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ON THE SOLIDITY OF GENERAL VARIETIES OF TREE LANGUAGES

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Abstract

For a class of hypersubstitutions $\mathcal{K}$, we define the $\mathcal{K}$-solidity of general varieties of tree languages (GVTLs) that contain tree languages over all alphabets, general varieties of finite algebras (GVFAs), and general varieties of finite congruences (GVFCs). We show that if $\mathcal{K}$ is a so-called category of substitutions, a GVTL is $\mathcal{K}$-solid exactly in case the corresponding GVFA, or the corresponding GVFC, is $\mathcal{K}$-solid. We establish the solidity status of several known GVTLs with respect to certain categories of substitutions derived from some important classes of tree homomorphisms.

Keywords: varieties of tree languages, solid varieties, hypersubstitutions, tree homomorphisms.

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1. Introduction

The solidity of varieties of algebras is an extensively studied topic. For general expositions and extensive bibliographies, the reader may consult Schweigert [32], Koppitz and Denecke [26], and Denecke and Wismath [10, 11], where also appropriate references to the important early work by people like J. Aczél, V.D. Belousov and W. Taylor can be found. A hypersubsitution is a mapping that replaces in terms each operation symbol with a term of the same arity, and a variety is said to be solid if every hypersubstitution turns each identity satisfied by the variety to an identity also satisfied by the variety. Solid varieties were introduced by Graczyńska and Schweigert [20] who also noted that the solidity of a variety can be defined in terms of the operator D that from a class of algebras $\mathbf{U}$ forms the class $D(\mathbf{U})$ of all derived algebras obtained from members of $\mathbf{U}$ by a hypersubstitution. The notion of M-solidity of Denecke and Reichel [9] is also very useful here since few of our varieties are fully solid; for a submonoid M of the monoid of all hypersubstitutions of the given type, a variety is M-solid if its set of identities is closed under all members of M. Although not used here, we should also mention the work on solid pseudovarieties by Graczyńska, Pöschel and Volkov [21] and Pibaljommee's study [27] of M-solid pseudovarieties.

Independently of these developments in algebra, Thatcher [37] defined tree homomorphisms as special tree transformations. Engelfriet's fundamental study [13] of compositions and decompositions of tree transformations clearly shows the importance of tree homomorphisms. They also appear in various models of syntax-directed translation. For such matters, cf. [2, 16, 17, 18]. When trees are defined as terms, as one usually does, it is obvious that tree homomorphisms and hypersubstitutions are closely related. We shall clarify this relationship.

It is well known [17, 18] that the preimage of a regular tree language under any tree homomorphism is also regular, but few known families of special regular tree languages share this property. Many of these families are so-called varieties of tree languages. There are a few different approaches to varieties of tree languages (cf. [36] for a survey). In [35] the theory is presented for general varieties of tree languages (GVTLs), which contain tree languages over all ranked alphabets, and the matching general varieties of finite algebras (GFVAs) and general varieties of finite congruences (GVFCs). This is a good framework here, too, as tree homomorphisms typically change the ranked alphabets of trees, and as many natural families of regular tree languages are known to be GVTLs. We may also note that the definition of GVTLs already involves a mild solidity condition.

Baltazar [4] considers the M-solidity of Almeida's [1] varieties of V-languages, pseudovarieties, and varieties of filters of V-congruences, where M is a monoid of hypersubstitutions and V is a given pseudovariety, and establishes some connections between the M-solidity of a pseudovariety and the M-solidity of the corresponding varieties of V-languages. A part of our paper parallels these results but we prefer an independent presentation that develops the needed conceptual framework for the theory of general varieties. In fact, it appeared counterproductive to try to translate the results of [4] to our setting. Denecke and Koppitz [7] consider the M-solidity of positive varieties.

Section 2 recalls a few basic concepts and fixes some notation. In Section 3 we clarify the relation between tree homomorphisms and hypersubstitutions. A natural correspondence is achieved by slightly restricting the class of tree homomorphisms considered. Indeed, each such tree homomorphism has an underlying hypersubstitution that determines how it transforms the inner nodes of trees, and each hypersubstitution yields a set of such tree homomorphisms. This restriction on the tree homomorphisms has no effect on our notions of general varieties of tree languages, finite algebras or finite congruences. The systems of hypersubstitutions that will correspond to the monoids of hypersubstitutions of the theory of M-solidity, we call categories of substitutions (without suggesting any uses of category theory). We shall consider several such categories that we derive from some well-known types of tree homomorphisms.

In Section 4 we recall from [35] some basic notions concerning GVFAs. In Section 5, the solidity of a GVFA is defined in the natural way: if $\mathcal{K}$ is a category of substitutions, a GVFA $\mathbf{U}$ is said to be $\mathcal{K}$-solid if $D_{\mathcal{K}}(\mathbf{U}) \subseteq \mathbf{U}$, where $D_{\mathcal{K}}(\mathbf{U})$ is the class of all derived algebras obtained from a member of $\mathbf{U}$ by a hypersubstitution from $\mathcal{K}$. We give some properties of the operators $D_{\mathcal{K}}$ and a representation for the $\mathcal{K}$-solid GVFA generated by a given class of finite algebras. We also define a new product of finite algebras based on a general hypersubstitution.

For a category of substitutions $\mathcal{K}$, we call a tree homomorphism a $\mathcal{K}$-morphism if its underlying hypersubstitution belongs to $\mathcal{K}$. In Section 6 a GVTL $\mathcal{V}$ is defined to be $\mathcal{K}$-solid if for any $\mathcal{K}$-morphism φ, the pre-image $T\varphi^{-1}$ of any tree language T in $\mathcal{V}$ is also in $\mathcal{V}$. We show that if a GVTL is $\mathcal{K}$-solid, then so is the corresponding GVFA, and conversely. These results have partial counterparts in [4].

In Section 7 we define the $\mathcal{K}$-solidity of a GVFC and show that if a GVFC is $\mathcal{K}$-solid, then so is the corresponding GVTL. However, instead of proving also the converse, we complete the picture by showing that if a GVFA is $\mathcal{K}$-solid, then so is the corresponding GVFC.

Section 8 forms the other main part of the paper. We settle the solidity status of several known general varieties of tree languages with respect to the categories of linear, non-deleting, strict, symbol-to-symbol and alphabetic substitutions as well as their intersections. The nontrivial GVTLs considered are those of nilpotent, definite, reverse definite, generalized definite, locally testable, aperiodic and piecewise testable tree languages and, in many cases, some sub-varieties of these. Due to the inclusion relations between the categories of substitutions, depicted in Figure 1, it suffices for each GVTL to prove just a couple of positive and negative solidity results. In Section 9 we make some concluding remarks.

