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ON SETS RELATED TO MAXIMAL CLONES

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Abstract

For an arbitrary h-ary relation ρ we are interested to express n-clone $Pol^n\rho$ in terms of some subsets of the set of all n-ary operations $O^n(A)$ on a finite set A, which are in general not clones but we can obtain $Pol^n\rho$ from these sets by using intersection and union. Therefore we specify the concept a function preserves a relation and moreover, we study the properties of this new concept and the connection between these sets and $Pol^n\rho$. Particularly we study $R^{n,k}_{\underline{a},b}$ for arbitrary partial order relations, equivalence relations and central relations.

Keywords: operations preserving relations, clones, semigroups.

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1. Introduction

Let A be an arbitrary set and let $O(A)$ be the set of all operations on the set A. A clone on the set A is a subset of $O(A)$ that are closed under superposition and contains all projections. Recall that a projection e_i^n maps every n-tuple $(a_1, \ldots, a^n) \in A^n$ to a_i. Clones have been widely studied by many authors for instance in [1, 2, 3, 4, 5, 6] and [10]. Among these clones, there are six well-known clones preserving six classes of relations, namely affine relations, bounded partial order relations, nontrivial equivalence relations, central relations, prime permutations, and h-regularly generated relations. It is well-known that these clones are maximal according to Rosenberg's characterization (see [3]). We are interested in the n-ary part of these clones which we call from now on n-clones. Moreover, we specify the well-known concept of that "a function preserves a relation" and show that we get the n-ary part of the maximal clones as union and intersection of sets of n-ary operations preserving (in our sense) a given relation. We show that these sets have particular interesting properties for other consideration in universal algebra.

First, we consider an arbitrary h-ary relation ρ and we want to represent $Pol^n\rho$ in terms of some subsets of $O^n(A)$ which in general are not clones, but from these subsets we can obtain $Pol^n\rho$ by using intersection and union. For this aim, for arbitrary $b \in A$ and $\underline{a} = (a_1, \ldots, a_n) \in A^n$, we define $\rho_k^b := \{(x_1, \ldots, x_{h-1}) \in A^{h-1} | (x_1, \ldots, x_{k-1}, b, x_k, \ldots, x_{h-1}) \in \rho\}$ and $\rho_k^{\underline{a}} := \{((x_{1,1}, \ldots, x_{h-1,1}), \ldots, (x_{1,n}, \ldots, x_{h-1,n})) \in (A^{h-1})^n | \ (x_{1,i}, \ldots, x_{k-1,i}, a_i, x_{k,i}, \ldots, \ x_{h-1,i}) \in \rho$ for every $i \in \{1, \ldots, n\}\}$. It is clear that $\rho_k^{\underline{a}}$ is a cartesian product of the $h-1$ relations $\rho_k^{a_1}, \ldots, \rho_k^{a_n}$, i.e., $\rho_k^{\underline{a}} = \rho_k^{a_1} \times \cdots \times \rho_k^{a_n}$. We say that $f \in O^n(A)$ $(\underline{a}, b)_k$-preserves ρ if and only if $(f(x_{1,1}, \ldots, x_{1,n}), \ldots, f(x_{h-1,1}, \ldots, x_{h-1,n})) \in \rho_k^b$ for every $((x_{1,1}, \ldots, x_{h-1,1}), \ldots, (x_{1,n}, \ldots, x_{h-1,n})) \in \rho_k^{\underline{a}}$. Then for every $k \in \{1, \ldots, h\}$, we define $R_{\underline{a},b}^{n,k} := \{f \in O^n(A) | f \ (\underline{a}, b)_k\text{-preserves } \rho\}$. We want to see the properties of $R_{\underline{a},b}^{n,k}$.

For further investigation, we recall the following concept of semigroup of n-ary operations on A. Let A be an arbitrary finite set and let $O^n(A)$ be the set of all n-ary operations on A. On $O^n(A)$, we define an operation $+$ by $f + g := f(g, \ldots, g)$ for arbitrary $f, g \in O^n(A)$, i.e., $(f+g)(\underline{x}) = f(g, \ldots, g)(\underline{x}) = f(g(\underline{x}), \ldots, g(\underline{x}))$ for every $\underline{x} \in A$. To simplify the notation, we use $\overline{x}$ for $(x, \ldots, x)$ and thus we have $(f+g)(\underline{x}) = f(\overline{g(\underline{x})})$. It is easy to see that the operation $+$ is associative giving a semigroup $(O^n(A); +)$ (see [7, 9] and [10]). It is clear that if C is an n-clone on A, then $(C; +)$ is a subsemigroup of $(O^n(A); +)$. Moreover, for every $f \in A^n$, by π_f we mean the unary operation on A such that $\pi_f(x) = f(\overline{x})$ for every $x \in A$. We use c_y^n for the constant n-ary operation with value y. Particularly for $A = \{0, 1\}$, we put $C_4^n := \{f \in O^n(A) | f(\overline{0}) = 0, f(\overline{1}) = 1\}$, $\neg C_4^n := \{f \in O^n(A) | f(\overline{0}) = 1, f(\overline{1}) = 0\}$, $K_0^n := \{f \in O^n(A) | f(\overline{0}) = f(\overline{1}) = 0\}$ and $K_1^n := \{f \in O^n(A) | f(\overline{0}) = f(\overline{1}) = 1\}$. Clearly, $C_4^n, \neg C_4^n, K_0^n$ and K_1^n are all disjoint and $O^n(A) = C_4^n \cup \neg C_4^n \cup K_0^n \cup K_1^n$. Moreover, the operation $+$ has the following properties:

$$f + g = \begin{cases} g & \text{if } f \in C_4^n \\ \neg g & \text{if } f \in \neg C_4^n \\ c_0^n & \text{if } f \in K_0^n \\ c_1^n & \text{if } f \in \neg K_0^n \end{cases}$$

(for more details see [9] and [10]).

2. PROPERTIES OF $R^{n,k}_{\underline{a},b}$

In this section, we study some properties of $R^{n,k}_{\underline{a},b}$ for arbitrary h-ary relation ρ on A. The following propositions hold in $O^n(A)$ for every $n \geq 1$.

Proposition 1. *Let A be an arbitrary finite set and let $n \geq 1$, $h \geq 2$ and $1 \leq k \leq h$ be natural numbers. Let ρ be an h-ary relation on A. Then the following propositions are true for every $\underline{a}, \underline{a'} \in A^n$ and $b, b', y \in A$.*

(i) *If $R^{n,k}_{\underline{a},b} \neq \emptyset$, then $\rho^{\underline{a}}_k = \emptyset$ or $\rho^b_k \neq \emptyset$.*

(ii) *If $\rho^b_k \subseteq \rho^{b'}_k$, then $R^{n,k}_{\underline{a},b} \subseteq R^{n,k}_{\underline{a},b'}$.*

(iii) *If $\rho^{\underline{a}}_k \subseteq \rho^{\underline{a'}}_k$, then $R^{n,k}_{\underline{a'},b} \subseteq R^{n,k}_{\underline{a},b}$.*

(iv) *Let $1 \leq i \leq n$. If $\rho^{\underline{a}}_k \neq \emptyset$, then $R^{n,k}_{\underline{a},b}$ contains a projection e^n_i if and only if $\rho^{a_i}_k \subseteq \rho^b_k$.*

(v) *$R^{n,k}_{\underline{a},b}$ contains a constant operation c^n_y if and only if $(y, \dots, y) \in \rho^b_k$.*

(vi) *If $f \in R^{n,k}_{\underline{b},b'}$ and $g \in R^{n,k}_{\underline{a},b}$, then $f + g \in R^{n,k}_{\underline{a},b'}$.*

Proof. (i) Let $R^{n,k}_{\underline{a},b} \neq \emptyset$ and let $f \in R^{n,k}_{\underline{a},b}$. Assume that $\rho^{\underline{a}}_k \neq \emptyset$. Then for every $((x_{1,1}, \dots, x_{h-1,1}), \dots, (x_{1,n}, \dots, x_{h-1,n})) \in \rho^{\underline{a}}_k$, we have $(f(x_{1,1}, \dots, x_{1,n}), \dots, f(x_{h-1,1}, \dots, x_{h-1,n})) \in \rho^b_k$ and thus $\rho^b_k \neq \emptyset$.

