=1}^n$ *in* E *is pairwise* $\Gamma_{\rm ex}$*-orthogonal iff it is* $\Gamma_{\rm ex}$*-orthogonal and then it is orthogonal with* $\oplus_{i=1}^n e_i = \bigvee_{i=1}^n e_i$.

(ii) *If an arbitrary family* $(e_i)_{i\in I} \in E$ *is* $\Gamma_{\rm ex}$*-orthogonal, then it is orthogonal and it is orthosummable iff its supremum exists in* E*, in which case* $\oplus_{i\in I} e_i = \bigvee_{i\in I} e_i$.

Proof. (i) Clearly, a subfamily of a $\Gamma_{\rm ex}$-orthogonal family is $\Gamma_{\rm ex}$-orthogonal. It is also clear from the definition, that every $\Gamma_{\rm ex}$-orthogonal family is pairwise $\Gamma_{\rm ex}$-orthogonal. We prove both the converse and orthogonality by induction on n. For $n = 1$ the assertion obviously holds. Suppose now the statement holds for $(n-1)$ elements, $n > 1$ and assume that $(e_i)_{i=1}^n$ is a pairwise $\Gamma_{\rm ex}$-orthogonal family. Then by the induction hypotheses, $(e_i)_{i=1}^{n-1}$ is orthogonal, $\oplus_{i=1}^{n-1} e_i = \bigvee_{i=1}^{n-1} e_i$, and there exist pairwise disjoint mappings $\xi_i \in \Gamma_{\rm ex}(E)$ with $\xi_i e_i = e_i$ for $i = 1, 2, \ldots, n-1$. Moreover, e_i and e_n are $\Gamma_{\rm ex}$-orthogonal for $i = 1, 2, \ldots, n-1$; hence there exist $\alpha_i, \beta_i \in \Gamma_{\rm ex}(E)$ with $\alpha_i \wedge \beta_i = 0$, $\alpha_i e_i = e_i$, and $\beta_i e_n = e_n$. For $i = 1, 2, \ldots, n-1$, put $\pi_i := \xi_i \wedge \alpha_i$ and put $\pi_n := \bigwedge_{i=1}^{n-1} \beta_i$. Then $\pi_i \in \Gamma_{\rm ex}(E)$ are pairwise disjoint and $\pi_i e_i = e_i$ for $i = 1, 2, \ldots, n$, so the family $(e_i)_{i=1}^n$ is $\Gamma_{\rm ex}$-orthogonal. We now put $\pi := \bigvee_{i=1}^{n-1} \pi_i$ to get $\pi \wedge \pi_n = 0$, $\pi(\oplus_{i=1}^{n-1} e_i) = \oplus_{i=1}^{n-1} \pi e_i = \oplus_{i=1}^{n-1} e_i$, and $\pi_n e_n = e_n$; hence by Lemma 3.6 (i), $(\oplus_{i=1}^{n-1} e_i) \perp e_n$ and $\oplus_{i=1}^n e_i = (\oplus_{i=1}^{n-1} e_i) \oplus e_n = (\vee_{i=1}^{n-1} e_i) \vee e_n = \vee_{i=1}^n e_i$.

(ii) If $(e_i)_{i\in I}$ is $\Gamma_{\rm ex}$-orthogonal, then every finite subfamily is $\Gamma_{\rm ex}$-orthogonal and by (i), $\oplus_{i\in F} e_i = \bigvee_{i\in F} e_i$, where F is any finite subset of I. Therefore $\bigvee_{i\in I} e_i = \bigvee_F (\bigvee_{i\in F} e_i) = \bigvee_F (\oplus_{i\in F} e_i) = \oplus_{i\in I} e_i$. ■

Lemma 5.3.

(i) $c, d \in \Gamma(E)$ *are* $\Gamma_{\rm ex}$*-orthogonal iff* $\pi_c \wedge \pi_d = 0$ *iff* $c \perp d$ *iff* $c \wedge d = 0$.

(ii) *A family of central elements is* $\Gamma_{\rm ex}$*-orthogonal iff it is orthogonal iff it is pairwise orthogonal iff it is pairwise disjoint.*

Proof. (i) If $\pi_c \wedge \pi_d = 0$, then c and d are $\Gamma_{\rm ex}$-orthogonal by definition. If c, d are $\Gamma_{\rm ex}$-orthogonal, then there exist $\pi, \xi \in \Gamma_{\rm ex}(E)$ such that $\pi c = c$, $\xi d = d$ and $\pi \wedge \xi = 0$. But $\pi_c \le \pi$ and $\pi_d \le \xi$ by Theorem 4.7 (vii), thus π_c and π_d are disjoint too. The remaining equivalences follow from Theorem 4.6 (v).

(ii) If the family $(c_i)_{i\in I}$ of central elements in E is $\Gamma_{\rm ex}$-orthogonal, then by Lemma 5.2 (ii) it is orthogonal. If it is orthogonal, then by the definition of orthogonality it is pairwise orthogonal. If it is pairwise orthogonal, then by Theorem 4.6 (v) it is pairwise disjoint. Finally, suppose that $(c_i)_{i\in I}$ is pairwise disjoint. Then by Theorem 4.6 (v) again, $(\pi_{c_i})_{i\in I}$ is a pairwise disjoint family in $\Gamma_{\rm ex}(E)$ such that $\pi_{c_i} c_i = c_i$ for all $i \in I$, so $(c_i)_{i\in I}$ is $\Gamma_{\rm ex}$-orthogonal. ■

Definition 5.4. The generalized pseudo-effect algebra E is *centrally orthocomplete* (COGPEA) iff it satisfies the following conditions:

(CO1) Every Γ_{ex}-orthogonal family in E is orthosummable, i.e. (Lemma 5.2 (ii)), it has a supremum) in E.

(CO2) If $e \in E$ is such that $e \oplus e_i$ (resp. $e_i \oplus e$) exists for every element of a Γ_{ex}-orthogonal family $(e_i)_{i\in I} \subset E$, then $e \oplus (\oplus_{i\in I} e_i)$ (resp. $(\oplus_{i\in I} e_i) \oplus e$) exists in E.

Theorem 5.5. *Let E be a COGPEA and $(\pi_i)_{i\in I}$ a pairwise disjoint family in $\Gamma_{\mathrm{ex}}(E)$. Let $(e_i)_{i\in I}$, $(f_i)_{i\in I}$ be families of elements in E such that $e_i \oplus f_i$ exists for all $i \in I$ and $e_i, f_i \in \pi_i(E)$. Then:*

(i) *$(e_i)_{i\in I}$, $(f_i)_{i\in I}$, and $(e_i \oplus f_i)_{i\in I}$ are Γ_{ex}-orthogonal, hence orthosummable.*

(ii) *$\oplus_{i\in I} e_i = \bigvee_{i\in I} e_i$, $\oplus_{i\in I} f_i = \bigvee_{i\in I} f_i$ and $\oplus_{i\in I}(e_i \oplus f_i) = \bigvee_{i\in I}(e_i \oplus f_i)$.*

(iii) *$(\oplus_{i\in I} e_i) \oplus (\oplus_{i\in I} f_i)$ exists.*

(iv) *$(\oplus_{i\in I} e_i) \oplus (\oplus_{i\in I} f_i) = \oplus_{i\in I}(e_i \oplus f_i) = \bigvee_{i\in I}(e_i \oplus f_i)$.*

Proof. Since e_i, f_i belong to $\pi_i(E)$ for every $i \in I$, so does $e_i \oplus f_i$. Thus (i) follows directly from (CO1) and the definition of Γ_{ex}-orthogonality, and (ii) is implied by Lemma 5.2 (ii).

(iii) Put $e := \oplus_{i\in I} e_i = \bigvee_{i\in I} e_i$ and $f := \oplus_{i\in I} f_i = \bigvee_{i\in I} f_i$. By hypotheses $e_i \oplus f_i$ exists for every $i \in I$, and for $i \neq j$, $e_i \oplus f_j$ also exists by Lemma 3.6 (i). Applying (CO2) we find that $e_i \oplus f$ exists for all $i \in I$, and applying (CO2) once more we conclude that $e \oplus f$ exists too.