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2. General preliminaries

We may write $A := B$ to emphasize that A is defined to be B. Similarly, $A :\Leftrightarrow B$ means that A is defined by the condition expressed by B. For any integer $n \geq 0$, let $[n] := \{1, \ldots, n\}$. For a relation $\rho \subseteq A \times B$, the fact that $(a, b) \in \rho$ is also expressed by $a\,\rho\,b$ or $a \equiv_\rho b$. For any $a \in A$, let $a\rho := \{b \mid a\rho b\}$. For an equivalence relation, we write $[a]_\rho$, or just $[a]$, for $a\rho$. For any $A' \subseteq A$, let $A'\rho := \{b \in B \mid (\exists a \in A')\, a\,\rho\,b\}$. The *converse* of ρ is the relation $\rho^{-1} := \{(b, a) \mid a\rho b\}$

($\subseteq B \times A$). The *composition* of two relations $\rho \subseteq A \times B$ and $\rho' \subseteq B \times C$ is the relation $\rho \circ \rho' := \{(a, c) \mid a \in A, c \in C, (\exists b \in B)\, a\rho b \text{ and } b\rho' c\}$.

For a mapping $\varphi : A \to B$, the image $\varphi(a)$ of an element $a \in A$ is also denoted by $a\varphi$. Especially homomorphisms will be treated this way as right operators and the composition of $\varphi : A \to B$ and $\psi : B \to C$ is written as $\varphi\psi$. For any sets $A_1, \ldots, A_n$ ($n \geq 1$) and any $i \in [n]$, we let π_i denote the i^{th} *projection* $A_1 \times \cdots \times A_n \to A_i$, $(a_1, \ldots, a_n) \mapsto a_i$,.

A *ranked alphabet* Σ is a finite set of symbols each of which has a unique positive integer *arity*. For any $m \geq 1$, the set of m-ary symbols in Σ is denoted by Σ_m. The *rank type* of Σ is the set $r(\Sigma) := \{m \mid \Sigma_m \neq \emptyset\}$. In examples we write $\Sigma = \{f_1/m_1, \ldots, f_k/m_k\}$ when Σ consists of the symbols $f_1, \ldots, f_k$ of the respective arities $m_1, \ldots, m_k$. Similarly as in the theory of hypersubstitutions (cf. [8, 26, 32]), we assume that ranked alphabets contain no nullary symbols. In what follows, Σ, Ω, Γ and Δ are ranked alphabets. In addition to ranked alphabets, we use ordinary finite nonempty alphabets $X, Y, Z, \ldots$ that we call *leaf alphabets*. These are assumed to be disjoint from the ranked alphabets. Furthermore, let $\Xi := \{\xi_1, \xi_2, \xi_3, \ldots\}$ be a countably infinite set of *variables* which do not appear in any of the other alphabets. For any $n \geq 1$, we set $\Xi_n := \{\xi_1, \ldots, \xi_n\}$.

For any ranked alphabet Σ and any set of symbols S such that $\Sigma \cap S = \emptyset$, the set $T_\Sigma(S)$ of Σ*-terms over* S is the smallest set T such that $S \subseteq T$, and $f(t_1, \ldots, t_m) \in T$ whenever $m \in r(\Sigma)$, $f \in \Sigma_m$ and $t_1, \ldots, t_m \in T$. If S is a leaf alphabet X, such terms are regarded in the usual way as representations of labeled trees, and we call them ΣX*-trees*. Subsets of $T_\Sigma(X)$ are called ΣX*-tree languages*. We may also speak simply about *trees* and *tree languages* without specifying the alphabets. The set of *subtrees* $\mathrm{sub}(t)$, the *height* $\mathrm{hg}(t)$ and the *root (symbol)* $\mathrm{root}(t)$ of a ΣX-tree t are defined as follows:

(1) $\mathrm{sub}(x) = \{x\}$, $\mathrm{hg}(t) = 0$ and $\mathrm{root}(t) = x$ for any $x \in X$;

(2) $\mathrm{sub}(t) = \{t\} \cup \mathrm{sub}(t_1) \cup \ldots \cup \mathrm{sub}(t_m)$, $\mathrm{hg}(t) = \max\{\mathrm{hg}(t_1), \ldots, \mathrm{hg}(t_m)\} + 1$ and $\mathrm{root}(t) = f$ for $t = f(t_1, \ldots, t_m)$.

For any $n \geq 1$, $T_\Sigma(\Xi_n)$ is the set of n*-ary* Σ*-terms*, and $T_\Sigma(\Xi) := \bigcup_{n \geq 1} T_\Sigma(\Xi_n)$ is the set of all Σ*-terms with variables.* If $t \in T_\Sigma(\Xi_n)$ and $t_1, \ldots, t_n$ are terms of any kind, $t[t_1, \ldots, t_n]$ denotes the term obtained from t by substituting for every occurrence of a variable $\xi_1, \ldots, \xi_n$ the respective term $t_1, \ldots, t_n$.

Let ξ be a special symbol not in any of our alphabets. A ΣX*-context* is a $\Sigma(X \cup \{\xi\})$-tree in which ξ appears exactly once. The set of all ΣX-contexts is denoted by $C_\Sigma(X)$. If $p, q \in C_\Sigma(X)$, then $p \cdot q = q(p)$ is the ΣX-context obtained from q by replacing the ξ in it with p. Similarly, if $t \in T_\Sigma(X)$ and $p \in C_\Sigma(X)$, then $t \cdot p = p(t)$ is the ΣX-tree obtained when the ξ in p is replaced with t. Clearly, $C_\Sigma(X)$ forms a monoid with $p \cdot q$ as the product and ξ as the unit. The powers p^n of a ΣX-context are defined thus: $p^0 = \xi$ and $p^n = p^{n-1} \cdot p$ ($n \geq 1$).

Any ranked alphabet Σ is also used as a set of operation symbols, and a Σ-*algebra* $\mathcal{A}$ consists of a nonempty set A and a Σ-indexed family of operations $(f^{\mathcal{A}} \mid f \in \Sigma)$ on A such that if $f \in \Sigma_m$, then $f^{\mathcal{A}} : A^m \to A$ is an m-ary operation on A. We write simply $\mathcal{A} = (A, \Sigma)$ without any symbol for the assignment $f \mapsto f^{\mathcal{A}}$. Note that by our above assumption about ranked alphabets, there are no nullary operations. Subalgebras, homomorphisms and direct products are defined as usual (cf. [5, 6, 11], for example). If $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic*, we write $\mathcal{A} \cong \mathcal{B}$, and if there is an epimorphism $\varphi : \mathcal{A} \to \mathcal{B}$, then $\mathcal{B}$ is an *image* of $\mathcal{A}$, $\mathcal{B} \twoheadleftarrow \mathcal{A}$ in symbols. If $\mathcal{A}$ is a subalgebra of $\mathcal{B}$, we write $\mathcal{A} \sqsubseteq \mathcal{B}$. Furthermore, $\mathcal{B}$ is said to *cover* $\mathcal{A}$, expressed by $\mathcal{A} \preceq \mathcal{B}$, if $\mathcal{A}$ is an image of a subalgebra of $\mathcal{B}$.