(ii) Let $\rho^b_k \subseteq \rho^{b'}_k$ and let $f \in R^{n,k}_{\underline{a},b}$. Then $(f(x_{1,1}, \dots, x_{1,n}), \dots, f(x_{h-1,1}, \dots, x_{h-1,n})) \in \rho^b_k$ for arbitrary $((x_{1,1}, \dots, x_{h-1,1}), \dots, (x_{1,n}, \dots, x_{h-1,n})) \in \rho^{\underline{a}}_k$. By assumption, $(f(x_{1,1}, \dots, x_{1,n}), \dots, f(x_{h-1,1}, \dots, x_{h-1,n})) \in \rho^{b'}_k$, i.e., $f \in R^{n,k}_{\underline{a},b'}$ and therefore $R^{n,k}_{\underline{a},b} \subseteq R^{n,k}_{\underline{a},b'}$.

(iii) Let $\rho^{\underline{a}}_k \subseteq \rho^{\underline{a'}}_k$ and let $f \in R^{n,k}_{\underline{a'},b}$. Then for every $((x_{1,1}, \dots, x_{h-1,1}), \dots, (x_{1,n}, \dots, x_{h-1,n})) \in \rho^{\underline{a}}_k$ we have $((x_{1,1}, \dots, x_{h-1,1}), \dots, (x_{1,n}, \dots, x_{h-1,n})) \in \rho^{\underline{a'}}_k$ and thus $(f(x_{1,1}, \dots, x_{1,n}), \dots, f(x_{h-1,1}, \dots, x_{h-1,n})) \in \rho^b_k$ by assumption, i.e., $f \in R^{n,k}_{\underline{a},b}$.

(iv) Let $\rho^{\underline{a}}_k \neq \emptyset$. Let e^n_i be in $R^{n,k}_{\underline{a},b}$ and let $(x_{1,i}, \dots, x_{h-1,i}) \in \rho^{a_i}_k$. By assumption, we can find $(x_{1,j}, \dots, x_{h-1,j}) \in \rho^{a_j}_k$, $i \neq j = 1, \dots, n$. Thus, we obtain $(x_{1,i}, \dots, x_{h-1,i}) = (e^n_i(x_{1,1}, \dots, x_{1,n}), \dots, e^n_i(x_{h-1,1}, \dots, x_{h-1,n})) \in \rho^b_k$ and hence $\rho^{a_i}_k \subseteq \rho^b_k$. Conversely, let $\rho^{a_i}_k \subseteq \rho^b_k$. Then for every $((x_{1,1}, \dots, x_{h-1,1}), \dots, (x_{1,n}, \dots, x_{h-1,n})) \in \rho^{\underline{a}}_k$ we have $(e^n_i(x_{1,1}, \dots, x_{1,n}), \dots, e^n_i(x_{h-1,1}, \dots, x_{h-1,n})) = (x_{1,i}, \dots, x_{h-1,i}) \in \rho^{a_i}_k \subseteq \rho^b_k$ and therefore $e^n_i \in R^{n,k}_{\underline{a},b}$.

(v) Let c^n_y be in $R^{n,k}_{\underline{a},b}$. Then for every $((x_{1,1},\ldots,x_{h-1,1}),\ldots,(x_{1,n},\ldots,x_{h-1,n}))$ $\in \rho^{\underline{a}}_k$ we get $(y,\ldots,y) = (c^n_y(x_{1,1},\ldots,x_{1,n}),\ldots,c^n_y(x_{h-1,1},\ldots,x_{h-1,n})) \in \rho^b_k$. Conversely, let $(y,\ldots,y) \in \rho^b_k$. Then for every $((x_{1,1},\ldots,x_{h-1,1}),\ldots,(x_{1,n},\ldots,x_{h-1,n})) \in \rho^{\underline{a}}_k$ we obtain $(c^n_y(x_{1,1},\ldots,x_{1,n}),\ldots,c^n_y(x_{h-1,1},\ldots,x_{h-1,n})) = (y,\ldots,y)$ $\in \rho^b_k$, i.e., $c^n_y \in R^{n,k}_{\underline{a},b}$.

(vi) Let $f \in R^{n,k}_{\overline{b},b'}$ and $g \in R^{n,k}_{\underline{a},b}$. Let $((x_{1,1},\ldots,x_{h-1,1}),\ldots,(x_{1,n},\ldots,x_{h-1,n}))$ $\in \rho^{\underline{a}}_k$. Then we have $(g(x_{1,1},\ldots,x_{1,n}),\ldots,g(x_{h-1,1},\ldots,x_{h-1,n})) \in \rho^b_k$ and hence $((g(x_{1,1},\ldots,x_{1,n}),\ldots,g(x_{h-1,1},\ldots,x_{h-1,n})),\ldots,(g(x_{1,1},\ldots,x_{1,n}),\ldots,g(x_{h-1,1},\ldots,x_{h-1,n}))) \in \rho^{\overline{b}}_k$. Thus $((f+g)(x_{1,1},\ldots,x_{1,n}),\ldots,(f+g)(x_{h-1,1},\ldots,x_{h-1,n}))$ $= (f(\overline{g(x_{1,1},\ldots,x_{1,n})}),\ldots,\ f(\overline{g(x_{h-1,1},\ldots,x_{h-1,n})})) \in \rho^{b'}_k$ by assumption. ■

Remark 2. By Proposition 1 (vi) it follows that $R^{n,k}_{\overline{b},b}$ forms subsemigroup of $(O^n(A);+)$ for every $b \in A$ and for every $1 \le k \le h$.

Recall that for every clone C, we call the n-ary part $C \cap O^n(A)$ an n-clone.

Proposition 3. *Let A be an arbitrary finite set and let ρ be an arbitrary h-ary relation on A. The following assertions hold for every natural number $n \ge 1$, $\underline{a} \in A^n$ and $b \in A$.*

(i) *If $R^{n,k}_{\underline{a},b}$ is an n-clone, then $\rho^{\underline{a}}_k \subseteq \rho^{\overline{b}}_k$.*

(ii) *If $\pi_f \in R^{1,k}_{b,b}$ for every $f \in R^{n,k}_{\underline{a},b}$, then $R^{n,k}_{\underline{a},b}$ forms a subsemigroup of $(O^n(A);+)$.*

(iii) *For $h = 2$, if $R^{n,k}_{\underline{a},b}$ forms a subsemigroup of $(O^n(A);+)$, then $\pi_f \in R^{1,k}_{b,b}$ for every $f \in R^{n,k}_{\underline{a},b}$.*

Proof. (i) If $R^{n,k}_{\underline{a},b}$ is an n-clone, then $R^{n,k}_{\underline{a},b}$ contains all projections. Therefore by Proposition 1 (iv), $\rho^{\underline{a}}_k \subseteq \rho^{\overline{b}}_k$.