(iv) As $e \oplus f$ exists, so does $e_i \oplus f$ for every $i \in I$, and therefore by Lemma 2.4, $e_i \oplus f = \bigvee_{j\in I}(e_i \oplus f_j)$. Therefore a second application of Lemma 2.4 yields

$$(1) \qquad e \oplus f = \Big(\bigvee_{i\in I} e_i\Big) \oplus f = \bigvee_{i\in I}(e_i \oplus f) = \bigvee_{i\in I}\bigvee_{j\in I}(e_i \oplus f_j) = \bigvee_{i,j\in I}(e_i \oplus f_j).$$

Also, by Lemma 3.6 (i), for all $i, j \in I$,

$$(2) \qquad i \neq j \Rightarrow e_i \oplus f_j = e_i \vee f_j \le (e_i \oplus f_i) \vee (e_j \oplus f_j) \le \bigvee_{i\in I}(e_i \oplus f_i).$$

Combining (1) and (2), and using (ii) above, we conclude that $e \oplus f = \bigvee_{i\in I}(e_i \oplus f_i) = \oplus_{i\in I}(e_i \oplus f_i)$. ∎

Theorem 5.6. *If E is a COGPEA and $(\pi_i)_{i\in I}$ is a pairwise disjoint family of elements in $\Gamma_{\rm ex}(E)$, then the supremum $\bigvee_{i\in I}\pi_i$ exists in the boolean algebra $\Gamma_{\rm ex}(E)$ and for every $e\in E$, $(\bigvee_{i\in I}\pi_i)e=\bigvee_{i\in I}\pi_i e=\oplus_{i\in I}\pi_i e$.*

Proof. Let $e,f\in E$ and $i,j\in I$. The family $(\pi_i)_{i\in I}$ is pairwise disjoint and $\pi_i(\pi_i e)=\pi_i e$ for every $i\in I$, whence $(\pi_i e)_{i\in I}$ is a $\Gamma_{\rm ex}$-orthogonal family in E. Thus by (CO1) $(\pi_i e)_{i\in I}$ is orthosummable with $\oplus_{i\in I}\pi_i e=\bigvee_{i\in I}\pi_i e$ (Lemma 5.2 (ii)). We define $\pi\colon E\to E$ by $\pi e:=\bigvee_{i\in I}\pi_i e=\oplus_{i\in I}\pi_i e$. It will be sufficient to prove that π is in $\Gamma_{\rm ex}(E)$ and that it is the supremum of $(\pi_i)_{i\in I}$ in $\Gamma_{\rm ex}(E)$.

Suppose $e\oplus f$ exists, so that $\pi_i(e\oplus f)=\pi_i e\oplus\pi_i f$ for all $i\in I$. In Theorem 5.5, put $e_i:=\pi_i e$ and $f_i:=\pi_i f$ for all $i\in I$ to infer that $\pi e\oplus\pi f$ exists and

$$\pi e\oplus\pi f=(\oplus_i\pi_i e)\oplus(\oplus_i\pi_i f)=\oplus_i(\pi_i e\oplus\pi_i f)=\oplus_i(\pi_i(e\oplus f))=\pi(e\oplus f),$$

which proves that π satisfies (EXC1). We also have $\pi_i(\pi e)=\pi_i\bigvee_{j\in I}\pi_j e=\bigvee_{j\in I}\pi_i\pi_j e=\pi_i e$ by Theorem 3.13 (i), whence $\pi(\pi e)=\bigvee_{i\in I}\pi_i(\pi e)=\bigvee_{i\in I}\pi_i e=\pi e$, proving (EXC2). Moreover, as $\pi_i e\le e$ for all $i\in I$, it follows that $\pi e=\bigvee_{i\in I}\pi_i e\le e$ and therefore (EXC3) holds. To prove (EXC4), suppose that $\pi e=e$ and $\pi f=0$. Then $\bigvee_{i\in I}\pi_i f=0$, so $\pi_i f=0$ for all $i\in I$. As $\pi_i(\pi_i e)=\pi_i e$, (EXC4) implies that $\pi_i e\perp f$ for every $i\in I$. But then, by (CO2), $e=\pi e\perp f$, and (EXC4) holds for π too.

Evidently, $\pi_i e\le\pi e$ for every $e\in E$, whence $\pi_i\le\pi$ for all $i\in I$. Also, if $\pi_i\le\xi\in\Gamma_{\rm ex}(E)$ for all $i\in I$, then $\pi_i e\le\xi e$, so $\pi e=\bigvee_{i\in I}\pi_i e\le\xi e$ for all $e\in E$ and thus $\pi\le\xi$. So $\pi=\bigvee_{i\in I}\pi_i$. ■

Since a boolean algebra is complete iff every pairwise disjoint subset has a supremum, Theorem 5.6 has the following corollary.

Corollary 5.7. *The exocenter $\Gamma_{\rm ex}(E)$ of a COGPEA E is a complete boolean algebra.*

We may now extend Theorem 5.6 in the same way as in [13, Theorem 6.9] for an arbitrary family $(\pi_i)_{i\in I}$ in the complete boolean algebra $\Gamma_{\rm ex}(E)$.

Theorem 5.8. *Suppose that E is a COGPEA, let $(\pi_i)_{i\in I}$ be a family in $\Gamma_{\rm ex}(E)$, and let $e\in E$. Then:*

(i) *$\bigvee_{i\in I}\pi_i e$ exists in E and $(\bigvee_{i\in I}\pi_i)e=\bigvee_{i\in I}\pi_i e$.*

(ii) *If $I\neq\emptyset$, then $\bigwedge_{i\in I}\pi_i e$ exists in E and $(\bigwedge_{i\in I}\pi_i)e=\bigwedge_{i\in I}\pi_i e$.*

The proof of the next theorem, which extends Theorem 3.9 to arbitrary direct sums, is analogous to the proof of [13, Theorem 6.10].

Theorem 5.9. *Suppose that E is a COGPEA, let $(\pi_i)_{i\in I}$ be a pairwise disjoint family in the complete boolean algebra $\Gamma_{\rm ex}(E)$ with $\pi := \bigvee_{i\in I} \pi_i$, and consider the cartesian product $X := \bigtimes_{i\in I}\pi_i(E)$. Then each element in X is a $\Gamma_{\rm ex}$-orthogonal (hence orthosummable) family $(e_i)_{i\in I}$ and $\oplus_{i\in I}e_i = \bigvee_{i\in I} e_i$. Define the mapping $\Phi : X \to \pi(E)$ by $\Phi((e_i)_{i\in I}) := \oplus_{i\in I}e_i$. Then Φ is a GPEA-isomorphism of X onto $\pi(E)$ and if $e \in \pi(E)$, then $\Phi^{-1}e = (\pi_i e)_{i\in I} \in X$.*

Corollary 5.10. *Let E be a COGPEA, let $(p_i)_{i\in I}$ be a nonempty $\Gamma_{\rm ex}$-orthogonal family in E with $p := \bigvee_{i\in I} p_i$, let $(\pi_i)_{i\in I}$ be a corresponding family of pairwise disjoint mappings in $\Gamma_{\rm ex}(E)$ such that $p_i = \pi_i p_i$ for all $i \in I$, and let X be the cartesian product $X := \bigtimes_{i\in I}E[0, p_i]$. Then:*

(i) *If $(e_i)_{i\in I} \in X$, then $e_i = \pi_i e_i$ for all $i \in I$, so $(e_i)_{i\in I}$ is a $\Gamma_{\rm ex}$-orthogonal, hence orthosummable family in E.*

(ii) *If $(e_i)_{i\in I} \in X$ with $e := \oplus_{i\in I}e_i$, then $\pi_i e = e_i$ for all $i \in I$. In particular, $\pi_i p = p_i$ for all $i \in I$.*

(iii) *If $e \in E[0, p]$, then $\pi_i e = e \wedge p_i$ for all $i \in I$, $(\pi_i e)_{i\in I} \in X$ and $\bigvee_{i\in I} \pi_i e = e$.*

(iv) *The mapping $\Phi : X \to E[0, p]$ defined by $\Phi((e_i)_{i\in I}) := \oplus_{i\in I}e_i = \bigvee_{i\in I} e_i$ is a PEA-isomorphism of X onto $E[0, p]$ and $\Phi^{-1}(e) = (\pi_i e)_i \in I \in X$ for all $e \in E[0, p]$.*

The following theorem can also be proved using the same arguments as in the proof of [13, Theorem 6.11]

Theorem 5.11. *Suppose that E is a COGPEA and $(c_i)_{i\in I}$ is a family of elements in the center $\Gamma(E)$ of E. Then:*

(i) *If $I \neq \emptyset$, then $c := \bigwedge_{i\in I} c_i$ exists in E, $c \in \Gamma(E)$, $\pi_c = \bigwedge_{i\in I} \pi_{c_i}$ and c is the infimum of $(c_i)_{i\in I}$ as calculated in $\Gamma(E)$.*

(ii) *If $(c_i)_{i\in I}$ is bounded above in E, then $d := \bigvee_{i\in I} c_i$ exists in E, $d \in \Gamma(E)$, $\pi_d = \bigvee_{i\in I} \pi_{c_i}$ and d is the supremum of $(c_i)_{i\in I}$ as calculated in $\Gamma(E)$.*

The next theorem extends the results obtained for centrally orthocomplete GEAs in [14, Lemma 7.5, Theorem 7.6]. Here we give a simplified proof.