For any Σ and X, the ΣX-*term algebra* $\mathcal{T}_\Sigma(X) = (T_\Sigma(X), \Sigma)$ is defined by putting $f^{\mathcal{T}_\Sigma(X)}(t_1, \ldots, t_m) = f(t_1, \ldots, t_m)$ for all $m \in r(\Sigma)$, $f \in \Sigma_m$ and $t_1, \ldots, t_m \in T_\Sigma(X)$. It is generated by X and any mapping $\alpha : X \to A$ of X into any Σ-algebra $\mathcal{A} = (A, \Sigma)$ has a unique homomorphic extension $\widehat{\alpha} : \mathcal{T}_\Sigma(X) \to \mathcal{A}$. If $t^{\mathcal{A}} : A^n \to A$ is the *term function* defined in $\mathcal{A}$ by a term $t \in T_\Sigma(\Xi_n)$, then $t^{\mathcal{A}}(a_1, \ldots, a_n) = t\widehat{\alpha}$ for all $(a_1, \ldots, a_n) \in A^n$, when $\widehat{\alpha} : \mathcal{T}_\Sigma(\Xi_n) \to \mathcal{A}$ is obtained from the map $\alpha : \Xi_n \to A$, $\xi_i \mapsto a_i$.

A mapping $p : A \to A$ is an *elementary translation* of $\mathcal{A} = (A, \Sigma)$ if there exist an $m \in r(\Sigma)$, an $f \in \Sigma_m$, an $i \in [m]$, and elements $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m \in A$ such that $p(a) = f^{\mathcal{A}}(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_m)$ for every $a \in A$. The set $\mathrm{Tr}(\mathcal{A})$ of all *translations* of $\mathcal{A}$ is the smallest set of unary operations on A that contains the identity map $1_A : A \to A, a \mapsto a$, and all the elementary translations, and is closed under composition. It is well known [5, 6] that an equivalence on A is a congruence of $\mathcal{A}$ exactly in case it is invariant with respect to every (elementary) translation of $\mathcal{A}$. The translations of the term algebra $\mathcal{T}_\Sigma(X)$ correspond to ΣX-contexts: for any $p \in \mathrm{Tr}(\mathcal{T}_\Sigma(X))$, there is a unique $q \in C_\Sigma(X)$ such that $p(t) = q(t)$ for every $t \in T_\Sigma(X)$, and conversely.

3. Tree homomorphisms and hypersubstitutions

We shall now clarify the relation between hypersubstitutions (cf. [26, 10, 11, 32]) and tree homomorphisms (cf. [37, 13, 2, 17, 18]). Then we introduce the systems of hypersubstitutions and tree homomorphisms to be used for defining our notions of solidity.

Definition 3.1. A *tree homomorphism* $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ is determined by a mapping $\varphi_X : X \to T_\Omega(Y)$ and a mapping $\varphi_m : \Sigma_m \to T_\Omega(Y \cup \Xi_m)$ for each $m \in r(\Sigma)$ as follows:

(1) $x\varphi = \varphi_X(x)$ for $x \in X$, and

(2) $t\varphi = \varphi_m(f)[t_1\varphi, \ldots, t_m\varphi]$ for $t = f(t_1, \ldots, t_m)$.

Definition 3.2. A $\Sigma\Omega$-*hypersubstitution* is a mapping $\varkappa : \Sigma \to T_\Omega(\Xi)$ such that if $f \in \Sigma_m$, then $\varkappa(f) \in T_\Omega(\Xi_m)$. We write $\varkappa : \Sigma \to \Omega$ and call these mappings simply $\Sigma\Omega$-*substitutions*, or just *substitutions* without specifying the ranked alphabets. Let $\mathcal{S}(\Sigma, \Omega)$ denote the set of $\Sigma\Omega$-substitutions

A substitution $\varkappa : \Sigma \to \Omega$ is extended to a mapping $\widehat{\varkappa} : T_\Sigma(\Xi) \to T_\Omega(\Xi)$ by setting $\xi_i\widehat{\varkappa} = \xi_i$ for $\xi_i \in \Xi$, and $t\widehat{\varkappa} = \varkappa(f)[t_1\widehat{\varkappa}, \ldots, t_m\widehat{\varkappa}]$ for $t = f(t_1, \ldots, t_m)$.

For each $n \geq 1$, we get a mapping $\widehat{\varkappa}_n : T_\Sigma(\Xi_n) \to T_\Omega(\Xi_n)$ as a restriction of $\widehat{\varkappa}$. We denote also $\widehat{\varkappa}$ and $\widehat{\varkappa}_n$ by $\varkappa$ if there is no danger of confusion.

The *composition* $\varkappa\lambda$ of a $\Sigma\Omega$-substitution $\varkappa$ and an $\Omega\Gamma$-substitution λ is the $\Sigma\Gamma$-substitution $\varkappa\lambda : \Sigma \to \Gamma$ such that $(\varkappa\lambda)(f) = \widehat{\lambda}(\varkappa(f))$ for every $f \in \Sigma$. For each ranked alphabet Σ, we define the *identity* $\Sigma\Sigma$-*substitution* $\iota_\Sigma : \Sigma \to \Sigma$ by setting $\iota_\Sigma(f) = f(\xi_1, \ldots, \xi_m)$ for all $m \in r(\Sigma)$ and $f \in \Sigma_m$.

For any Σ, the $\Sigma\Sigma$-substitutions are just the ordinary hypersubstitutions of type Σ. It is also obvious that $\widehat{\varkappa\lambda} = \widehat{\varkappa}\widehat{\lambda}$, $(\varkappa\lambda)\mu = \varkappa(\lambda\mu)$, and $\iota_\Sigma\varkappa = \varkappa = \varkappa\iota_\Omega$ for any substitutions $\varkappa : \Sigma \to \Omega$, $\lambda : \Omega \to \Gamma$ and $\mu : \Gamma \to \Delta$.

Any $\Sigma\Omega$-substitution $\varkappa$ yields a tree homomorphism $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ such that $\varphi_m(f) = \varkappa(f)$ for all $m \in r(\Sigma)$ and $f \in \Sigma_m$, when we introduce a mapping $\varphi_X : X \to T_\Omega(Y)$. The converse construction is not always possible since for a tree homomorphism $\varphi : T_\Sigma(X) \to T_\Omega(Y)$, the terms $\varphi_m(f)$ may include also symbols from Y. We eliminate this discrepancy as follows.

Definition 3.3. A tree homomorphism $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ is *pure* if $\varphi_m(f) \in T_\Omega(\Xi_m)$ for all $m \in r(\Sigma)$ and $f \in \Sigma_m$. The *underlying substitution* $\dot{\varphi} : \Sigma \to \Omega$ of a pure tree homomorphism $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ is defined by setting $\dot{\varphi}(f) = \varphi_m(f)$ for all $m \in r(\Sigma)$ and $f \in \Sigma_m$.