(ii) Let $f, g \in R^{n,k}_{\underline{a},b}$. For every $((x_{1,1},\ldots,x_{h-1,1}),\ldots,(x_{1,n},\ldots,x_{h-1,n})) \in \rho^{\underline{a}}_k$ we have $(g(x_{1,1},\ldots,x_{1,n}),\ldots,g(x_{h-1,1},\ldots,x_{h-1,n})) \in \rho^b_k$ and hence

$$\begin{aligned} & ((f+g)(x_{1,1},\ldots,x_{1,n}),\ldots,(f+g)(x_{h-1,1},\ldots,x_{h-1,n})) \\ = \ & (f(\overline{g(x_{1,1},\ldots,x_{1,n})}),\ldots,f(\overline{g(x_{h-1,1},\ldots,x_{h-1,n})})) \\ = \ & (\pi_f(g(x_{1,1},\ldots,x_{1,n})),\ldots,\pi_f(g(x_{h-1,1},\ldots,x_{h-1,n}))) \in \rho^b_k \end{aligned}$$

by assumption. Therefore $f + g \in R^{n,k}_{\underline{a},b}$, i.e., $R^{n,k}_{\underline{a},b}$ forms a subsemigroup of $(O^n(A);+)$.

(iii) Let $h = 2$. Let $f \in R^{n,k}_{\underline{a},b}$ and let $y \in \rho^b_k$. By Proposition 1 (v), $c^n_y \in R^{n,k}_{\underline{a},b}$ and hence $f + c^n_y \in R^{n,k}_{\underline{a},b}$ by assumption. Thus for every $(x_1, \ldots, x_n) \in \rho^{\underline{a}}_k$ we have $(f + c^n_y)(x_1, \ldots, x_n) \in \rho^b_k$. Therefore $\pi_f(y) = \pi_f(c^n_y(x_1, \ldots, x_n)) = f(\overline{c^n_y(x_1, \ldots, x_n)}) = (f + c^n_y)(x_1, \ldots, x_n) \in \rho^b_k$, i.e., $\pi_f \in R^{n,k}_{b,b}$. ■

Theorem 4. *Let A be an arbitrary finite set and let ρ be a binary relation on A. For arbitrary $n \geq 1$, $\underline{a} \in A^n$ and $b \in A$, it follows that $R^{n,k}_{\underline{a},b}$ forms a subsemigroup of $(O^n(A); +)$ if and only if $\pi_f \in R^{1,k}_{b,b}$ for every $f \in R^{n,k}_{\underline{a},b}$.*

Proof. is clear by Proposition 3 (ii) and Proposition 3 (iii). ■

Corollary 5. *Let A be an arbitrary finite set and let ρ be a binary relation on A. For arbitrary $a, b \in A$, it follows that $R^{1,k}_{a,b}$ forms a subsemigroup of $(O^1(A); \circ)$ if and only if $R^{1,k}_{a,b} \subseteq R^{1,k}_{b,b}$.*

Proof. It is clear by Theorem 4 and the fact that $f = \pi_f$ for $n = 1$. ■

Proposition 6. *Let A be an arbitrary finite set and let ρ be an arbitrary h-ary relation on A. For every natural number $n \geq 1$ it follows $Pol^n\rho \subseteq \bigcap_{\underline{a}\in A^n} \bigcup_{b\in A} \bigcap_{k=1}^{h} R^{n,k}_{\underline{a},b}$.*

Proof. Let $f \in Pol^n\rho$ and let $\underline{a} \in A^n$ be arbitrary. We will show that $f \in \bigcap_{k=1}^{h} R^{n,k}_{\underline{a},b}$ for some $b \in A$. Let $((x_{1,1}, \ldots, x_{h-1,1}), \ldots, (x_{1,n}, \ldots, x_{h-1,n})) \in \rho^{\underline{a}}_k$, i.e., $(x_{1,i}, \ldots, x_{k-1,i}, a_i, x_{k+1,i}, \ldots, x_{h-1,1}) \in \rho$ for every $i \in \{1, \ldots, n\}$. Then $(f(x_{1,1}, \ldots, x_{1,n}), \ldots, f(x_{k-1,1}, \ldots, x_{k-1,n}), f(\underline{a}), f(x_{k+1,1}, \ldots, x_{k+1,n}), \ldots, f(x_{h-1,1}, \ldots, x_{h-1,n})) \in \rho$ by assumption and therefore $(f(x_{1,1}, \ldots, x_{1,n}), \ldots, f(x_{h-1,1}, \ldots, x_{h-1,n})) \in \rho^{f(\underline{a})}_k$, i.e., $f \in R^{n,k}_{\underline{a},f(\underline{a})}$. ■

Example 7. Let A be an arbitrary finite set and let $n \geq 1$ be a natural number and let ρ be an h-ary relation on A. For every $\underline{a} \in A^n$ and $b \in A$, if $\rho = \{\overline{a} \in A^h | a \in A\}$, then $R^{n,k}_{\underline{a},b}$ is an n-clone if and only if $\underline{a} = \overline{b}$ if and only if $R^{n,k}_{\underline{a},b}$ forms a subsemigroup of $(O^n(A); +)$. This can be explained as follows. It is clear that when $R^{n,k}_{\underline{a},b}$ is an n-clone, then $R^{n,k}_{\underline{a},b}$ forms a subsemigroup of $(O^n(A); +)$ and moreover, contains all projections. Since $\rho = \{\overline{a} \in A^h | a \in A\}$, it follows that $\rho^{\underline{a}}_k$ contains only $(\overline{a_1}, \ldots, \overline{a_n}) \in \rho^n$. Thus, $a_i = e^n_i(\underline{a}) = b$ for every $i = 1, \ldots, n$ and hence $\underline{a} = \overline{b}$. Conversely, if $\underline{a} = \overline{b}$, then for every $b \in A$, ρ^b_k contains only $\overline{b}$. Thus $e^n_i(\overline{b}) = b$, i.e., $e^n_i \in R^{n,k}_{\overline{b},b}$ for every $1 \leq i \leq n$. Now, for every $f, g \in R^{n,k}_{\overline{b},b}$, we have $(f + g)(\overline{b}) = f(\overline{b}) = b$ that implies $f + g \in R^{n,k}_{\overline{b},b}$. Hence, $R^{n,k}_{\overline{b},b}$ is an n-clone. Furthermore, let $R^{n,k}_{\underline{a},b}$ form a subsemigroup of $(O^n(A); +)$. For every $f \in R^{n,k}_{\underline{a},b}$, it follows that $f(\underline{a}) = b$. Thus, if $g_1, \ldots, g_n \in R^{n,k}_{\underline{a},b}$, then we have $f(g_1, \ldots, g_n)(\underline{a}) = f(g_1, \ldots, g_1)(\underline{a}) = (f + g_1)(\underline{a}) = b$ and thus $R^{n,k}_{\underline{a},b}$ is an n-clone.

3. Properties of $R^{n,k}_{\underline{a},b}$ for some particular relations

In this section, we study some properties of $R^{n,k}_{\underline{a},b}$ for some particular relations, i.e., partial order relation, equivalence relation and central relation. Instead of $R^{n,k}_{\underline{a},b}$, we use $P^{n,k}_{\underline{a},b}$ for partial order relation $\leq$. The following propositions are true for arbitrary partial order relation $\leq$ on A.

Proposition 8. *Let $(A;\leq)$ be an arbitrary finite partially ordered set and let $n \geq 1$ be a natural number. Then the following properties hold for every $\underline{a}, \underline{a}' \in A^n$, $b, b' \in A$ and $k = 1, 2$.*

(i) $P^{n,k}_{\underline{a},b} \neq \emptyset$.