Theorem 5.12. *Let E be a COGPEA. Then:*

(i) *There exists a largest element $u \in \Gamma(E)$ and $\Gamma(E) \subseteq \pi_u(E) = E[0, u]$.*

(ii) *The center $\Gamma(E)$ is a complete boolean algebra.*

Proof. (i) We apply Zorn's lemma to obtain a maximal pairwise disjoint family of nonzero elements $(c_i)_{i\in I} \subseteq \Gamma(E)$. (Note that $(c_i)_{i\in I}$ could be the empty family.) By Lemma 5.3, $(c_i)_{i\in I}$ is $\Gamma_{\rm ex}$-orthogonal, and since E is a COGPEA, $u := \bigvee_{i\in I} c_i = \oplus_{i\in I} c_i$ exists in E. Thus the family $(c_i)_{i\in I}$ is bounded above by u in E, and we infer from Theorem 5.11 (ii) that $u \in \Gamma(E)$. Let $c \in \Gamma(E)$. Working in the generalized boolean algebra $\Gamma(E)$ (Theorem 4.7 (iv)), we have $c = (c \wedge u) \vee d$, where $d := c \ominus (c \wedge u) \in \Gamma(E)$. As $d \wedge u = 0$ and $c_i \leq u$, it follows that $d \wedge c_i = 0$ for all $i \in I$, whence $d = 0$ by the maximality of $(c_i)_{i\in I}$, and it follows that $c = c \wedge u \leq u$. Consequently, $\pi_c \leq \pi_u$, and therefore $c \in E[0, c] = \pi_c(E) \subseteq \pi_u(E) = E[0, u]$.

(ii) Since the generalized boolean algebra $\Gamma(E)$ has a unit (largest element), it is a boolean algebra, and it is complete by Theorem 5.11. ■

Theorem 5.13. *Let u be the unit (largest element) in the complete boolean algebra $\Gamma(E)$ of the COGPEA E. Then:*

(i) *The PEA $E[0, u] = \pi_u(E)$ is a direct summand of E and the complementary direct summand is $(\pi_u)'(E) = \{f \in E : f \perp u\} = \{e \ominus (u \wedge e) : e \in E\}$.*

(ii) *The center of $E[0, u]$ is $\Gamma(E)$, the complementary direct summand $(\pi_u)'(E)$ is centerless (i.e., its center is $\{0\}$), and no nonzero direct summand of $(\pi_u)'(E)$ is a PEA.*

(iii) *If $E = H \oplus K$ where the direct summand H is a PEA and K is centerless, then $H = E[0, u]$ and $K = \{f \in E : f \perp u\}$.*

Proof. As $u \in \Gamma(E)$, we have $\pi_u \in \Gamma_{\rm ex}(E)$ as per Definition 4.4, by Theorem 4.3, the PEA $E[0, u] = \pi_u(E)$ is a direct summand of E, and its complementary direct summand is $(\pi_u)'(E) = \{f \in E : f \perp u\}$. If $e \in E$, then by Theorem 4.6 (i), $\pi_u e = u \wedge e$, whence $(\pi_u)'e = \pi_u e/e = e\backslash\pi_u e = e \ominus \pi_u e = e \ominus (u \wedge e)$, and it follows that $(\pi_u)'(E) = \{e \ominus (u \wedge e) : e \in E\}$.

(ii) As a consequence of Theorem 5.12 (i), we have $\Gamma(E) \subseteq \pi_u(E) = E[0, u]$. Therefore, by Theorem 4.8 (vi), $\Gamma(E[0, u]) = \Gamma(\pi_u(E)) = \Gamma(E) \cap \pi_u(E) = \Gamma(E)$. Also by Theorem 4.8 (vi), $\Gamma((\pi_u)'(E)) = \Gamma(E) \cap (\pi_u)'(E) \subseteq \pi_u(E) \cap (\pi_u)'(E) = \{0\}$.

(iii) Assume the hypotheses of (iii). By Theorem 3.10, there exists $\pi \in \Gamma_{\rm ex}(E)$ with $\pi(E) = H$, so $K = \pi'(E)$. Since H is a PEA, there is a largest element $c \in H = \pi(E)$; hence by Corollary 4.5, $H = \pi(E) = E[0, c]$, $c \in \Gamma(E)$, $\pi = \pi_c$, and $K = (\pi_c)'(E) = \{f \in E : f \perp c\}$. Also, since u is the largest element in $\Gamma(E)$, we have $c \leq u$, whence $u \ominus c \in \Gamma(E)$ by Theorem 4.6 (x). Furthermore, $(u \ominus c) \perp c$, therefore $u \ominus c \in K$, and by Theorem 4.8 (vi) we have $u \ominus c \in \Gamma(E) \cap K = \Gamma(E) \cap (\pi_c)'(E) = \Gamma((\pi_c)'(E)) = \Gamma(K)$. Consequently, as K is centerless, $u \ominus c = 0$, so $c = u$, $H = E[0, u]$, and $K = \{f \in E : f \perp u\}$. ■

6. The exocentral cover

Definition 6.1. If $e \in E$, and if there is the smallest mapping in the set $\{\pi \in \Gamma_{\text{ex}}(E) : \pi e = e\}$, we will refer to it as *exocentral cover* of e and denote it by γ_e. If every element of E has an exocentral cover, we say that the family $(\gamma_e)_{e\in E}$ is the *exocentral cover system* for E, and in this case, we also denote the set of all mappings in the exocentral cover system by $\Theta_\gamma := \{\gamma_e : e \in E\}$. (We note that it is quite possible to have $\gamma_e = \gamma_f$ with $e \neq f$.)

Theorem 6.2. *If E is a COGPEA, then the exocentral cover γ_e exists for every $e \in E$ and $\gamma_e = \bigwedge\{\pi \in \Gamma_{\text{ex}}(E) : \pi e = e\} \in \Gamma_{\text{ex}}(E)$.*

Proof. Let $e \in E$ and put $\gamma := \bigwedge\{\pi : \pi \in \Gamma_{\text{ex}}(E), \pi e = e\}$. As the identity mapping 1 is in the set $\{\pi \in \Gamma_{\text{ex}}(E) : \pi e = e\}$, it is nonempty, and by Theorem 5.8 (ii),

$$\gamma e = \Big(\bigwedge\{\pi : \pi \in \Gamma_{\text{ex}}(E), \pi e = e\}\Big)e = \bigwedge\{\pi e : \pi \in \Gamma_{\text{ex}}(E), \pi e = e\} = e.$$

Therefore, γ is the smallest mapping in the set $\{\pi \in \Gamma_{\text{ex}}(E) : \pi e = e\}$, so $\gamma_e = \gamma$. ■

Theorem 6.3. *Let E be a COGPEA and $e, f \in E$. Then:*

(i) $\gamma_0 = 0$.

(ii) $\gamma_e e = e$.

(iii) $e \leq f \Rightarrow \gamma_e \leq \gamma_f$.

(iv) *If $e \oplus f$ exists, then* $\gamma_{e\oplus f} = \gamma_e \vee \gamma_f$.

(v) $\gamma_{\gamma_e f} = \gamma_e \circ \gamma_f = \gamma_e \wedge \gamma_f$.

(vi) $\gamma_{(\gamma_e)' f} = (\gamma_e)' \circ \gamma_f = (\gamma_e)' \wedge \gamma_f$.

(vii) $\gamma_e \wedge \gamma_f \in \Theta_\gamma$.

(viii) $(\gamma_e)' \wedge \gamma_f \in \Theta_\gamma$.

Proof. Parts (i) and (ii) are obvious from Definition 6.1.

(iii) If $e \leq f = \gamma_f f$, then by Theorem 3.3 (iii), $\gamma_f e = e$. But since γ_e is the smallest mapping in $\Gamma_{\text{ex}}(E)$ that fixes e, it follows that $\gamma_e \leq \gamma_f$.

(iv) Suppose that $e \oplus f$ exists. We have $(\gamma_e \vee \gamma_f)e = \gamma_e e \vee \gamma_f e = e \vee \gamma_f e = e$ because $\gamma_f e \leq e$. Similarly $(\gamma_e \vee \gamma_f)f = f$. Thus $(\gamma_e \vee \gamma_f)(e \oplus f) = (\gamma_e \vee \gamma_f)e \oplus (\gamma_e \vee \gamma_f)f = e \oplus f$, and so $\gamma_{e\oplus f} \leq \gamma_e \vee \gamma_f$. On the other hand, $e, f \leq e \oplus f$, so by (iii), $\gamma_e, \gamma_f \leq \gamma_{e\oplus f}$ and thus $\gamma_e \vee \gamma_f \leq \gamma_{e\oplus f}$.

(v) Since $\gamma_e \in \Gamma_{\rm ex}(E)$, $\gamma_e(\gamma_e f) = \gamma_e f$ and $\gamma_f(\gamma_e f) = \gamma_e(\gamma_f f) = \gamma_e f$. Therefore $\gamma_{\gamma_e f} \le \gamma_e \wedge \gamma_f = \gamma_e \circ \gamma_f$. To prove the reverse inequality, consider $f = \gamma_e f \oplus (\gamma_e)' f$ and (iv) to obtain $\gamma_f = \gamma_{\gamma_e f} \vee \gamma_{(\gamma_e)' f}$. Also $(\gamma_e)'((\gamma_e)' f) = (\gamma_e)' f$, and as $\gamma_{(\gamma_e)' f}$ is the smallest mapping in $\Gamma_{\rm ex}(E)$ that fixes $(\gamma_e)' f$, we have $\gamma_{(\gamma_e)' f} \le (\gamma_e)'$. But then $\gamma_e \wedge \gamma_{(\gamma_e)' f} = 0$ and thus $\gamma_e \circ \gamma_f = \gamma_e \wedge \gamma_f = (\gamma_e \wedge \gamma_{\gamma_e f}) \vee (\gamma_e \wedge \gamma_{(\gamma_e)' f}) = \gamma_e \wedge \gamma_{\gamma_e f} \le \gamma_{\gamma_e f}$.