Clearly, a pure tree homomorphism $\varphi : T_\Sigma(X) \to T_\Sigma(Y)$ is a homomorphism of Σ-algebras $\mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma(Y)$ if and only if $\dot{\varphi} = \iota_\Sigma$. Moreover, $(\dot{\varphi\psi}) = \dot{\varphi}\dot{\psi}$ for any pure tree homomorphisms $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ and $\psi : T_\Omega(Y) \to T_\Gamma(Z)$.

Any pure tree homomorphism has a unique underlying substitution, but many pure tree homomorphisms $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ belong to the same $\Sigma\Omega$-substitution because the map $\varphi_X : X \to T_\Omega(Y)$ can be freely chosen.

The pure tree homomorphisms are also obtained from the following notion introduced by Głazek [19] and Kolibiar [25] (cf. also [11, 26, 32]).

Definition 3.4. A mapping $\varphi : A \to B$ is a *semi-weak homomorphism* from an algebra $\mathcal{A} = (A, \Sigma)$ to an algebra $\mathcal{B} = (B, \Omega)$, if for all $m \in r(\Sigma)$ and $f \in \Sigma_m$, there is a term $t \in T_\Omega(\Xi_m)$ such that $f^{\mathcal{A}}(a_1, \ldots, a_m)\varphi = t^{\mathcal{B}}(a_1\varphi, \ldots, a_m\varphi)$ for all $a_1, \ldots, a_m \in A$.

The following observation has a straightforward proof.

Proposition 3.5. *A mapping $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ is a semi-weak homomorphism from $\mathcal{T}_\Sigma(X)$ to $\mathcal{T}_\Omega(Y)$ if and only if it is a pure tree homomorphism.*

From now on, we shall assume that all tree homomorphisms considered are pure even when this is not explicitly said. The following classes of substitutions correspond to some well-known types of tree homomorphisms.

Definition 3.6. A $\Sigma\Omega$-substitution $\kappa : \Sigma \to \Omega$ is

(1) *linear* if for all $m \in r(\Sigma)$ and $f \in \Sigma_m$, each ξ_i $(i \in [m])$ appears at most once in $\varkappa(f)$, and otherwise it is *nonlinear*;

(2) *non-deleting* if for all $m \in r(\Sigma)$ and $f \in \Sigma_m$, every ξ_i $(i \in [m])$ appears at least once in $\varkappa(f)$, and otherwise it is *deleting*;

(3) *strict* if $\varkappa(f) = \xi_i$ for no $m \in r(\Sigma)$, $f \in \Sigma_m$ and $i \in [m]$;

(4) *symbol-to-symbol* if for all $m \in r(\Sigma)$ and $f \in \Sigma_m$, $\varkappa(f) = g(\xi_{i_1}, \ldots, \xi_{i_k})$ for some $k \in r(\Omega)$, $g \in \Omega_k$ and $i_1, \ldots, i_k \in [m]$;

(5) *alphabetic* if for all $m \in r(\Sigma)$ and $f \in \Sigma_m$, $\varkappa(f) = g(\xi_1, \ldots, \xi_m)$ for some $g \in \Omega_m$.

Let $l\mathcal{S}(\Sigma,\Omega)$, $n\mathcal{S}(\Sigma,\Omega)$, $s\mathcal{S}(\Sigma,\Omega)$, $ss\mathcal{S}(\Sigma,\Omega)$ and $a\mathcal{S}(\Sigma,\Omega)$ denote the sets of all linear, non-deleting, strict, symbol-to-symbol and alphabetic $\Sigma\Omega$-substitutions, respectively. Intersections of these sets are denoted by combining prefixes. For example, $ln\mathcal{S}(\Sigma,\Omega)$ is the set of all linear non-deleting $\Sigma\Omega$-substitutions.

Strict substitutions are also called *pre-hypersubstitutions*, and (linear) non-deleting substitutions are sometimes said to be *regular*. Following [35] we call alphabetic substitutions also *assignments*.

By a *family of substitutions* we mean a map $\mathcal{K}$ that assigns to each pair Σ, Ω of ranked alphabets a set $\mathcal{K}(\Sigma,\Omega)$ of $\Sigma\Omega$-substitutions. We write $\mathcal{K} = \{\mathcal{K}(\Sigma,\Omega)\}$ with the understanding that Σ and Ω range over all ranked alphabets. The inclusion relation and intersections of these families are defined in the natural way: $\mathcal{K} \subseteq \mathcal{K}'$ iff $\mathcal{K}(\Sigma,\Omega) \subseteq \mathcal{K}'(\Sigma,\Omega)$ for all Σ and Ω, and $\mathcal{K} \cap \mathcal{K}' = \{\mathcal{K}(\Sigma,\Omega) \cap \mathcal{K}'(\Sigma,\Omega)\}$.

The largest family of substitutions is $\mathcal{S} := \{\mathcal{S}(\Sigma,\Omega)\}$. From Definition 3.6 we get the families $l\mathcal{S} := \{l\mathcal{S}(\Sigma,\Omega)\}$, $n\mathcal{S} := \{n\mathcal{S}(\Sigma,\Omega)\}$, $ln\mathcal{S} := \{ln\mathcal{S}(\Sigma,\Omega)\}$, etc. Moreover, let $\mathcal{I} = \{\mathcal{I}(\Sigma,\Omega)\}$ be such that for any Σ and Ω, $\mathcal{I}(\Sigma,\Sigma) = \{\iota_\Sigma\}$ but $\mathcal{I}(\Sigma,\Omega) = \emptyset$ if $\Sigma \neq \Omega$. Clearly, $\mathcal{I} \subset a\mathcal{S} \subset ss\mathcal{S} \subset s\mathcal{S}$ and $a\mathcal{S} \subset l\mathcal{S} \cap n\mathcal{S} \cap ss\mathcal{S}$.

If $\mathcal{K}$ is a family of substitutions, a tree homomorphism $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ is called a $\mathcal{K}$-*morphism* if its underlying substitution $\dot\varphi$ is in $\mathcal{K}(\Sigma,\Omega)$.

Remark 3.7.

a. Any $\mathcal{I}$-morphism $\varphi : T_\Sigma(X) \to T_\Sigma(Y)$ is also a homomorphism $\varphi : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma(Y)$ of Σ-algebras, and conversely.
b. The $l\mathcal{S}$-, $n\mathcal{S}$- and $s\mathcal{S}$-morphisms are, respectively, exactly the linear, non-deleting and strict (pure) tree homomorphisms.
c. The $a\mathcal{S}$-morphisms are in essence the *inner alphabetic tree homomorphisms* of [24]. In [35] they were obtained as the g-morphisms of term algebras. The *alphabetic tree homomorphisms* (cf. [17, 18]) are the $a\mathcal{S}$-morphisms that map every leaf symbol to a leaf symbol. Similarly, the $ss\mathcal{S}$-morphism are generalized symbol-to-symbol tree homomorphisms.

Definition 3.8. A family of substitutions $\mathcal{K} = \{\mathcal{K}(\Sigma, \Omega)\}$ is called a *category of substitutions* if the following conditions hold for all Σ, Ω and Γ.