(ii) $P^{n,k}_{\underline{a},b} \subseteq P^{n,k}_{\underline{a},b'}$ *if and only if* $b \in \leq^{b'}_{k}$.

(iii) *If* $\leq^{\underline{a}}_{k} \subseteq \leq^{\underline{a}'}_{k}$, *then* $P^{n,k}_{\underline{a}',b} \subseteq P^{n,k}_{\underline{a},b}$.

(iv) *Let* $1 \leq i \leq n$. $P^{n,k}_{\underline{a},b}$ *contains the projection* e^n_i *if and only if* $a_i \in \leq^{b}_{k}$.

(v) $P^{n,k}_{\underline{a},b}$ *contains the constant operation* c^n_y *if and only if* $y \in \leq^{b}_{k}$. *Moreover,* $P^{n,k}_{\underline{a},b}$ *contains exactly* $\mid \leq^{b}_{k} \mid$ *constant operations.*

(vi) *If* $g \in P^{n,k}_{\underline{a},b}$ *and* $f \in P^{n,k}_{\overline{b},b'}$, *then* $f + g \in P^{n,k}_{\underline{a},b'}$.

Proof. (i) Since $b \in \leq^{b}_{k}$ for every $b \in A$ and $k = 1, 2$, then by Proposition 1 (v), $c^n_b \in P^{n,k}_{\underline{a},b}$.

(ii) Let $P^{n,k}_{\underline{a},b} \subseteq P^{n,k}_{\underline{a},b'}$. Since $b \in \leq^{b}_{k}$ then by (i) $c^n_b \in P^{n,k}_{\underline{a},b} \subseteq P^{n,k}_{\underline{a},b'}$ and hence for every $(x_1, \ldots, x_n) \in \leq^{\underline{a}}_{k}$ we have $b = c^n_b(x_1, \ldots, x_n) \in \leq^{b'}_{k}$. The opposite direction is clear by Proposition 1 (ii).

(iii), (iv), (v) and (vi) are clear respectively by (iii), (iv), (v) and (vi) of Proposition 1. ■

If A has the least element and the greatest element, then we have the following properties.

Proposition 9. *Let $(A;\leq)$ be an arbitrary finite partially ordered set and let $n \geq 1$ be a natural number. If A has the least and the greatest element and $\bigwedge_A$ is the least element and $\bigvee_A$ is the greatest element in A, then for every $\underline{a} \in A^n$ and $b \in A$ the following propositions are true.*

(i) $P^{n,1}_{\underline{a},b} = O^n(A)$ *if and only if* $b = \bigwedge_A$ ($P^{n,2}_{\underline{a},b} = O^n(A)$ *if and only if* $b = \bigvee_A$).

(ii) $P^{n,1}_{\underline{a},b} = \{c^n_b\}$ *if and only if* $\underline{a} = \overline{\bigwedge}_A$ *and* $b = \bigvee_A$ ($P^{n,2}_{\underline{a},b} = \{c^n_b\}$ *if and only if* $\underline{a} = \overline{\bigvee}_A$ *and* $b = \bigwedge_A$).

Proof. We prove for $k = 2$ and similar way for $k = 1$.

(i) Let $P^{n,2}_{\underline{a},b} = O^n(A)$. Then $c^n_y \in P^{n,2}_{\underline{a},b}$ for all $y \in A$ and hence for all $(x_1, \ldots, x_n) \in$
$\leq^{\underline{a}}_2$ we obtain $y = c^n_y(x_1, \ldots, x_n) \in \leq^b_2$ for every $y \in A$, i.e., $b = \bigvee_A$. Conversely, let $b = \bigvee_A$ and let $f \in O^n(A)$. Then for all $(x_1, \ldots, x_n) \in \leq^{\underline{a}}_2$, we have $f(x_1, \ldots, x_n) \in \leq^b_2$, i.e., $O^n(A) = P^{n,2}_{\underline{a},b}$.

(ii) Let $P^{n,2}_{\underline{a},b} = \{c^n_b\}$. Assume $\underline{a} \neq \overline{\bigvee}_A$. If $b = \bigvee_A$, then by (i), $P^{n,2}_{\underline{a},b} = O^n(A)$, a contradiction. If $b \neq \bigvee_A$, then consider an n-ary operation $f \in O^n(A)$ satisfying $f(x_1, \ldots, x_n) = b$ for every $(x_1, \ldots, x_n) \neq \overline{\bigvee}_A$ and $f(\overline{\bigvee}_A) = \bigvee_A$. This f is not equal to c^n_b and is in $P^{n,2}_{\underline{a},b}$. Hence $P^{n,2}_{\underline{a},b} \neq \{c^n_b\}$, a contradiction. Assume now $b \neq \bigwedge_A$. Then by Proposition 8 (v), $\{c^n_{\bigwedge_A}, c^n_b\} \subseteq P^{n,2}_{\underline{a},b}$, a contradiction. Conversely, let $f \in P^{n,2}_{\underline{a},b} = P^{n,2}_{\overline{\bigvee}_A, \bigwedge_A}$ and let $(x_1, \ldots, x_n) \in A^n$. Since $(x_1, \ldots, x_n) \in \leq^{\overline{\bigvee}_A}_2$, then $f(x_1, \ldots, x_n) \in \leq^{\bigwedge_A}_2$, i.e., $f(x_1, \ldots, x_n) = \bigwedge_A$, i.e., $f = c^n_{\bigwedge_A}$. Hence $P^{n,2}_{\underline{a},b} = \{c^n_b\}$. ∎

Theorem 10. *Let* $(A; \leq)$ *be an arbitrary finite partially ordered set and let* $n \geq 1$ *be a natural number. For arbitrary* $\underline{a} \in A^n$ *and* $b \in A$ *the following propositions are equivalent for* $k = 1, 2$.

(i) $P^{n,k}_{\underline{a},b}$ *is an* n*-clone.*

(ii) $P^{n,k}_{\underline{a},b} = P^{n,k}_{\overline{b},b}$.

(iii) $\underline{a} \in \leq^{\overline{b}}_k$ *and* $P^{n,k}_{\underline{a},b} \subseteq P^{n,k}_{\overline{b},b}$.

Proof. (i)⇒(ii) Let $P^{n,k}_{\underline{a},b}$ be an n-clone. Then by Proposition 3 (i), $\leq^{\underline{a}}_k \subseteq \leq^{\overline{b}}_k$ and thus by Proposition 8 (iii), $P^n_{\overline{b},b} \subseteq P^n_{\underline{a},b}$. Now, let $f \in P^n_{\underline{a},b}$ and let $(y_1, \ldots, y_n) \in \leq^{\overline{b}}_k$, i.e., $y_i \in \leq^b_k$ for every $i = 1, 2, \ldots, n$. Then by Proposition 8 (v), $c^n_{y_i} \in P^{n,k}_{\underline{a},b}$ and hence $f(c^n_{y_1}, \ldots, c^n_{y_n}) \in P^{n,k}_{\underline{a},b}$ for $1 \leq i \leq n$. Therefore, for every $(x_1, \ldots, x_n) \in \leq^{\underline{a}}_k$, $f(c^n_{y_1}, \ldots, c^n_{y_n})(x_1, \ldots, x_n) \in \leq^b_k$ and hence $f(y_1, \ldots, y_n) = f(c^n_{y_1}(x_1, \ldots, x_n), \ldots, c^n_{y_n}(x_1, \ldots, x_n)) = f(c^n_{y_1}, \ldots, c^n_{y_n})(x_1, \ldots, x_n) \in \leq^b_k$, i.e., $f \in P^{n,k}_{\overline{b},b}$. Thus $P^{n,k}_{\underline{a},b} \subseteq P^{n,k}_{\overline{b},b}$ and hence $P^{n,k}_{\underline{a},b} = P^{n,k}_{\overline{b},b}$.