(vi) By (v), $(\gamma_e)' \wedge \gamma_{\gamma_e f} = (\gamma_e)' \wedge \gamma_e \wedge \gamma_f = 0$. Also, as in the proof of (v), we have $\gamma_f = \gamma_{(\gamma_e)' f} \vee \gamma_{\gamma_e f}$ and $\gamma_{(\gamma_e)' f} \le (\gamma_e)'$. Therefore, $(\gamma_e)' \wedge \gamma_f = [(\gamma_e)' \wedge \gamma_{(\gamma_e)' f}] \vee [(\gamma_e)' \wedge \gamma_{\gamma_e f}] = \gamma_{(\gamma_e)' f} \vee 0 = \gamma_{(\gamma_e)' f}$.

Parts (vii) and (viii) follow immediately from parts (v) and (vi). ■

Corollary 6.4. *With the partial order inherited from* $\Gamma_{\rm ex}(E)$, $\Theta_\gamma = \{\gamma_e : e \in E\}$ *is a generalized boolean algebra.*

Proof. By [15, Theorem 3.2] with $B := \Gamma_{\rm ex}(E)$ and $L := \Theta_\gamma$, it will be sufficient to prove that, for all $e, f \in E$, (i) $\Theta_\gamma \ne \emptyset$, (ii) $e, f \in E \Rightarrow (\gamma_e)' \wedge \gamma_f \in \Theta_\gamma$, and (iii) $\gamma_e \wedge \gamma_f = 0 \Rightarrow \gamma_e \vee \gamma_f \in \Theta_\gamma$. Condition (i) is obvious and (ii) follows from Theorem 6.3 (viii). To prove (iii), suppose that $\gamma_e \wedge \gamma_f = 0$. Then, as $e = \gamma_e e$ and $f = \gamma_f f$, Lemma 3.6 (i) implies that $e \perp f$; hence by Theorem 6.3 (iv), $\gamma_e \vee \gamma_f = \gamma_{e \oplus f} \in \Theta_\gamma$, proving (iii). ■

The following definition, originally formulated for a generalized effect algebra (GEA) [13, Definition 7.1] as a generalization of the notion of a hull mapping on an effect algebra [12, Definition 3.1], extends to the GPEA E the notion of a so-called hull system.

Definition 6.5. A family $(\eta_e)_{e \in E}$ is a *hull system* for E iff (1) $\eta_0 = 0$, (2) $e \in E \Rightarrow \eta_e e = e$, and (3) $e, f \in E \Rightarrow \eta_{\eta_e f} = \eta_e \circ \eta_f$. If $(\eta_e)_{e \in E}$ is a hull system for E, then an element $e \in E$ is *η-invariant* iff $\eta_e f = e \wedge f$ for all $f \in E$.

Theorem 6.6. *If E is a COGPEA, then $(\gamma_e)_{e \in E}$ is a hull system for E, the center $\Gamma(E)$ is precisely the set of γ-invariant elements in E, and for $c \in \Gamma(E)$, $\gamma_c = \pi_c$.*

Proof. That $(\gamma_e)_{e \in E}$ is a hull system for E follows from parts (i), (ii), and (v) of Theorem 6.3, and the remainder of the theorem follows from parts (i), (vii), and (viii) of Theorem 4.6. ■

Theorem 6.7. *Let E be a COGPEA and $(e_i)_{i \in I} \subseteq E$. Then the family $(e_i)_{i \in I}$ is $\Gamma_{\rm ex}$-orthogonal iff $\gamma_{e_i} \wedge \gamma_{e_j} = 0$ for all $i, j \in I$, $i \ne j$.*

Proof. If $(\gamma_{e_i})_{i \in I}$ is pairwise disjoint, then since $\gamma_{e_i} e_i = e_i$, it follows that $(e_i)_{i \in I}$ is $\Gamma_{\rm ex}$-orthogonal. Conversely, suppose that $(e_i)_{i \in I}$ is $\Gamma_{\rm ex}$-orthogonal. Then there exists a pairwise disjoint family $(\pi_i)_{i \in I} \in \Gamma_{\rm ex}(E)$ such that $\pi_i e_i = e_i$ for all $i \in I$.

But then $\gamma_{e_i} \leq \pi_i$ for all $i \in I$, and therefore the family $(\gamma_{e_i})_{i\in I}$ is also pairwise disjoint. ■

In view of Theorem 6.7, a Γ_{ex}-orthogonal family of elements of the COGPEA E will also be called *γ-orthogonal.*

7. TYPE DETERMINING SETS

Definition 7.1. Let E be a COGPEA and $Q, K \subseteq E$. Then we consider four closure operators on the set of all subsets Q of E:

(1) $[Q]_\gamma$ is the set of all orthosums (suprema) of γ-orthogonal families in Q, with the understanding that $[\emptyset]_\gamma = \{0\}$.

(2) $Q^\gamma := \{\gamma_e q : e \in E, q \in Q\}$.

(3) $Q^\downarrow := \bigcup_{q\in Q} E[0, q]$.

(4) $Q'' := (Q')'$, where $Q' := \{e \in E : q \wedge e = 0 \text{ for all } q \in Q\}$.

We say that

(5) K is *type-determining* (TD) set iff $K = [K]_\gamma = K^\gamma$.

(6) K is *strongly type-determining* (STD) set iff $K = [K]_\gamma = K^\downarrow$.

We note that $Q \subseteq Q''$, $P \subseteq Q \Rightarrow Q' \subseteq P'$, and $Q' = Q'''$.

Theorem 7.2. *Let E be a COGPEA and let $Q, K \subseteq E$. Then:*

(i) *If $q \in [Q]_\gamma$, then there is a γ-orthogonal family $(q_i)_{i\in I}$ in Q such that $q = \oplus_{i\in I} q_i = \bigvee_{i\in I} q_i$; moreover, if $e \leq q$, then $(e \wedge q_i)_{i\in I}$ is a γ-orthogonal family in $Q^\downarrow$ and $e = \oplus_{i\in I}(e \wedge q_i) = \bigvee_{i\in I}(e \wedge q_i)$.*

(ii) *$[K^\gamma]_\gamma$ is the smallest TD subset of E containing K.*

(iii) *$[K^\downarrow]_\gamma$ is the smallest STD subset of E containing K.*

(iv) *$K' = (K')^\downarrow = (K^\downarrow)'$ is STD.*

(v) *$K' = ([K^\gamma]_\gamma)' = ([K^\downarrow]_\gamma)'$.*

Proof. (i) By the definition of $[Q]_\gamma$, there exists a family $(q_i)_{i\in I}$ in Q such that $(\gamma_{q_i})_{i\in I}$ is a pairwise disjoint family in $\Gamma_{\rm ex}(E)$ and $q = \oplus_{i\in I} q_i = \bigvee_{i\in I} q_i$. By Theorem 3.13 (i), for each $i \in I$, $\gamma_{q_i} q = \gamma_{q_i}(\bigvee_{j\in I} q_j) = \bigvee_{j\in I} \gamma_{q_i} q_j = \bigvee_{j\in I} \gamma_{q_i}(\gamma_{q_j} q_j) = q_i$. Therefore, as $e \le q$, we can apply Theorem 3.3 (iv) to obtain $\gamma_{q_i} e = e \wedge \gamma_{q_i} q = e \wedge q_i \in Q^{\downarrow}$. By Theorem 6.3 (iii), $\gamma_{e\wedge q_i} \le \gamma_{q_i}$, so the family $(e \wedge q_i)_{i\in I}$ is γ-orthogonal. Let us define $\pi := \bigvee_{i\in I} \gamma_{q_i}$ in the complete boolean algebra $\Gamma_{\rm ex}(E)$. Then by Theorem 5.6, $\pi q = \bigvee_{i\in I} \gamma_{q_i} q = \bigvee_{i\in I} q_i = q$, hence, as $e \le q \in \pi(E)$, it follows by Theorems 3.3 (iii) and 5.6 that $e = \pi e = \bigvee_{i\in I} \gamma_{q_i} e = \bigvee_{i\in I}(e \wedge q_i)$.