(C1) $a\mathcal{S}(\Sigma, \Omega) \subseteq \mathcal{K}(\Sigma, \Omega)$.

(C2) If $\varkappa \in \mathcal{K}(\Sigma, \Omega)$ and $\lambda \in \mathcal{K}(\Omega, \Gamma)$, then $\varkappa\lambda \in \mathcal{K}(\Sigma, \Gamma)$.

(C3) If $\varkappa \in \mathcal{S}(\Sigma, \Omega)$ and $\varkappa\iota \in \mathcal{K}(\Sigma, \Gamma)$ for some $\iota \in a\mathcal{S}(\Omega, \Gamma)$, then $\varkappa \in \mathcal{K}(\Sigma, \Omega)$.

Requirement (C3) means that any substitution $\varkappa : \Sigma \to \Omega$ that becomes a $\mathcal{K}$-substitution by an alphabetic relabeling of the images $\varkappa(f)$, is also itself in $\mathcal{K}$.

Lemma 3.9. *Let $\mathcal{K} = \{\mathcal{K}(\Sigma, \Omega)\}$ be a category of substitutions.*

(C4) *$\iota_\Sigma \in \mathcal{K}(\Sigma, \Sigma)$ for every ranked alphabet Σ.*

(C5) *Every projection $\pi_i : \Sigma^1 \times \cdots \times \Sigma^n \to \Sigma^i$ is in $\mathcal{K}(\Sigma^1 \times \cdots \times \Sigma^n, \Sigma^i)$ $(i \in [n])$.*

(C6) *If $\varkappa \in \mathcal{K}(\Sigma, \Omega)$ and $\Omega \subseteq \Gamma$, then $\varkappa \in \mathcal{K}(\Sigma, \Gamma)$ when we view any $\Sigma\Omega$-substitution in the natural way also as a $\Sigma\Gamma$-substitution.*

Proof. (C4) and (C5) follow from (C1). The embedding $\iota_{\Omega,\Gamma} : \Omega \to \Gamma$, $f \mapsto f$, of Ω into Γ is an alphabetic substitution, and $\varkappa$ viewed as a $\Sigma\Gamma$-substitution is just the composition $\varkappa\iota_{\Omega,\Gamma}$. Hence (C6) follows from (C1) and (C2). ∎

The following are our most important examples of categories of substitutions.

Proposition 3.10. *The families $\mathcal{S}$, $l\mathcal{S}$, $n\mathcal{S}$, $s\mathcal{S}$, $ss\mathcal{S}$, $a\mathcal{S}$ and their intersections (such as $ln\mathcal{S} = l\mathcal{S} \cap n\mathcal{S}$) are categories of substitutions. Moreover, $\mathcal{S}$ is the greatest category of substitutions while $a\mathcal{S}$ is the least category of substitutions.*

Proof. It is clear that $\mathcal{S}$ satisfies all three conditions (C1)–(C3), and it is easy to verify conditions (C1) and (C2) for $l\mathcal{S}$, $n\mathcal{S}$, $s\mathcal{S}$, $ss\mathcal{S}$ and $a\mathcal{S}$.

Let $\varkappa : \Sigma \to \Omega$ be any $\Sigma\Omega$-substitution and let $\iota \in a\mathcal{S}(\Omega, \Gamma)$. Consider any $m \in r(\Sigma)$ and $f \in \Sigma_m$. Since ι is alphabetic, $\iota(\varkappa(f))$ is obtained from $\varkappa(f)$ by relabeling each inner node while preserving all variables. This 'isomorphism' implies that $\iota(\varkappa(f))$ is linear, nondeleting, in $T_\Omega(\Xi_m)\backslash\Xi_m$, of the form $g(\xi_{i_1}, \ldots, \xi_{i_k})$, or of the form $g(\xi_1, \ldots, \xi_m)$, if and only if $\varkappa(f)$ has the same respective form. It follows that if $\varkappa\iota$ is linear, nondeleting, strict, symbol-to-symbol or alphabetic, then so is $\varkappa$. Hence, $l\mathcal{S}$, $n\mathcal{S}$, $s\mathcal{S}$, $ss\mathcal{S}$ and $a\mathcal{S}$ satisfy (C3), too. Finally, we note that if two families satisfy one of the conditions (Ci), then also their intersection satisfies (Ci). The assertions concerning $\mathcal{S}$ and $a\mathcal{S}$ are obvious. ■

4. General varieties of finite algebras

We shall now recall some notions and facts from Section 3 of [35]. The prefix g appearing in some names stands for "generalized".

We call Ω a *subalphabet* of Σ and write $\Omega \subseteq \Sigma$, if $\Omega_m \subseteq \Sigma_m$ for every $m > 0$. If $\Omega \subseteq \Sigma$, an Ω-algebra $\mathcal{B} = (B, \Omega)$ is an Ω-*subalgebra* of a Σ-algebra $\mathcal{A} = (A, \Sigma)$ if $B \subseteq A$ and $f^{\mathcal{B}}(b_1, \ldots, b_m) = f^{\mathcal{A}}(b_1, \ldots, b_m)$ for all $m \in r(\Omega)$, $f \in \Omega_m$ and $b_1, \ldots, b_m \in B$. Then we also call $\mathcal{B}$ a *g-subalgebra* of $\mathcal{A}$ without specifying Ω.

A *g-morphism* $(\iota, \varphi) : \mathcal{A} \to \mathcal{B}$ from a Σ-algebra $\mathcal{A} = (A, \Sigma)$ to an Ω-algebra $\mathcal{B} = (B, \Omega)$ consists of an assignment $\iota : \Sigma \to \Omega$ and a mapping $\varphi : A \to B$ such that $f^{\mathcal{A}}(a_1, \ldots, a_m)\varphi = \iota(f)^{\mathcal{B}}(a_1\varphi, \ldots, a_m\varphi)$ for all $m \in r(\Sigma)$, $f \in \Sigma_m$ and $a_1, \ldots, a_m \in A$. It is a *g-epimorphism*, a *g-monomorphism* or a *g-isomorphism* if the maps ι and φ are surjective, injective or bijective, respectively. We call $\mathcal{B}$ a *g-image* of $\mathcal{A}$, if there exists a g-epimorphism $(\iota, \varphi) : \mathcal{A} \to \mathcal{B}$, and $\mathcal{A}$ and $\mathcal{B}$ are *g-isomorphic*, $\mathcal{A} \cong_g \mathcal{B}$ in symbols, if there is a g-isomorphism $(\iota, \varphi) : \mathcal{A} \to \mathcal{B}$.

In a g-morphism $(\iota, \varphi) : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Omega(Y)$ of term algebras, φ is the $a\mathcal{S}$-morphism $T_\Sigma(X) \to T_\Omega(Y)$ such that $\varphi_X(x) = x\varphi$ for $x \in X$, and $\varphi_m(f) = \iota(f)(\xi_1, \ldots, \xi_m)$ for $m \in r(\Sigma), f \in \Sigma_m$. Moreover, ι is fully determined by φ. Conversely, any $a\mathcal{S}$-morphism $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ yields a unique g-morphism $(\iota, \varphi) : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Omega(Y)$, where $\iota(f) = \varphi_m(f)$ for all $m \in r(\Sigma)$ and $f \in \Sigma_m$. Hence, we may replace g-morphisms of term algebras by $a\mathcal{S}$-morphisms.