(ii)⇒(iii) By Proposition 8 (iv), $P^{n,k}_{\overline{b},b}$ contains all projections. Therefore $P^n_{\underline{a},b} = P^n_{\overline{b},b}$ contains all projections and hence by Proposition 8 (iv), $a_i \in \leq^b_k$ for every $i \in \{1, \ldots, n\}$, i.e., $\underline{a} \in \leq^{\overline{b}}_k$.

(iii)⇒(i) By assumption and Proposition 8 (iv), $P^n_{\underline{a},b}$ contains all projections. Moreover, let $f, g_1, \ldots, g_n$ be in $P^n_{\underline{a},b} \subseteq P^n_{\overline{b},b}$. Then, $g_i(x_1, \ldots, x_n) \in \leq^b_k$ for every $(x_1, \ldots, x_n) \in \leq^a_k$ and $i = 1, 2, \ldots, n$. Thus, $(g_1(x_1, \ldots, x_n), \ldots, g_n(x_1, \ldots, x_n)) \in \leq^{\overline{b}}_k$. Hence for every $(x_1, \ldots, x_n) \in \leq^a_k$, $f(g_1, \ldots, g_n)(x_1, \ldots, x_n) = f(g_1(x_1, \ldots, x_n), \ldots, g_n(x_1, \ldots, x_n)) \in \leq^b_k$. Therefore, $f(g_1, \ldots, g_n) \in P^{n,k}_{\underline{a},b}$ and hence $P^{n,k}_{\underline{a},b}$ is an n-clone. ■

Theorem 11. *Let $(A; \leq)$ be an arbitrary finite partially ordered set and let $n \geq 1$ be a natural number. For every $\underline{a} \in A^n$, $b \in A$ it follows that $Pol^n \leq = \bigcap_{\underline{a}\in A^n} \bigcup_{b\in A}(P^{n,1}_{\underline{a},b} \cap P^{n,2}_{\underline{a},b})$.*

Proof. $(\subseteq)$ is clear by Proposition 6.
$(\supseteq)$ Let $f \in \bigcap_{\underline{a}\in A^n} \bigcup_{b\in A}(P^{n,1}_{\underline{a},b} \cap P^{n,2}_{\underline{a},b})$. Let $(u_i, v_i) \in \leq$, $i = 1, 2, \ldots, n$. Now, take $\underline{a} = \underline{u}$. By assumption, for this $\underline{a} \in A^n$, we can find $b \in A$ such that $f \in P^{n,1}_{\underline{a},b} \cap P^{n,2}_{\underline{a},b}$. Therefore $f(\underline{u}) \in \leq^b_2$ and $f(\underline{v}) \in \leq^b_1$, i.e., $(f(\underline{u}), f(\underline{v})) \in \leq$. Hence $f \in Pol^n \rho_A$. ■

Example 12. Let $A = \{0, 1\}$. By Proposition 9, $P^{n,2}_{\overline{1},0} = \{c^n_0\}$ and $P^{n,1}_{\overline{0},1} = \{c^n_1\}$. Moreover, $P^{n,1}_{\underline{a},0} = O^n(\{0,1\}) = P^{n,2}_{\underline{a},1}$. Therefore $P^{n,1}_{\underline{a},0} \cap P^{n,2}_{\underline{a},0} = P^{n,2}_{\underline{a},0}$ and $P^{n,1}_{\underline{a},1} \cap P^{n,2}_{\underline{a},1} = P^{n,1}_{\underline{a},1}$ and hence $\bigcup_{b\in A}(P^{n,1}_{\underline{a},b} \cap P^{n,2}_{\underline{a},b}) = P^{n,1}_{\underline{a},1} \cup P^{n,2}_{\underline{a},0}$ and we get $Pol^n \leq = \bigcap_{\underline{a}\in A^n} \bigcup_{b\in A}(P^{n,1}_{\underline{a},b} \cap P^{n,2}_{\underline{a},b}) = \bigcap_{\underline{a}\in A^n}(P^{n,1}_{\underline{a},1} \cup P^{n,2}_{\underline{a},0})$ by Theorem 11. Now consider all operations on $O^2(\{0,1\})$ as follows

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}
$(0,0)$	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
$(0,1)$	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
$(1,0)$	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
$(1,1)$	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1.

Remark 13. From above example, we have $f_1 = c^2_0$, $f_{16} = c^2_1$ and we obtain $K^2_0 = \{c^2_0, f_2, f_3, f_4\}$, $C^2_4 = \{f_5, f_6, f_7, f_8\}$, $\neg C^2_4 = \{f_9, f_{10}, f_{11}, f_{12}\}$ and $K^2_1 = \{f_{13}, f_{14}, f_{15}, c^2_1\}$. By Proposition 9, $P^{2,2}_{(1,1),0} = \{c^2_0\}$ and $P^{2,1}_{(0,0),1} = \{c^2_1\}$. Moreover, it is easy to see that

$$P^{2,1}_{(0,1),1} = \{f_7, f_8, f_{15}, c^2_1\} \quad P^{2,2}_{(0,0),0} = K^n_0 \cup C^2_4$$

$$P^{2,1}_{(1,0),1} = \{f_6, f_8, f_{14}, c^n_1\} \quad P^{2,2}_{(0,1),0} = \{c^2_0, f_2, f_5, f_6\}$$

$$P^{2,1}_{(1,1),1} = C^2_4 \cup K^2_1 \qquad P^{2,2}_{(1,0),0} = \{c^2_0, f_3, f_5, f_7\}$$

and hence we obtain $Pol^2 \leq = C^2_4 \cup \{c^2_0, c^2_1\}$.

Generally, for $A = \{0 \le 1\}$ and for every $n \geq 1$, by applying Proposition 8, we have some properties on $P^{n,1}_{\underline{a},b}$ and $P^{n,2}_{\underline{a},b}$ as follows

(i) $c^n_1 \in P^{n,1}_{\underline{a},b}$ and $c^n_0 \in P^{n,2}_{\underline{a},b}$ since $1 \in \le^b_1$ and $0 \in \le^b_2$.

(ii) If $\underline{a} \neq \overline{0}$, then $P^{n,1}_{\underline{a},b} \cap C^n_4 \neq \emptyset$. This fact holds since $1 \in \le^b_1$ which implies $e^n_i \in C^n_4$ is contained in $P^{n,1}_{\underline{a},b}$ by Proposition 8 (iv) for some i such that $a_i \neq 0$. Similarly, if $\underline{a} \neq \overline{1}$, then $P^{n,2}_{\underline{a},b} \cap C^n_4 \neq \emptyset$.

(iii) If $b = 1$, then $P^{n,1}_{\underline{a},b} \cap (K^n_0 \cup \neg C^n_4) = \emptyset$. It is clear that $\overline{1} \in \le^{\underline{a}}_1$. But for all $f \in K^n_0 \cup \neg C^n_4$, we have $f(\overline{1}) = 0 \notin \le^1_1$ and hence $f \notin P^{n,1}_{\underline{a},b}$. Similarly if $b = 0$, then $P^{n,2}_{\underline{a},b} \cap (K^n_1 \cup \neg C^n_4) = \emptyset$.

(iv) $Pol^n \le = C \cup \{c^n_0, c^n_1\}$ for some $C \subseteq C^n_4$.