(ii) From the definition it is clear that $[K^\gamma]_\gamma$ is contained in every TD set containing K. It is also easily seen that $K \subseteq [K^\gamma]_\gamma$ and $[[K^\gamma]_\gamma]_\gamma \subseteq [K^\gamma]_\gamma$. To prove that $([K^\gamma]_\gamma)^\gamma \subseteq [K^\gamma]_\gamma$, let $e \in ([K^\gamma]_\gamma)^\gamma$. Then there exists $h \in E$ and $q \in [K^\gamma]_\gamma$ with $e = \gamma_h q \le q$. By (i) with $Q := K^\gamma$, we find that there exists a γ-orthogonal family $(q_i)_{i\in I}$ in K^γ such that $q = \bigvee_{i\in I} q_i$ and $e = \bigvee_{i\in I}(e \wedge q_i)$. Thus, as $q_i \le q$ for all $i \in I$, Theorem 3.3 (iv) implies that $\gamma_h q_i = q_i \wedge \gamma_h q = q_i \wedge e$ for all $i \in I$. Also, as $q_i \in K^\gamma$ for every $i \in I$, there exist $h_i \in E$ and $k_i \in K$ such that $q_i = \gamma_{h_i} k_i$, and we have $e \wedge q_i = \gamma_h q_i = \gamma_h \gamma_{h_i} k_i = (\gamma_h \wedge \gamma_{h_i}) k_i = \gamma_{\gamma_h h_i} k_i \in K^\gamma$. Therefore the elements of the γ-orthogonal family $(e \wedge q_i)_{i\in I}$ all belong to K^γ and so $e \in [K^\gamma]_\gamma$.

We omit the proof of (iii) as it is similar to the proof of (ii).

(iv) Evidently, $K' = (K')^{\downarrow} = (K^{\downarrow})'$. It remains to prove that $[K']_\gamma \subseteq K'$. Let $q \in [K']_\gamma$, $k \in K$, and $e \in E$ with $e \le q, k$. By (i) with $Q := K'$, there are γ-orthogonal families $(q_i)_{i\in I} \subseteq K'$ and $(e \wedge q_i)_{i\in I}$ such that $q = \bigvee_{i\in I} q_i$ and $e = \bigvee_{i\in I}(e \wedge q_i)$. Since $e \le k$ and $k \wedge q_i = 0$, it follows that $e \wedge q_i = 0$ for all $i \in I$, so $e = 0$. Thus $q \wedge k = 0$, whence $q \in K'$.

(v) We have $K \subseteq K''$ and as $K'' = (K')'$, it is STD by (iv), hence it is TD. But then by (ii), $[K^\gamma]_\gamma \subseteq K''$, therefore $K' \subseteq ([K^\gamma]_\gamma)'$. We also get $([K^\gamma]_\gamma)' \subseteq K'$ because $K \subseteq [K^\gamma]_\gamma$. Similarly, $K \subseteq [K^{\downarrow}]_\gamma$, whence $([K^{\downarrow}]_\gamma)' \subseteq K'$ and by (iv) and (iii), $[K^{\downarrow}]_\gamma \subseteq K''$; hence $K' = K''' \subseteq ([K^{\downarrow}]_\gamma)'$. ■

Corollary 7.3. *If A (which may be empty) is the set of all atoms in E, then the STD set A' is the set of all elements in E that dominate no atom in E, and the STD set A'' is the set of all elements $p \in E$ such that either $p = 0$ or the PEA $E[0,p]$ is atomic.*

Theorem 7.4. *The set $\Gamma(E)$ of central elements of a COGPEA E is a TD subset of E.*

Proof. Obviously $\Gamma(E) \subseteq [\Gamma(E)]_\gamma$ and by theorem 5.12 (ii), $[\Gamma(E)]_\gamma \subseteq \Gamma(E)$. To prove that $\Gamma(E)^\gamma \subseteq \Gamma(E)$, let $c_1 \in \Gamma(E)^\gamma$, so that $c_1 := \gamma_e c$ for some $e \in E$ and $c \in \Gamma(E)$. We claim that c_1 is the greatest element of $\gamma_{c_1}(E)$; hence by Corollary 4.5, it is a central element of E. Indeed, if $f \in \gamma_{c_1}(E)$, then $f = \gamma_{c_1} f = \gamma_{\gamma_e c} f = \gamma_e(\gamma_c f) = \gamma_e(c \wedge f) \le \gamma_e c = c_1$ by Theorem 6.3 (v) and Theorem 4.6 (i). ■

Definition 7.5. A nonempty class $\mathcal{K}$ of PEAs is called a *type class* iff the following conditions are satisfied: (1) $\mathcal{K}$ is closed under the passage to direct summands. (2) $\mathcal{K}$ is closed under the formation of arbitrary nonempty direct products. (3) If E_1 and E_2 are isomorphic PEAs and E_1 is in $\mathcal{K}$, then $E_2 \in \mathcal{K}$. If, in addition to (2) and (3), $\mathcal{K}$ satisfies (1′) $H \in \mathcal{K}, h \in H \Rightarrow H[0,h] \in \mathcal{K}$, then $\mathcal{K}$ is called a *strong type class.*

Theorem 7.6. *Let $\mathcal{K}$ be a type class of PEAs and define $K := \{k \in E : E[0,k] \in \mathcal{K}\}$. Then K is a TD subset of E, and if $\mathcal{K}$ is a strong type class, then K is STD.*

Proof. Suppose $k \in K$ and $e \in E$. Then $E[0,k] \in \mathcal{K}$, $\gamma_e \in \Gamma_{\rm ex}(E)$, and by Lemma 4.9, $\gamma_e|_{E[0,k]} \in \Gamma_{\rm ex}(E[0,k])$. Thus by Theorem 4.8 (v) and Definition 7.5 (1), $E[0,\gamma_e k] = \gamma_e(E[0,k]) = \gamma_e|_{E[0,k]}(E[0,k]) \in \mathcal{K}$, so $K^\gamma \subseteq K$. If $\mathcal{K}$ is a strong type class, it is clear, that $K^\downarrow \subseteq K$. Finally, suppose that $k \in [K]_\gamma$. Then there exists a γ-orthogonal family $(k_i)_{i\in I}$ in K such that $k = \bigvee_{i\in I} k_i$. Thus by Definition 7.5 (2), $X := \bigtimes_{i\in I} E[0,k_i] \in \mathcal{K}$ and by Corollary 5.10, X is PEA-isomorphic to $E[0,k]$, whence by Definition 7.5 (3), $E[0,k] \in \mathcal{K}$, and therefore $k \in K$. ∎

Example 7.7. The class $\mathcal{K}$ of all EAs is a strong type class of PEAs; hence by Theorem 7.6, the set K of all elements $k \in E$ such that $E[0,k]$ is an EA is an STD subset of E.

Standing Assumption:
From now on we will assume that K is a TD subset of the COGPEA E.

Definition 7.8. $\widetilde{K} := K \cap \Gamma(E)$.

Theorem 7.9. *There exists $k^* \in K$ such that γ_{k^*} is the largest mapping in $\{\gamma_k : k \in K\} = \{\gamma_e : e \in E, e \le k^*\} = \Theta_\gamma[0,\gamma_{k^*}]$, which is a sublattice of $\Gamma_{\rm ex}(E)$, and as such, it is a boolean algebra. Moreover, $\widetilde{K}$ is a TD subset of E, there exists $\widetilde{k} \in \widetilde{K}$ such that $\gamma_{\widetilde{k}}$ is the largest mapping in $\{\gamma_k : k \in \widetilde{K}\} = \{\gamma_e : e \in E, e \le \widetilde{k}\} = \Theta_\gamma[0,\gamma_{\widetilde{k}}]$, which is a sublattice of $\Gamma_{\rm ex}(E)$, and as such, it is a boolean algebra.*

Proof. Let us take a maximal γ-orthogonal family $(k_i)_{i\in I} \subseteq K$ and set $k^* := \bigvee_{i\in I} k_i$. Then $k^* \in K$, because K is TD subset of E. Let $k \in K$. As $\gamma_k k = k$ and $(\gamma_{k^*})'k \le k$, we have $(\gamma_k \wedge (\gamma_{k^*})')k = \gamma_k k \wedge (\gamma_{k^*})'k = k \wedge (\gamma_{k^*})'k = (\gamma_{k^*})'k$. Also, by Theorem 6.3 (vi), $\gamma_k \wedge (\gamma_{k^*})' = \gamma_d$, where $d := (\gamma_{k^*})'k$, and since $K^\gamma \subset K$, it follows that $\widehat{k} := (\gamma_{k^*})'k = \gamma_d k \in K$ with $\gamma_{k^*}\widehat{k} = \gamma_{k^*}((\gamma_{k^*})'k) = 0$. Therefore, by Theorem 6.3 (v), $\gamma_{\widehat{k}} \wedge \gamma_{k^*} = \gamma_{\gamma_{k^*}\widehat{k}} = 0$, and since $k_i \le k^*$, it follows that $\gamma_{\widehat{k}} \wedge \gamma_{k_i} = 0$ for all $i \in I$. Consequently, $(\gamma_{k^*})'k = \widehat{k} = 0$ by the maximality of $(k_i)_{i\in I}$, therefore $k = \gamma_{k^*}k$, whence $\gamma_k \le \gamma_{k^*}$.