An *equivalence* on a ranked alphabet Σ is an equivalence σ on the set Σ such that if $f \,\sigma\, g$ for some $f, g \in \Sigma$, then f and g have the same arity. Let $\mathrm{Er}(\Sigma)$ denote the set of these equivalences. For any $\sigma \in \mathrm{Er}(\Sigma)$, the *quotient ranked alphabet* Σ/σ is defined by $(\Sigma/\sigma)_m := \{[f]_\sigma \mid f \in \Sigma_m\}$ $(m > 0)$. A *g-congruence* of an algebra $\mathcal{A} = (A, \Sigma)$ is a pair $(\sigma, \theta) \in \mathrm{Er}(\Sigma) \times \mathrm{Eq}(A)$ such that for all $m \in r(\Sigma)$, $f, g \in \Sigma_m$ and $a_1, \ldots, a_m, b_1, \ldots, b_m \in A$, if $f \,\sigma\, g$ and $a_1 \,\theta\, b_1, \ldots, a_m \,\theta\, b_m$, then $f^{\mathcal{A}}(a_1, \ldots, a_m)\theta g^{\mathcal{A}}(b_1, \ldots, b_m)$. Let $\mathrm{GCon}(\mathcal{A})$ denote the set of all g-congruences of $\mathcal{A}$. It is clear that if $(\sigma, \theta) \in \mathrm{GCon}(\mathcal{A})$, then $\theta \in \mathrm{Con}(\mathcal{A})$. The *g-quotient algebra* of $\mathcal{A}$ with respect to $(\sigma, \theta) \in \mathrm{GCon}(\mathcal{A})$ is the Σ/σ-algebra $\mathcal{A}/(\sigma,\theta) = (A/\theta, \Sigma/\sigma)$ such that $[f]_\sigma^{\mathcal{A}/(\sigma,\theta)}([a_1]_\theta, \ldots, [a_m]_\theta) = [f^{\mathcal{A}}(a_1, \ldots, a_m)]_\theta$ for all $m \in r(\Sigma)$, $f \in \Sigma_m$ and $a_1, \ldots, a_m \in A$.

The usual relations between homomorphisms, congruences and quotient algebras hold also between g-morphisms, g-congruences and g-quotients. In particular, the kernel $\ker(\iota,\varphi) := (\ker\iota, \ker\varphi)$ of any g-morphism $(\iota,\varphi) : \mathcal{A} \to \mathcal{B}$ is a g-congruence of $\mathcal{A}$, and if (ι,φ) is a g-epimorphism, then $\mathcal{A}/\ker(\iota,\varphi) \cong_g \mathcal{B}$.

For our purposes it suffices to define the generalized direct products for finite families of algebras only. The *product* $\Sigma^1 \times \cdots \times \Sigma^n$ of ranked alphabets $\Sigma^1, \ldots, \Sigma^n$ is the ranked alphabet Σ such that $\Sigma_m = \Sigma^1_m \times \cdots \times \Sigma^n_m$ for every $m > 0$. Obviously, $r(\Sigma) = r(\Sigma^1) \cap \ldots \cap r(\Sigma^n)$. Let $\varkappa : \Gamma \to \Sigma^1 \times \cdots \times \Sigma^n$ be an assignment for some ranked alphabet Γ. For each $i \in [n]$, the composition $\varkappa_i := \varkappa\pi_i$ of $\varkappa$ and the i^{th} projection $\pi_i : \Sigma^1 \times \cdots \times \Sigma^n \to \Sigma^i$ is an assignment $\Gamma \to \Sigma^i$. The *$\varkappa$-product* of any algebras $\mathcal{A}_1 = (A_1, \Sigma^1), \ldots, \mathcal{A}_n = (A_n, \Sigma^n)$ is the Γ-algebra $\varkappa(\mathcal{A}_1, \ldots, \mathcal{A}_n) = (A_1 \times \cdots \times A_n, \Gamma)$ such that

$$f^{\varkappa(\mathcal{A}_1,\ldots,\mathcal{A}_n)}(\mathbf{a}_1, \ldots, \mathbf{a}_m) = (\varkappa_1(f)^{\mathcal{A}_1}(a_{11}, \ldots, a_{m1}), \ldots, \varkappa_n(f)^{\mathcal{A}_n}(a_{1n}, \ldots, a_{mn})),$$

for all $m \in r(\Gamma)$, $f \in \Gamma_m$ and $\mathbf{a}_i = (a_{i1}, \ldots, a_{in}) \in A_1 \times \cdots \times A_n$ $(i = 1, \ldots, m)$. For $n = 0$, we define the product to be the trivial Γ-algebra. Without specifying the assignment $\varkappa$, we call such products jointly *g-products*.

A class of finite Σ-algebras $\mathbf{U}$ is a *variety of finite Σ-algebras* (Σ-VFA), or a *pseudovariety*, if it is closed under the formation of subalgebras, homomorphic images and finite direct products, i.e., if $S(\mathbf{U}), H(\mathbf{U}), P_f(\mathbf{U}) \subseteq \mathbf{U}$. These Σ-VFAs correspond bijectively to varieties of tree languages over the given ranked alphabet Σ (cf. [1, 33, 34, 36]). To obtain such an Eilenberg-type correspondence for varieties of tree languages that contain tree languages over all ranked alphabets, we have to consider varieties of finite algebras that contain algebras of all finite types. Thus, when we now say that $\mathbf{U}$ is a *class of finite algebras*, $\mathbf{U}$ may include finite Σ-algebras for any Σ. The class of Σ-algebras in $\mathbf{U}$ is denoted by $\mathbf{U}_\Sigma$.

For any class $\mathbf{U}$ of finite algebras, let $S_g(\mathbf{U})$ be the class of all g-subalgebras of members of $\mathbf{U}$, $H_g(\mathbf{U})$ the class of all g-images of members of $\mathbf{U}$, and $P_{gf}(\mathbf{U})$ be the class of all algebras isomorphic to a g-product of members of $\mathbf{U}$. We call $\mathbf{U}$ a *generalized variety of finite algebras (GVFA)* if $S_g(\mathbf{U}), H_g(\mathbf{U}), P_{gf}(\mathbf{U}) \subseteq \mathbf{U}$. The *GVFA generated* by a class $\mathbf{U}$ of finite algebras is denoted by $V_{gf}(\mathbf{U})$.