Now, we come to the properties of $R^{n,k}_{\underline{a},b}$ for an arbitrary equivalence relation θ on A. By symmetry property of θ, it follows that $\theta^b_1 = \theta^b_2$ and $\theta^{\underline{a}}_1 = \theta^{\underline{a}}_2$ for every $b \in A$ and $\underline{a} \in A^n$ and these imply $R^{n,1}_{\underline{a},b} = R^{n,2}_{\underline{a},b}$. Therefore, we can omit the number k. Moreover, since $\theta^b_1 = \theta^b_2$ is actually an equivalence class that contains b, then we use $[b]_\theta$ instead θ^b and we use $E^{n,\theta}_{\underline{a},b}$ instead of $R^{n,k}_{\underline{a},b}$ since we might have various equivalence relations on A. Thus, by $f \in E^{n,\theta}_{\underline{a},b}$ we mean an n-ary operation such that $(f(x_1, \dots, x_n), b) \in \theta$ for every $(x_1, \dots, x_n)$ satisfying $(x_i, a_i) \in \theta$ for every $i = 1, \dots, n$.

Proposition 14. *Let A be an arbitrary finite set and $n \geq 1$ be an arbitrary natural number. For an arbitrary equivalence relation $\theta_A \neq A \times A$ on A, the following properties are true for arbitrary $\underline{a}, \underline{a}' \in A^n$ and $b, b' \in A$.*

(i) $E^{n,\theta}_{\underline{a},b} \neq \emptyset$.

(ii) $[b]_\theta = [b']_\theta$ *if and only if* $E^{n,\theta}_{\underline{a},b} \cap E^{n,\theta}_{\underline{a},b'} \neq \emptyset$ *if and only if* $E^{n,\theta}_{\underline{a},b} = E^{n,\theta}_{\underline{a},b'}$.

(iii) $[\underline{a}]_{\theta^n} = [\underline{a}']_{\theta^n}$ *if and only if* $E^{n,\theta}_{\underline{a},b} = E^{n,\theta}_{\underline{a}',b}$.

(iv) *If* $[\underline{a}]_{\theta^n} \neq [\underline{a}']_{\theta^n}$, *then* $E^{n,\theta}_{\underline{a},b} \cap E^{n,\theta}_{\underline{a}',b'} \neq \emptyset$.

(v) *Let* $1 \le i \le n$. $E^{n,\theta}_{\underline{a},b}$ *contains a projection* e^n_i *if and only if* $[b]_\theta = [a_i]_\theta$.

(vi) $E^{n,\theta}_{\underline{a},b}$ *contains a constant operation* c^n_y *if and only if* $y \in [b]_\theta$. *Moreover,* $E^{n,\theta}_{\underline{a},b}$ *contains precisely* $|[b]_\theta|$ *constant operations.*

(vii) *If* $f \in E^{n,\theta}_{\bar{b},b'}$ *and* $g \in E^{n,\theta}_{\underline{a},b}$, *then* $f + g \in E^{n,\theta}_{\underline{a},b'}$.

Proof. (i) By reflexivity of θ and Proposition 1 (v), $c^n_b \in E^{n,\theta}_{\underline{a},b}$, i.e., $E^{n,\theta}_{\underline{a},b} \neq \emptyset$.

(ii) By Proposition 1 (ii), if $[b]_\theta = [b']_\theta$, then $E^{n,\theta}_{\underline{a},b} = E^{n,\theta}_{\underline{a},b'}$. Thus if $[b]_\theta = [b']_\theta$, then $E^{n,\theta}_{\underline{a},b} \cap E^{n,\theta}_{\underline{a},b'} \neq \emptyset$ since $E^{n,\theta}_{\underline{a},b} \neq \emptyset$ by (i). Conversely, let $E^{n,\theta}_{\underline{a},b} \cap E^{n,\theta}_{\underline{a},b'} \neq \emptyset$ and let $f \in E^{n,\theta}_{\underline{a},b} \cap E^{n,\theta}_{\underline{a},b'}$. Then for every $(x_1, \ldots, x_n) \in [\underline{a}]_{\theta^n}$ we have $f(x_1, \ldots, x_n) \in [b]_\theta \cap [b']_\theta$ and hence $[b]_\theta = [b']\theta$.

(iii) By Proposition 1 (iii), if $[\underline{a}]_{\theta^n} = [\underline{a'}]_{\theta^n}$, then $E^{n,\theta}_{\underline{a},b} = E^{n,\theta}_{\underline{a'},b}$. Conversely, let $E^{n,\theta}_{\underline{a},b} = E^{n,\theta}_{\underline{a'},b}$. Assume that $[\underline{a}]_{\theta^n} \neq [\underline{a'}]_{\theta^n}$, i.e., $\underline{a} \notin [\underline{a'}]_{\theta^n}$ and $\underline{a'} \notin [\underline{a}]_{\theta^n}$. Since $\theta_A \neq A \times A$, then there is $b' \in A$ such that $[b]_\theta \neq [b']_\theta$. Now consider an n-ary operation f on A such that $f(x_1, \ldots, x_n) = b$ for all $(x_1, \ldots, x_n) \in [\underline{a}]_{\theta^n}$ and $f(\underline{a'}) = b'$. Then it is clear that $f \in E^{n,\theta}_{\underline{a},b}$ but $f \notin E^{n,\theta}_{\underline{a'},b}$ and hence $E^{n,\theta}_{\underline{a},b} \neq E^{n,\theta}_{\underline{a'},b}$, a contradiction.

(iv) Let $[\underline{a}]_{\theta^n} \neq [\underline{a'}]_{\theta^n}$. Consider an $f \in O^n(A)$ such that $f(x_1, \ldots, x_n) = b$ for every $(x_1, \ldots, x_n) \in [\underline{a}]_{\theta^n}$ and $f(x_1, \ldots, x_n) = b'$ for every $(x_1, \ldots, x_n) \in [\underline{a'}]_{\theta^n}$. It is clear that $f \in E^{n,\theta}_{\underline{a},b} \cap E^{n,\theta}_{\underline{a'},b}$ and hence $E^{n,\theta}_{\underline{a},b} \cap E^{n,\theta}_{\underline{a'},b'} \neq \emptyset$.

(v), (vi) and (vii) are clear by Proposition 1 (iv), (v) and (vi). ∎

Theorem 15. *Let* A *be an arbitrary finite set and* $n \geq 1$ *be an arbitrary natural number and let* $\theta \neq A \times A$ *be an arbitrary equivalence relation on* A. *Then for arbitrary* $\underline{a} \in A^n$ *and* $b \in A$ *it follows that* $E^{n,\theta}_{\underline{a},b}$ *is an* n*-clone if and only if* $[\underline{a}]_{\theta^n} = [\bar{b}]_{\theta^n}$.