Suppose $k \in K$ and put $e := \gamma_k k^*$. Then $e \leq k^*$ with $\gamma_k = \gamma_k \wedge \gamma_{k^*} = \gamma_{\gamma_k k^*} = \gamma_e$, whence $\{\gamma_k : k \in K\} \subseteq \{\gamma_e : e \in E, e \leq k^*\}$. If $e \in E$ and $e \leq k^*$, then $\gamma_e \leq \gamma_{k^*}$, so $\{\gamma_e : e \in E, e \leq k^*\} \subseteq \{\gamma_e : e \in E, \gamma_e \leq \gamma_{k^*}\} = \Theta_\gamma[0, \gamma_{k^*}]$. Finally, suppose $e \in E$ with $\gamma_e \leq \gamma_{k^*}$, and put $k := \gamma_e k^*$. Since K is TD, we have $k \in K$; moreover, $\gamma_e = \gamma_e \wedge \gamma_{k^*} = \gamma_k$, so $\Theta_\gamma[0, \gamma_{k^*}] \subseteq \{\gamma_k : k \in K\}$.

By Corollary 6.4, Θ_γ is a generalized boolean algebra; hence the interval $\Theta_\gamma[0, \gamma_{k^*}] = \{\pi \in \Theta_\gamma : 0 \leq \pi \leq \gamma_{k^*}\}$ is a boolean algebra with unit γ_{k^*}.

That $\widetilde{K}$ is a TD subset, follows from Theorem 7.4 and the fact that $\widetilde{K} = K \cap \Gamma(E)$. Thus we obtain the second part of the theorem by applying the first part to $\widetilde{K}$. ∎

Since $\gamma_{k^*} \in \Gamma_{\text{ex}}(E)$ is the largest element in $\{\gamma_k : k \in K\}$, it is uniquely determined by the TD set K. Likewise, $\widetilde{k}$ is uniquely determined by $\widetilde{K} = K \cap \Gamma(E)$, hence it also is uniquely determined by K, and we may formulate the following definition.

Definition 7.10. With the notation of Theorem 7.9, (1) $\gamma_K := \gamma_{k^*}$ and (2) $\gamma_{\widetilde{K}} := \gamma_{\widetilde{k}}$.

Corollary 7.11. $\Theta_\gamma[0, \gamma_K]$ *is a boolean algebra and we have*:

(i) $\gamma_{\widetilde{K}} \leq \gamma_K \in \Theta_\gamma[0, \gamma_K] \subseteq \Gamma_{\text{ex}}(E)$.

(ii) $\gamma_K = \bigvee_{k \in K} \gamma_k$.

(iii) γ_K *is the smallest mapping* $\pi \in \Gamma_{\text{ex}}(E)$ *such that* $K \subseteq \pi(E)$.

(iv) $\gamma_{\widetilde{K}} = \bigvee_{k \in \widetilde{K}} \gamma_k \in \Gamma_{\text{ex}}(E)$.

(v) $\gamma_{\widetilde{K}}$ *is the smallest mapping* $\pi \in \Theta_\gamma$ *such that* $\widetilde{K} \subseteq \pi(E)$.

Proof. (i) This is clear by Theorem 7.9, because $\{\gamma_k : k \in \tilde{K}\} \subseteq \{\gamma_k : k \in K\}$.

(ii) By Theorem 7.9, γ_K is the largest mapping in $\{\gamma_k : k \in K\}$, from which (ii) follows immediately.

(iii) First we show that $K \subseteq \gamma_K(E)$. Indeed, if $k \in K$, then $\gamma_k \leq \gamma_K$, so $k = \gamma_k k \leq \gamma_K k \leq k$, and therefore $k = \gamma_K k \in \gamma_K(E)$. Suppose $K \subseteq \pi(E)$ for some $\pi \in \Gamma_{\text{ex}}(E)$. Then, $k^* \in K \subseteq \pi(E)$, so $k^* = \pi k^*$. But γ_{k^*} is the smallest mapping in $\Gamma_{\text{ex}}(E)$ with the latter property, whence $\gamma_{k^*} \leq \pi$.

Proofs of (iv) and (v) are similar to (ii) and (iii) with $\tilde{K}$ instead of K. ∎

Definition 7.12. Let $\pi \in \Gamma_{\text{ex}}(E)$. Then:

(1) π is *type-K* iff there exists $k \in \widetilde{K}$ such that $\pi = \gamma_k$.

(2) π is *locally type-K* iff there exists $k \in K$ such that $\pi = \gamma_k$.

(3) π is *purely non-K* iff $\pi \wedge \gamma_K = 0$, i.e., iff $\pi \leq (\gamma_K)'$.

(4) π is *properly non-K* iff $\pi \wedge \gamma_{\widetilde{K}} = 0$, i.e., iff $\pi \leq (\gamma_{\widetilde{K}})'$.

Remark 7.13. Directly from Definition 7.12 and Corollary 7.11, we have the following for all $\pi, \xi \in \Gamma_{\rm ex}(E)$:

(i) If π is type-K, then π is locally type-K.

(ii) If π is purely non-K, then π is properly non-K.

(iii) If π is both type-K and properly non-K, then $\pi = 0$.

(iv) If π is both locally type-K and purely non-K, then $\pi = 0$.

(v) If $\xi \in \Theta_\gamma$ and π is type-K or locally type-K then so is $\pi \wedge \xi$.

(vi) If π is purely non-K or properly non-K, then so is $\pi \wedge \xi$.

(vii) If both π and ξ are type K, locally type K, purely non-K, or properly non-K, then so is $\pi \vee \xi$.

Theorem 7.14. *Let $\pi \in \Gamma_{\rm ex}(E)$. Then:*

(i) *π is type-K iff $\pi \in \Theta_\gamma$ and $\pi \leq \gamma_{\widetilde{K}}$.*

(ii) *If K is STD and π is type-K, then $\pi(E) \subseteq K$.*

(iii) *π is locally type-K iff $\pi \in \Theta_\gamma$ and $\pi \leq \gamma_K$.*

(iv) *If π is purely non-K, then $K \cap \pi(E) = \{0\}$*

(v) *if π is properly non-K, then $\widetilde{K} \cap \pi(E) = \{0\}$.*

Proof. (i) By Theorem 7.9 and Definition 7.10, $\{\gamma_k : k \in \widetilde{K}\} = \Theta_\gamma[0, \gamma_{\widetilde{K}}] = \{\gamma_e : e \in E, \gamma_e \leq \gamma_{\widetilde{K}}\}$, from which (i) follows immediately.

(ii) If π is type-K, then $\pi = \gamma_k$ for some $k \in K \cap \Gamma(E)$, whence by Theorem 6.6, $\pi = \gamma_k = \pi_k$, and therefore, since K is STD, $\pi(E) = E[0,k] \subseteq K$.

(iv) Suppose that π is purely non-K, i.e., $\pi \wedge \gamma_K = 0$. Thus if $k \in K$, then $\gamma_k \leq \gamma_K$, whence $\pi \wedge \gamma_k = 0$. Therefore, if $k \in K \cap \pi(E)$, then $k = k \wedge k = \pi k \wedge \gamma_k k = (\pi \wedge \gamma_k)k = 0$.

The proofs of (iii) and (v) are analogous to those of (i) and (iv). ■

Definition 7.15. An element $f \in E$ is *faithful* iff $\gamma_f = 1$.

As is easily seen, if $\pi \in \Gamma_{\rm ex}(E)$, then an element $f \in \pi(E)$ is faithful in the GPEA $\pi(E)$ iff $\gamma_f = \pi$.

Theorem 7.16. *Let $\pi \in \Theta_\gamma$ and put $k^\sharp := \pi k^*$, where $k^* \in K$ is the element in Theorem 7.9. Then $k^\sharp \in K \cap \pi(E)$ and the following conditions are mutually equivalent:*

(i) *π is locally type-K.*

(ii) *$k^\sharp$ is faithful in the direct summand $\pi(E)$ of E (i.e., $\gamma_{k^\sharp} = \pi$).*

(iii) *If $\xi \in \Theta_\gamma$ with $\xi \wedge \pi \neq 0$, then $k^\sharp$ has a nonzero component $0 \neq \xi k^\sharp$ in the direct summand $\xi(\pi(E))$ of the GPEA $\pi(E)$, and $\xi k^\sharp \in K$.*

Proof. As $\pi \in \Theta_\gamma$, there exists $d \in E$ with $\pi = \gamma_d$. Since K is TD and $k^* \in K$, we have $k^\sharp = \pi k^* = \gamma_d k^* \in K$. Also, $k^\sharp = \pi k^* \in \pi(E)$, whence $k^\sharp \in K \cap \pi(E)$.

(i) $\Rightarrow$ (ii): If $\pi = \gamma_d$ is locally type-K, then $\gamma_d \leq \gamma_K = \gamma_{k^*}$ so $\gamma_{k^\sharp} = \gamma_{\gamma_d k^*} = \gamma_d \wedge \gamma_{k^*} = \gamma_d = \pi$.