Let Q and R be any algebra class operators such as S_g, H_g or P_{gf}. As usual, QR is the operator such that $QR(\mathbf{U}) = Q(R(\mathbf{U}))$ for each class $\mathbf{U}$, and we write $Q \leq R$ iff $Q(\mathbf{U}) \subseteq R(\mathbf{U})$ for every $\mathbf{U}$. We shall use the operators S, H and P_f also in an extended sense by applying them to general classes of finite algebras. The obvious relations $S \leq S_g$, $H \leq H_g$ and $P_f \leq P_{gf}$ are frequently used without comment. As shown in [35], $S_gS_g = S_g$, $H_gH_g = H_g$, and $P_{gf}P_{gf} = P_{gf}$, and $S_gH_g \leq H_gS_g$, $P_{gf}H_g \leq HP_{gf} \leq H_gP_{gf}$, $P_{gf}S_g \leq SP_{gf} \leq S_gP_{gf}$, and hence $V_{gf}(\mathbf{U}) = H_gS_gP_{gf}(\mathbf{U})$ for any $\mathbf{U}$. In fact, it was shown that a finite algebra $\mathcal{A}$

belongs to $V_{gf}(\mathbf{U})$ iff $\mathcal{A} \preceq \varkappa(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ for a g-product $\varkappa(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ of some members $\mathcal{A}_1, \ldots, \mathcal{A}_n$ ($n \geq 0$) of $\mathbf{U}$, that is to say, $V_{gf}(\mathbf{U}) = HSP_{gf}(\mathbf{U})$.

Finally, let us note that if $\mathbf{U}$ is a GVFA, then $\mathbf{U}_\Sigma$ is a Σ-VFA for every Σ.

5. The solidity of general varieties of finite algebras

The $\mathcal{K}$-solid varieties of finite algebras to be defined in this section extend the notion of M-solid pseudovarieties of [21] or [4] to general varieties of finite algebras. We use the following variant of a notion considered in [20, 21, 32], for example.

Definition 5.1. For any $\Sigma\Omega$-substitution $\varkappa : \Sigma \to \Omega$ and any Ω-algebra $\mathcal{B} = (B, \Omega)$, the Σ-algebra $\varkappa(\mathcal{B}) = (B, \Sigma)$ such that $f^{\varkappa(\mathcal{B})} = \varkappa(f)^{\mathcal{B}}$ for all $f \in \Sigma$, is called a *derived algebra* of $\mathcal{B}$. If $\varkappa \in \mathcal{K}(\Sigma, \Omega)$ for a category of substitutions $\mathcal{K}$, we call $\varkappa(\mathcal{B})$ also a $\mathcal{K}$-*derived algebra* of $\mathcal{B}$. For any class $\mathbf{U}$ of algebras, let $D_{\mathcal{K}}(\mathbf{U})$ denote the class of all $\mathcal{K}$-derived algebras of members of $\mathbf{U}$.

Clearly, $\varkappa(\lambda(\mathcal{A})) = (\varkappa\lambda)(\mathcal{A})$ for any substitutions $\varkappa : \Sigma \to \Omega$ and $\lambda : \Omega \to \Gamma$ and any Γ-algebra $\mathcal{A}$. If $\varkappa$ and λ belong to a category $\mathcal{K}$, then so does $\varkappa\lambda$. Hence the following fact.

Lemma 5.2. *$D_{\mathcal{K}} D_{\mathcal{K}} = D_{\mathcal{K}}$ for every category of substitutions $\mathcal{K}$.*

Obviously, $\varphi : A \to B$ is a semi-weak homomorphism from $\mathcal{A} = (A, \Sigma)$ to $\mathcal{B} = (B, \Omega)$ if and only if there is a $\Sigma\Omega$-substitution $\varkappa$ such that φ is a homomorphism of Σ-algebras from $\mathcal{A}$ to $\varkappa(\mathcal{B})$. Also the following facts have easy proofs.

Lemma 5.3. *Let $\varkappa \in \mathcal{S}(\Sigma, \Omega)$, and let $\mathcal{B}$ and $\mathcal{C}$ be Ω-algebras.*

(a) *$t^{\varkappa(\mathcal{B})} = \varkappa(t)^{\mathcal{B}}$ for every $t \in T_\Sigma(\Xi_k), k \geq 1$.*

(b) *Any homomorphism $\varphi : \mathcal{B} \to \mathcal{C}$ of Ω-algebras is also a homomorphism $\varphi : \varkappa(\mathcal{B}) \to \varkappa(\mathcal{C})$ of Σ-algebras.*

(c) *If $\theta \in \mathrm{Con}(\mathcal{B})$, then $\theta \in \mathrm{Con}(\varkappa(\mathcal{B}))$ and $\varkappa(\mathcal{B})/\theta = \varkappa(\mathcal{B}/\theta)$.*

(d) *Any tree homomorphism $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ defines a homomorphism of Σ-algebras $\varphi : \mathcal{T}_\Sigma(X) \to \dot{\varphi}(\mathcal{T}_\Omega(Y))$.*

The following facts are quite obvious.

Lemma 5.4. *Let $\varkappa \in \mathcal{S}(\Sigma, \Omega)$, and let $\mathcal{A}$ and $\mathcal{B}$ be Ω-algebras. If $\mathcal{A} \sqsubseteq \mathcal{B}$, then $\varkappa(\mathcal{A}) \sqsubseteq \varkappa(\mathcal{B})$, and if $\mathcal{A} \twoheadleftarrow \mathcal{B}$, then $\varkappa(\mathcal{A}) \twoheadleftarrow \varkappa(\mathcal{B})$. Also, $\varkappa(\mathcal{A} \times \mathcal{B}) = \varkappa(\mathcal{A}) \times \varkappa(\mathcal{B})$.*

Definition 5.5. For any category of substitutions $\mathcal{K}$, a class $\mathbf{U}$ of finite algebras is said to be $\mathcal{K}$-*solid* if $D_{\mathcal{K}}(\mathbf{U}) \subseteq \mathbf{U}$. In particular, $\mathbf{U}$ is *solid* if it is $\mathcal{S}$-solid. The $\mathcal{K}$-solid GVFA generated by $\mathbf{U}$ is denoted by $V_{\mathcal{K}}(\mathbf{U})$.

In what follows, $\mathcal{K} = \{\mathcal{K}(\Sigma, \Omega)\}$ is any given category of substitutions.

In [20, 32], the relations $DS \leq SD$, $DH \leq HD$ and $DP \leq PD$ were shown for the fixed-type operator D. Hence, the solid variety generated by a class $\mathbf{U}$ of algebras of a given type is $HSPD(\mathbf{U})$. By restricting products to finite families, we get the representation $HSP_fD(\mathbf{U})$ for the solid Σ-VFA generated by a class $\mathbf{U}$ of finite Σ-algebras. We derive a similar description for the GVFAs $V_{\mathcal{K}}(\mathbf{U})$.

Lemma 5.6.

(a) $D_{\mathcal{K}}S \leq D_{\mathcal{K}}S_g \leq SD_{\mathcal{K}} \leq S_gD_{\mathcal{K}}$ *and*

(b) $D_{\mathcal{K}}H \leq D_{\mathcal{K}}H_g \leq HD_{\mathcal{K}} \leq H_gD_{\mathcal{K}}$.