Proof. If $E^{n,\theta}_{\underline{a},b}$ is an n-clone, then by Proposition 3 (i), $\underline{a} \in [\bar{b}]_{\theta^n}$, i.e., $[\underline{a}]_{\theta^n} = [\bar{b}]_{\theta^n}$. Conversely, let $[\underline{a}]_{\theta^n} = [\bar{b}]_{\theta^n}$. By Proposition 1 (iv), $E^{n,\theta}_{\underline{a},b}$ contains all projections. Now, let $f, g_1, \ldots, g_n \in E^{n,\theta}_{\underline{a},b}$ and $(x_1, \ldots, x_n) \in [\underline{a}]_{\theta^n}$ be arbitrary. Then $g_1(x_1, \ldots, x_n), \ldots, g_n(x_1, \ldots, x_n) \in [b]_\theta$ and therefore $(g_1(x_1, \ldots, x_n), \ldots, g_n(x_1, \ldots, x_n)) \in [\bar{b}]_{\theta^n} = [\underline{a}]_{\theta^n}$. Thus, $f(g_1, \ldots, g_n)(x_1, \ldots, x_n) = f(g_1(x_1, \ldots, x_n), \ldots, g_n(x_1, \ldots, x_n)) \in [b]_\theta$, i.e., $f(g_1, \ldots, g_n) \in E^{n,\theta}_{\underline{a},b}$. ∎

Now, if we define $E^{n,\theta}_{\underline{a}} := \{f \in O^n(A) | f(x_1, \ldots, x_n) \in [f(\underline{a})]_\theta$ for every $(x_1, \ldots, x_n) \in [\underline{a}]_{\theta^n}\}$, then we have the following proposition.

Proposition 16. *Let* A *be an arbitrary finite set and* $n \geq 1$ *be an arbitrary natural number. For an arbitrary equivalence relation* $\theta \neq A \times A$ *on* A *and an arbitrary* $\underline{a} \in A^n$ *it follows that* $E^{n,\theta}_{\underline{a}} = \bigcup_{b \in A} E^{n,\theta}_{\underline{a},b}$.

Proof. ($\subseteq$) It is clear by definition.

($\supseteq$) Let $f \in \bigcup_{b \in A} E^{n,\theta}_{\underline{a},b}$. Then, we can find $b \in A$ such that $f \in E^{n,\theta}_{\underline{a},b}$ and thus for every $(x_1, \ldots, x_n) \in [\underline{a}]_{\theta^n}$, we have $f(x_1, \ldots, x_n), f(\underline{a}) \in [b]_\theta$, i.e.,

$f(x_1,\dots,x_n) \in [f(\underline{a})]_\theta$, i.e., $f \in E^{n,\theta}_{\underline{a}}$. Hence $\bigcup_{b\in A} E^{n,\theta}_{\underline{a},b} \subseteq E^{n,\theta}_{\underline{a}}$ and therefore $E^{n,\theta}_{\underline{a}} = \bigcup_{b\in A} E^{n,\theta}_{\underline{a},b}$. ■

Theorem 17. *Let A be an arbitrary finite set and $n \geq 1$ be an arbitrary natural number. For an arbitrary equivalence relation $\theta \neq A \times A$ on A, it follows that $Pol^n\theta = \bigcap_{\underline{a}\in A^n} \bigcup_{b\in A} E^{n,\theta}_{\underline{a},b} = \bigcap_{\underline{a}\in A^n} E^{n,\theta}_{\underline{a}}$.*

Proof. ($\subseteq$) is clear by Proposition 6 and by Proposition 16.

($\supseteq$) Let $f \in \bigcap_{\underline{a}\in A^n} E^{n,\theta}_{\underline{a}}$. For every $(x_i, y_i) \in \theta$, $i = 1,\dots,n$ we have $(x_1,\dots,x_n) \in [(y_1,\dots,y_n)]_{\theta^n}$. By assumption, we know that $f \in E^{n,\theta}_{(y_1,\dots,y_n)}$. Therefore $f(x_1,\dots,x_n) \in [f(y_1,\dots,y_n)]_\theta$ and thus $(f(x_1,\dots,x_n), f(y_1,\dots,y_n)) \in \theta$, i.e., $f \in Pol^n\theta$. ■

Recall that an h-ary relation ζ on A is called a central relation if ζ satisfies these three properties: (i) totally symmetric, i.e., if $(a_1,\dots,a_h) \in \zeta$, then $(a_{\sigma(1)},\dots, a_{\sigma(h)}) \in \zeta$ for all permutations σ on $\{1,\dots,h\}$ (ii) totally reflexive, i.e., $\kappa^h_A \subseteq \zeta$ for $\kappa^h_A := \{(a_1 \dots, a_h) | \exists i \exists j\, (i \neq j \wedge a_i = a_j)\}$ and (iii) there exists $\emptyset \neq C \subseteq A$ such that $(c, a_2,\dots,a_h) \in \zeta$ for every $c \in C$ and for all $a_2,\dots,a_h \in A$. We call the set C as the central of ζ. Now, we consider an arbitrary h-ary central relation ζ, $h \geq 2$. We use $h \leq |A|$ since otherwise we would have $\zeta = A^h$ and the center of ζ would be trivial. Moreover, by totally symmetry property of ζ, we have $\zeta^b_k = \zeta^b_l$ and $\zeta^{\underline{a}}_k = \zeta^{\underline{a}}_l$ for every $\underline{a} \in A^n$, $b \in A$ and $k \neq l \in \{1,\dots,h\}$. Therefore, we can again omit the number k and since we might have many central relation on A, we then use $C^{n,\zeta}_{\underline{a},b}$ instead of $R^{n,k}_{\underline{a},b}$. Without lost of generality, we use implicitly $k = h$, i.e., by $f \in C^{n,\zeta}_{\underline{a},b}$ we mean n-ary operation satisfying $(f(x_{1,1},\dots,x_{1,n}),\dots,f(x_{h-1,1},\dots,x_{h-1,n}), b) \in \zeta$ for every $(x_{1,i},\dots,x_{h-1,i}, a_i) \in \zeta$, $i = 1,\dots,n$. By the third property of ζ, i.e., for every $(x_1,\dots,x_{h-1}) \in A^{h-1}$ and for all $c \in C$, it follows $(x_1,\dots,x_{h-1}, c) \in \zeta$ and we have the following simple properties.

Proposition 18. *Let ζ be an h-ary relation on A with central C. For every $b \in A$ and $\underline{a} \in A^n$ we have the following properties.*

(i) *$\zeta^b \subseteq \zeta^c$ for every $c \in C$.*

(ii) *$\zeta^{\underline{a}} \subseteq \zeta^{\underline{c}}$ for every $\underline{c} \in C^n$.*

By Proposition 1, we have the following properties for an arbitrary h-ary central relation ζ on A.

Proposition 19. *Let A be an arbitrary finite set and let $n \geq 1$, $h \geq 2$ be natural numbers. Let ζ be an h-ary central relation on A with C as the central. Then the following propositions are true for every $\underline{a}, \underline{a'} \in A^n$ and $b, b', y \in A$.*

(i) $C^{n,\zeta}_{\underline{a},b} \neq \emptyset$.

(ii) *If* $\zeta^b \subseteq \zeta^{b'}$, *then* $C^{n,\zeta}_{\underline{a},b} \subseteq C^{n,\zeta}_{\underline{a},b'}$.

(iii) *If* $\zeta^{\underline{a}} \subseteq \zeta^{\underline{a'}}$, *then* $C^{n,\zeta}_{\underline{a'},b} \subseteq C^{n,\zeta}_{\underline{a},b}$.

(iv) *Let* $1 \leq i \leq n$. $C^{n,\zeta}_{\underline{a},b}$ *contains a projection* e^n_i *if and only if* $\zeta^{a_i} \subseteq \zeta^b$.

(v) $C^{n,\zeta}_{\underline{a},b}$ *contains the constant operation* c^n_y *if and only if* $(y, \dots, y) \in \zeta^b$.

(vi) *If* $f \in C^{n,\zeta}_{\underline{b},b'}$ *and* $g \in C^{n,\zeta}_{\underline{a},b}$, *then* $f + g \in C^{n,\zeta}_{\underline{a},b'}$.