(ii) $\Rightarrow$ (iii): Assume (ii) and the hypotheses of (iii). Then $\xi k^\sharp = \xi \pi k^* \in \xi(\pi(E))$, $\gamma_{k^\sharp} = \pi$, there exists $e \in E$ with $\xi = \gamma_e$, and $0 \neq \xi \wedge \pi = \gamma_e \wedge \gamma_{k^\sharp} = \gamma_{\gamma_e k^\#} = \gamma_{\xi k^\#}$, so $\xi k^\# \neq 0$. Also, since K is TD and $k^\# \in K$, we have $\xi k^\# = \gamma_e k^\# \in K$.

(iii) $\Rightarrow$ (i): Assume (iii). We have $\pi = \gamma_d$, and since $k^\# \in K$, we also have $\gamma_{k^\#} \leq \gamma_K$; hence, by Theorem 7.14 (iii), it will be sufficient to show that $\gamma_d \leq \gamma_{k^\#}$. Aiming for a contradiction, we assume that $\gamma_d \not\leq \gamma_{k^\#}$, i.e., by Theorem 6.3 (vi), $\xi := \gamma_e = (\gamma_{k^\sharp})' \wedge \gamma_d \neq 0$, where $e := \gamma_{k^\#} d$. Then $\xi \leq \gamma_d = \pi$, so $\xi \wedge \pi = \xi \neq 0$. But $\xi \leq (\gamma_{k^\sharp})'$ implies $\xi k^\sharp = 0$, contradicting (iii). ■

Corollary 7.17. *If $\pi \in \Gamma_{\rm ex}(E)$ is locally type-K and $\xi \in \Theta_\gamma$ with $\xi \wedge \pi \neq 0$, then the direct summand $\xi(\pi(E))$ of $\pi(E)$ contains a nonzero element of K.*

Proof. The nonzero element $\xi k^\sharp \in K$ in Theorem 7.16 belongs to $\xi(\pi(E))$. ■

Lemma 7.18.

(i) *There exists a unique mapping $\pi \in \Theta_\gamma$, namely $\pi = \gamma_K$, such that π is locally type-K and π' is purely non-K.*

(ii) *There exists a unique mapping $\xi \in \Theta_\gamma$, namely $\xi = \gamma_{\widetilde{K}}$, such that ξ is type-K and ξ' is properly non-K.*

Proof. By Theorem 7.14 (iii), π is locally type-K iff $\pi \leq \gamma_K$ and by Definition 7.12 (3), π' is purely non-K iff $\pi' \wedge \gamma_K = 0$, i.e., iff $\gamma_K \leq \pi$, from which (i) follows. Similarly, (ii) follows from Theorem 7.14 (i) and Definition 7.12 (4). ■

8. TYPE-DECOMPOSITION OF COGPEA

We maintain our standing hypothesis that K is a TD subset of the COGPEA E. According to Lemma 7.18, we have two bipartite direct decompositions $E = \pi(E) \oplus \pi'(E)$ and $E = \xi(E) \oplus \xi'(E)$, corresponding to $\pi = \gamma_K$ and $\xi = \gamma_{\widetilde{K}}$. Thus we may decompose E into four direct summands:

$$E = (\pi \wedge \xi)(E) \oplus (\pi \wedge \xi')(E) \oplus (\pi' \wedge \xi)(E) \oplus (\pi' \wedge \xi')(E)$$

one of which, namely $(\pi' \wedge \xi)(E)$ is necessarily $\{0\}$, because by Corollary 7.11 (i), $\xi \leq \pi$. Therefore we have the following *fundamental direct decomposition theorem for a COGPEA E with a TD set $K \subseteq E$.*

Theorem 8.1. *There exist unique pairwise disjoint mappings $\pi_1, \pi_2, \pi_3 \in \Gamma_{\rm ex}(E)$, namely $\pi_1 = \gamma_{\widetilde{K}}, \pi_2 = \gamma_K \wedge (\gamma_{\widetilde{K}})'$, and $\pi_3 = (\gamma_K)'$, such that:*

(i) $\pi_1 \vee \pi_2 \vee \pi_3 = 1$ *so that* $E = \pi_1(E) \oplus \pi_2(E) \oplus \pi_3(E)$, *and*

(ii) π_1 *is type-K,* π_2 *is locally type-K but properly non-K, and* π_3 *is purely non-K.*

Proof. For the existence part of the theorem, put $\pi_1 = \gamma_{\widetilde{K}}, \pi_2 = \gamma_K \wedge (\gamma_{\widetilde{K}})'$, and $\pi_3 = (\gamma_K)'$. Obviously, π_1, π_2, and π_3 are pairwise disjoint, and since $\gamma_{\widetilde{K}} \leq \gamma_K$, it is clear that $\pi_1 \vee \pi_2 \vee \pi_3 = 1$. Evidently, $\pi_1 \in \Theta_\gamma$, and by Theorem 6.3 (viii), $\pi_2 \in \Theta_\gamma$. Thus, by Theorem 7.14 (i), π_1 is type K, and by Theorem 7.14 (iii) π_2 is locally type K. Also, by parts (3) and (4) of Definition 7.12, π_3 is purely non-K and π_2 is properly non-K.

To prove uniqueness, suppose that $\pi_{1a}, \pi_{2a}, \pi_{3a}$ are pairwise disjoint mappings in the boolean algebra $\Gamma_{\rm ex}(E)$ satisfying (i) and (ii). Then $\pi_{1a} \leq \gamma_{\tilde{K}}$ by Theorem 7.14 (i), $\pi_{2a} \leq \gamma_K \wedge (\gamma_{\tilde{K}})'$ by Theorem 7.14 (iii) and Definition 7.12 (4), and $\pi_{3a} \leq (\gamma_K)'$ by Definition 7.12 (3). Thus after an elementary boolean computation, we finally get $\pi_1 = \pi_{1a}$, $\pi_2 = \pi_{2a}$ and $\pi_3 = \pi_{3a}$. ■

In what follows we will obtain a decomposition of the COGPEA E into types I, II and III analogous to the type decomposition of a von Neumann algebra. We shall be dealing with two TD subsets K and F of E such that $K \subseteq F$. For the case in which E is the projection lattice of a von Neumann algebra, one takes K to be the set of abelian elements and F to be the set of finite elements in E.

Thus, in what follows, assume that K and F are TD subsets of the COGPEA E such that $K \subseteq F$. By Theorem 8.1, we decompose E as

$E = \pi_1(E) \oplus \pi_2(e) \oplus \pi_3(E)$ and also as $E = \xi_1(E) \oplus \xi_2(E) \oplus \xi_3(E)$ where

$$\pi_1 = \gamma_{\widetilde{K}},\ \pi_2 = \gamma_K \wedge (\gamma_{\widetilde{K}})',\ \pi_3 = (\gamma_K)',$$

$$\xi_1 = \gamma_{\widetilde{F}},\ \xi_2 = \gamma_F \wedge (\gamma_{\widetilde{F}})',\ \xi_3 = (\gamma_F)'.$$

As $K \subseteq F$, it is clear that $\gamma_K \leq \gamma_F$, $\gamma_{\widetilde{K}} \leq \gamma_{\widetilde{F}}$, $(\gamma_F)' \leq (\gamma_K)'$, and $(\gamma_{\widetilde{F}})' \leq (\gamma_{\widetilde{K}})'$.

Applying Theorem 8.1, we obtain a direct sum decomposition

$$E = \tau_{11}(E) \oplus \tau_{21}(E) \oplus \tau_{22}(E) \oplus \tau_{31}(E) \oplus \tau_{32}(E) \oplus \tau_{33}(E),$$

where $\tau_{ij} = \pi_i \wedge \xi_j$, for $i, j = 1, 2, 3$. Evidently, $\tau_{11} = \pi_1$, $\tau_{33} = \xi_3$ and $\tau_{12} = \tau_{13} = \tau_{23} = 0$.

Definition 8.2. [([12, Definition 6.3], [15, Definition 13.3])] Let $\pi \in \Gamma_{\text{ex}}(E)$. For the TD sets K and F with $K \subseteq F$:

- π is *type-I* iff it is locally type-K, i.e., iff $\pi \in \Theta_\gamma$ and $\pi \leq \gamma_K$.
- π is *type-II* iff it is locally type-F, but purely non-K, i.e., iff $\pi \in \Theta_\gamma$ and $\pi \leq \gamma_F \wedge (\gamma_K)'$.
- π is *type-III* if it is purely non-F, i.e., iff $\pi \leq (\gamma_F)'$.
- π is *type-I$_F$* (respectively, *type-II$_F$*) iff it is type-I (respectively, type-II) and also type-F, i.e., iff $\pi \in \Theta_\gamma$ and $\pi \leq \gamma_K \wedge \gamma_{\tilde{F}}$ (respectively, $\pi \in \Theta_\gamma$ and $\pi \leq \gamma_F \wedge (\gamma_K)' \wedge \gamma_{\tilde{F}}$).
- π is *type-I$_{\neg F}$* (respectively, *type-II$_{\neg F}$*) iff it is type-I (respectively, type-II) and also properly non-F, i.e., iff $\pi \in \Theta_\gamma$ and $\pi \leq \gamma_K \wedge (\gamma_{\tilde{F}})'$ (respectively, iff $\pi \in \Theta_\gamma$ and $\pi \leq \gamma_F \wedge (\gamma_K)' \wedge (\gamma_{\tilde{F}})'$).