As a consequence of Proposition 18 and Proposition 19, we have

Proposition 20. *Let* A *be an arbitrary finite set and let* $n \geq 1$, $h \geq 2$ *be natural numbers. Let* ζ *be an* h*-ary central relation on* A *with* C *as the central. Then the following propositions are true for every* $\underline{a} \in A^n$ *and* $b \in A$.

(i) $C^{n,\zeta}_{\underline{a},b} \subseteq C^{n,\zeta}_{\underline{a},c}$ *for every* $c \in C$.

(ii) $C^{n,\zeta}_{\underline{c},b} \subseteq C^{n,\zeta}_{\underline{a},b}$ *for every* $\underline{c} \in C^n$.

(iii) *If* $b \in C$, *then* $C^{n,\zeta}_{\underline{a},b}$ *contains all projections, contains all constant operations and moreover is an* n*-clone.*

(iv) *If* $C^{n,\zeta}_{\underline{a'},b}$ *contains all projections for all* $\underline{a'} \in A^n$, *then* $b \in C$.

(v) *If* $h = 2$ *and* $C^{n,\zeta}_{\underline{a},b}$ *contains all constant operations, then* $b \in C$.

(vi) *If* $h \geq 3$, *then* $C^{n,\zeta}_{\underline{a},b}$ *contains all constant operations.*

Proof. (i) By Proposition 18 (i) and Proposition 19 (ii).

(ii) By Proposition 18 (ii) and Proposition 19 (iii).

(iii) By Proposition 18 (i), Proposition 19 (iv) and Proposition 19 (v). Moreover, since for every $x_1, \dots, x_{h-1} \in A$ it follows that $(x_1, \dots, x_{h-1}) \in \zeta^b$, then for every $(x_{1,1}, \dots, x_{h-1,1}), \dots, (x_{1,n}, \dots, x_{h-1,n}) \in \zeta^{\underline{a}}$ and for every $f, g_1, \dots, g_n \in C^{n,\zeta}_{\underline{a},b}$, we have $(f(g_1, \dots, g_n)(x_{1,1}, \dots, x_{1,n}), \dots, f(g_1, \dots, g_n)(x_{h-1,1}, \dots, x_{h-1,n})) \in \zeta^b$, i.e., $f(g_1, \dots, g_n) \in C^{n,\zeta}_{\underline{a},b}$. Thus $C^{n,\zeta}_{\underline{a},b}$ is an n-clone.

(iv) Let $(x_1, \dots, x_{h-1}) \in A^{h-1}$ be arbitrary. Then $(x_1, \dots, x_{h-1}, x_{h-1}) \in \zeta$. By assumption, for every $\underline{a'} \in A^n$ such that $a_i{}' = x_{h-1}$, we have that $C^{n,\zeta}_{\underline{a'},b}$ contains all projections and hence by Proposition 19 (iv), $\zeta^{a_i{}'} \subseteq \zeta^b$. Therefore, $(x_1, \dots, x_{h-1}) \in \zeta^{x_{h-1}} = \zeta^{a_i{}'} \subseteq \zeta^b$. Thus $(x_1, \dots, x_{h-1}, b) \in \zeta$ and hence $b \in C$ since $(x_1, \dots, x_{h-1})$ is arbitrary.

(v) Let $h = 2$ and let $y \in A$. By assumption, $C^{n,\zeta}_{\underline{a},b}$ contains c^n_y. Therefore, by Proposition 19 (v), $y \in \zeta^b$, i.e., $(y, b) \in \zeta$. Since y is arbitrary we have $b \in C$.

(vi) By totally reflexive property of ζ it follows that for every $(y, \dots, y) \in A^{h-1}$, $h \geq 3$ we have $(y, \dots, y, b) \in \zeta$, i.e., $c^n_y \in C^{n,\zeta}_{\underline{a},b}$ by Proposition 19 (v). ■

Proposition 21. *Let A be an arbitrary finite set and let $n \geq 1$, $h \geq 2$ be natural numbers. Let ζ be an h-ary central relation on A with C as the central. Then for every $\underline{a} \in A^n$ and $b \in A$, $C^{n,\zeta}_{\underline{a},b}$ contains all constant operations if and only if $b \in C$ or $h \geq 3$.*

Proof. If $C^{n,\zeta}_{\underline{a},b}$ contains all constant operations and $h < 3$, i.e., $h = 2$, then $b \in C$ by Proposition 20 (v). The converse is clear by Proposition 20 (iii) and Proposition 20 (vi). ■

Proposition 22. *Let A be an arbitrary finite set and let $n \geq 1$, $h \geq 2$ be natural numbers. Let ζ be an h-ary central relation on A with C as the central. For $b \in A$, the following propositions are equivalent.*

(i) *$C^{n,\zeta}_{\underline{a},b}$ contains all projections for all $\underline{a} \in A^n$.*

(ii) *$C^{n,\zeta}_{\underline{a},b}$ is an n-clone for all $\underline{a} \in A^n$.*

(iii) *$b \in C$.*

Proof. (i)⇒(iii) is clear by Proposition 20 (iv).

(iii)⇒(ii) is clear by Proposition 20 (iii).

(ii)⇒(i) is obvious by definition. ■

The following property is clear by Proposition 6.

Proposition 23. *Let A be an arbitrary finite set and let $n \geq 1$, $h \geq 2$ be natural numbers. Let ζ be an h-ary central relation on A with C as the central. Then $Pol^n \zeta \subseteq \bigcap_{\underline{a} \in A^n} \bigcup_{b \in A} C^{n,\zeta}_{\underline{a},b}$.*

References

[1] A. Fearnley, *Clones on Three Elements Preserving a Binary Relation*, Algebra Universalis **56** (2007) 165–177. doi:10.1007/s00012-007-1985-5

[2] Á. Szendrei, Clones in Universal Algebra (Les Presses de L' Université de Montréal, 1986).

[3] I.G. Rosenberg, *Über die Funktionale Vollständigkeit in den Mehrwertigen Logiken*, Rozpravy Ćeskoslovenské Akad. véd, Ser. Math. Nat. Sci. **80** (1970) 3–93.

[4] K. Denecke, D. Lau, R. Pöschel and D. Schweigert, *Hyperidentities, Hyperequational Classes and Clone Congruences*, Contributions to General Algebra 7, Verlag Hölder-Pichler-Tempsky, Wien (1991) 97–118.

[5] K. Denecke and S.L. Wismath, Hyperidentities and Clones (Gordon and Breach Science Publisher, 2000).

[6] K. Denecke and S.L. Wismath, Universal Algebra and Applications in Theoretical Computer Science (Chapman and Hall, 2002).

[7] K. Denecke and Y. Susanti, *Semigroups of n-ary Operations on Finite Sets*, in: Proceedings of International Conference on Algebra on Algebra 2010 Advances in Algebraic Structures, W. Hemakul, S. Wahyuni and P.W. Sy (Ed(s)), (World Scientific, 2012) 157–176. doi:10.1142/9789814366311$_-$0011

[8] K. Denecke and Y. Susanti, *On Sets Related to Clones of Quasilinear Operations*, in: Proceedings of the 6th SEAMS-GMU International Conference on Mathematics and Its Application 2011, S. Wahyuni, I.E. Wijayanti and D. Rosadi (Ed(s)), (University of Gadjah Mada, 2012) 145–158.

[9] R. Butkote and K. Denecke, *Semigroup Properties of Boolean Operations*, Asian-Eur. J. Math. **1** (2008) 157–176.

[10] R. Butkote, Universal-algebraic and Semigroup-theoretical Properties of Boolean Operations (Dissertation Universität Potsdam, 2009).

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