If π is type-I, type-II, etc. we also say that the direct summand $\pi(E)$ is type-I, type-II, etc.

The following theorem is the I/II/III - decomposition theorem for COGPEAs.

Theorem 8.3. *Let E be COGPEA and let K and F be TD sets in E with $K \subseteq F$. Then there are pairwise disjoint mappings $\pi_I, \pi_{II}, \pi_{III} \in \Gamma_{\text{ex}}(E)$ of types* I, II *and* III, *respectively, such that E decomposes as a direct sum*

$$E = \pi_I(E) \oplus \pi_{II}(E) \oplus \pi_{III}(E).$$

Such a direct sum decomposition is unique and

$$\pi_I = \gamma_K,\ \pi_{II} = \gamma_F \wedge (\gamma_K)',\ \pi_{III} = (\gamma_F)'.$$

Moreover, there are further decompositions

$$\pi_I(E) = \pi_{I_F}(E) \oplus \pi_{I_{\neg F}}(E),\ \pi_{II}(E) = \pi_{II_F}(E) \oplus \pi_{II_{\neg F}}(E),$$

where $\pi_{I_F}, \pi_{I_{\neg F}}, \pi_{II_F}, \pi_{II_{\neg F}}$ are of types $I_F, I_{\neg F}, II_F, II_{\neg F}$, respectively. These decompositions are also unique and

$$\pi_{I_F} = \gamma_K \wedge \gamma_{\widetilde{F}},\ \pi_{I_{\neg F}} = \gamma_K \wedge (\gamma_{\widetilde{F}})',$$

$$\pi_{II_F} = \gamma_{\widetilde{F}} \wedge (\gamma_K)',\ \pi_{II_{\neg F}} = \gamma_F \wedge (\gamma_{\widetilde{F}})' \wedge (\gamma_K)'.$$

Proof. For the existence part of the theorem, we put $\pi_I := \tau_{11} \vee \tau_{21} \vee \tau_{22}$, $\pi_{II} := \tau_{31} \vee \tau_{32}$, $\pi_{III} := \tau_{33}$, $\pi_{I_F} := \tau_{11} \vee \tau_{21}$, $\pi_{I_{\neg F}} := \tau_{22}$, $\pi_{II_F} := \tau_{31}$, and $\pi_{II_{\neg F}} := \tau_{32}$. Evidently, all the required conditions are satisfied. The proof of uniqueness is also straightforward. ∎

Acknowledgement

The second and third authors were supported by by ERDF OP R & D metaQUTE ITMS 26240120022, grant VEGA 2/0059/12 and by Science and Technology Assistance Agency under the contract no. APVV-0178-11.

REFERENCES

[1] P. Bush, P. Lahti and P. Mittelstaedt, The Quantum Theory of Measurement (Lecture Notes in Physics, Springer, Berlin-Heidelberg-New York, 1991).

[2] J.C Carrega, G. Chevalier and R. Mayet, *Direct decomposition of orthomodular lattices*, Alg. Univers. **27** (1990) 480–496. doi:10.1007/DF01188994

[3] A. Dvurečenskij, *Central elements and Cantor-Bernstein's theorem for pseudo-effect algebras*, J. Aust. Math. Soc. **74** (2003) 121–143.

[4] A. Dvurečenskij and S. Pulmannová, New Trends in Quantum Structures (Kluwer, Dordrecht, 2000).

[5] A. Dvurečenskij and T. Vetterlein, *Pseudoeffect algebras I. Basic properties*, Int. J. Theor. Phys. **40** (2001) 685–701. doi:10.1023/A:1004192715509

[6] A. Dvurečenskij and T. Vetterlein, *Pseudoeffect algebras II. Group representations*, Int. J. Theor. Phys. **40** (2001) 703–726. doi:10.1023/A:1004144832348

[7] A. Dvurečenskij and T. Vetterlein, *Algebras in the positive cone of po-groups*, Order **19** (2002) 127–146. doi:10.1023A:1016551707-476

[8] A. Dvurečenskij and T. Vetterlein, *Generalized pseudo-effect algebras*, in: Lectures on Soft Computing and Fuzzy Logic; Adv. Soft Comput, (Ed(s)), (Physica, Heidelberg, 2001) 89–111.

[9] D.J. Foulis and M.K. Bennett, *Effect algebras and unsharp quantum logics*, Found. Phys. **24** (1994) 1331–1352. doi:10.1007/BF02283036

[10] D.J. Foulis and S. Pulmannová, *Type-decomposition of an effect algebra*, Found. Phys. **40** (2010) 1543–1565. doi:10.1007/s10701-009-9344-3

[11] D.J Foulis and S. Pulmannová, *Centrally orthocomplete effect algebras*, Algebra Univ. **64** (2010) 283–307. doi:10.1007/s00012-010-0100-5

[12] D.J. Foulis and S. Pulmannová, *Hull mappings and dimension effect algebras*, Math. Slovaca **61** (2011) 485–522. doi:10.2478/s12175-011-0025-2

[13] D.J. Foulis and S. Pulmannová, *The exocenter of a generalized effect algebra*, Rep. Math. Phys. **61** (2011) 347–371.

[14] D.J. Foulis and S. Pulmannová, *The center of a generalized effect algebra*, to appear in Demonstratio Math.

[15] D.J. Foulis and S. Pulmannová, *Hull determination and type decomposition for a generalized effect algebra*, Algebra Univers. **69** (2013) 45–81. doi:10.1007/s00012-012-0214-z

[16] D.J. Foulis, S. Pulmannová and E. Vinceková, *Type decomposition of a pseudoeffect algebra*, J. Aust. Math. Soc. **89** (2010) 335–358. doi:10.1017/S1446788711001042

[17] K.R. Goodearl and F. Wehrung, *The Complete Dimension Theory of Partially Ordered Systems with Equivalence and Orthogonality*, Mem. Amer. Math. Soc. **831** (2005).

[18] R.J. Greechie, D.J. Foulis and S. Pulmannová, *The center of an effect algebra*, Order **12** (1995) 91–106. doi:10.1007/BF01108592

[19] G. Grätzer, General Lattice Theory (Academic Press, New York, 1978).

[20] J. Hedlíková and S. Pulmannová, *Generalized difference posets and orthoalgebras*, Acta Math. Univ. Comenianae **45** (1996) 247–279.

[21] M.F. Janowitz, *A note on generalized orthomodular lattices*, J. Natur. Sci and Math. **8** (1968) 89–94.

[22] G. Jenča, *Subcentral ideals in generalized effect algebras*, Int. J. Theor. Phys. **39** (2000) 745–755. doi:10.1023?A:1003610426013

[23] G. Kalmbach, Measures and Hilbert Lattices (World Scientific Publishing Co., Singapore, 1986).

[24] G. Kalmbach and Z. Riečanová, *An axiomatization for abelian relative inverses*, Demonstratio Math. **27** (1994) 535–537.

[25] F. Kôpka and F. Chovanec, *D-posets*, Math. Slovaca **44** (1994) 21–34.

[26] L.H. Loomis, The Lattice-Theoretic Background of the Dimension Theory of Operator Algebras (Mem. Amer. Math. Soc. No. 18, 1955).

[27] S. Maeda, *Dimension functions on certain general lattices*, J. Sci. Hiroshima Univ **A 19** (1955) 211–237.

[28] A. Mayet-Ippolito, *Generalized orthomodular posets*, Demonstratio Math. **24** (1991) 263–274.

[29] F.J. Murray and J. von Neumann, On Rings of Operators, J. von Neumann colected works, vol. III, Pergamon Press, Oxford, 1961, 6–321.

[30] S. Pulmannová and E. Vinceková, *Riesz ideals in generalized effect algebras and in their unitizations*, Algebra Univ. **57** (2007) 393–417. doi:10.1007/s00012-007-2043-z

[31] S. Pulmannová and E. Vinceková, *Abelian extensions of partially ordered partial monoids*, Soft Comput. **16** (2012) 1339–1346. doi:10.1007/s00500-012-0814-8

[32] A. Ramsay, *Dimension theory in complete weakly modular orthocomplemented lattices*, Trans. Amer. Math. Soc. **116** (1965) 9–31.

[33] Z. Riečanová, *Subalgebras, intervals, and central elements of generalized effect algebras*, Int. J. Theor. Phys. **38** (1999) 3209–3220. doi:10.1023/A:1026682215765

[34] M.H. Stone, *Postulates for Boolean algebras and generalized Boolean algebras*, Amer. J. Math. **57** (1935) 703–732.

[35] A. Wilce, *Perspectivity and congruence in partial Abelian semigroups*, Math. Slovaca **48** (1998) 117–135.

[36] Y. Xie and Y. Li, *Riesz ideals in generalized pseudo effect algebras and in their unitizations*, Soft Comput. **14** (2010) 387–398. doi:10.1007/s00500-009-0412-6

Received 19 October 2012
Revised 18 December 2012