Proof. In both cases, the first and the third inequality are obvious. Let $\mathbf{U}$ be a class of finite algebras. Any member of $D_{\mathcal{K}}S_g(\mathbf{U})$ is of the form $\varkappa(\mathcal{B}) = (B, \Gamma)$, where $\mathcal{B} = (B, \Omega)$ is a g-subalgebra of some $\mathcal{A} = (A, \Sigma)$ in $\mathbf{U}$ and $\varkappa : \Gamma \to \Omega$ is in $\mathcal{K}(\Gamma, \Omega)$. By Lemma 3.9, $\varkappa$ is also in $\mathcal{K}(\Gamma, \Sigma)$ and it is clear $\varkappa(\mathcal{B}) \sqsubseteq \varkappa(\mathcal{A})$. Hence $\varkappa(\mathcal{B}) \in SD_{\mathcal{K}}(\mathbf{U})$ which proves the second inequality of (a).

Any member of $D_{\mathcal{K}}H_g(\mathbf{U})$ has the form $\varkappa(\mathcal{B}) = (B, \Gamma)$, where $\mathcal{B} = (B, \Omega)$ is in $H_g(\mathbf{U})$ and $\varkappa \in \mathcal{K}(\Gamma, \Omega)$. Hence, there exists a g-epimorphism $(\iota, \varphi) : \mathcal{A} \to \mathcal{B}$ from some $\mathcal{A} = (A, \Sigma)$ in $\mathbf{U}$. Since $\iota : \Sigma \to \Omega$ is alphabetic and surjective, there exists for every $t \in T_\Omega(\Xi_m)$ $(m > 0)$ an $s \in T_\Sigma(\Xi_m)$ such that $\iota(s) = t$. This means that we can define a $\Gamma\Sigma$-substitution $\lambda : \Gamma \to \Sigma$ such that $\lambda\iota = \varkappa$. Moreover, $\lambda \in \mathcal{K}(\Gamma, \Sigma)$ by (C3), and hence $\lambda(\mathcal{A}) = (A, \Gamma)$ is in $D_{\mathcal{K}}(\mathbf{U})$. It is straightforward to verify that $\varphi : \lambda(\mathcal{A}) \to \varkappa(\mathcal{B})$ is a homomorphism of Γ-algebras. Since $\varphi : A \to B$ is surjective, this means that $\varkappa(\mathcal{B}) \in HD_{\mathcal{K}}(\mathbf{U})$. ∎

For any substitution $\varkappa : \Gamma \to \Sigma^1 \times \cdots \times \Sigma^n$ and each $i \in [n]$, let $\varkappa_i$ again be the $\Gamma\Sigma^i$-substitution $\varkappa\pi_i : \Gamma \to \Sigma^i$.

Definition 5.7. For any substitution $\varkappa : \Gamma \to \Sigma^1 \times \cdots \times \Sigma^n$, the *$\varkappa$-product* of any algebras $\mathcal{A}_1 = (A_1, \Sigma^1), \ldots, \mathcal{A}_n = (A_n, \Sigma^n)$ is the Γ-algebra $\varkappa(\mathcal{A}_1, \ldots, \mathcal{A}_n) = (A_1 \times \cdots \times A_n, \Gamma)$ such that

$$f^{\varkappa(\mathcal{A}_1,\ldots,\mathcal{A}_n)}(\mathbf{a}_1, \ldots, \mathbf{a}_m) = (\varkappa_1(f)^{\mathcal{A}_1}(a_{11}, \ldots, a_{m1}), \ldots, \varkappa_n(f)^{\mathcal{A}_n}(a_{1n}, \ldots, a_{mn})),$$

for all $m \in r(\Gamma)$, $f \in \Gamma_m$ and $\mathbf{a}_i = (a_{i1}, \ldots, a_{in}) \in A_1 \times \cdots \times A_n$ $(i = 1, \ldots, m)$. For $n = 0$, let $\varkappa(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ be a trivial Γ-algebra. If $\varkappa$ belongs to a category of substitutions $\mathcal{K}$, we call $\varkappa(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ a *$\mathcal{K}$-product*. For any class $\mathbf{U}$ of finite algebras, $P_{\mathcal{K}}(\mathbf{U})$ denotes the class of all $\mathcal{K}$-products of members of $\mathbf{U}$.

The g-products are precisely the $a\mathcal{S}$-products. Hence the following lemma.

Lemma 5.8. $P_{gf} = P_{a\mathcal{S}} \leq P_{\mathcal{K}}$.

For any $\Sigma\Omega$-substitution $\varkappa$ and any Ω-algebra $\mathcal{B} = (B, \Omega)$, the one-component $\varkappa$-product $\varkappa(\mathcal{B})$ is the same algebras as the derived algebra of $\varkappa(\mathcal{B})$ (when we identify (b) and b for each $b \in B$). Hence the following lemma.

Lemma 5.9. $D_{\mathcal{K}} \leq P_{\mathcal{K}}$.

From Lemmas 5.8 and 5.9 we get the following result.

Proposition 5.10. *Every GVFA is $a\mathcal{S}$-solid.*

The following lemma is a direct consequence of Definition 5.7.

Lemma 5.11. *Let $\varkappa : \Gamma \to \Sigma^1 \times \cdots \times \Sigma^n$ be a substitution. Then*

$$\varkappa(\mathcal{A}_1, \ldots, \mathcal{A}_n) = \varkappa_1(\mathcal{A}_1) \times \cdots \times \varkappa_n(\mathcal{A}_n)$$

for all algebras $\mathcal{A}_1 = (A_1, \Sigma^1), \ldots, \mathcal{A}_n = (A_n, \Sigma^n)$.

If $\varkappa \in \mathcal{K}(\Gamma, \Sigma^1 \times \cdots \times \Sigma^n)$, then $\varkappa_i \in \mathcal{K}(\Gamma, \Sigma^i)$ for every $i \in [n]$. Hence the following corollary of Lemma 5.11.

Corollary 5.12. $P_{\mathcal{K}} \leq P_f D_{\mathcal{K}} \leq P_{gf} D_{\mathcal{K}}$.

Lemma 5.13. $D_{\mathcal{K}} P_{\mathcal{K}} = P_{\mathcal{K}}$.

Proof. To prove $D_{\mathcal{K}} P_{\mathcal{K}} \leq P_{\mathcal{K}}$, it suffices to show that $\varkappa(\lambda(\mathcal{A}_1, \ldots, \mathcal{A}_n)) = (\varkappa\lambda)(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ for any substitutions $\varkappa : \Sigma \to \Gamma$ and $\lambda : \Gamma \to \Sigma^1 \times \cdots \times \Sigma^n$ and any algebras $\mathcal{A}_1 = (A_1, \Sigma^1), \ldots, \mathcal{A}_n = (A_n, \Sigma^n)$; if $\varkappa$ and λ are in $\mathcal{K}$, then so is $\varkappa\lambda$. Both sides of the claimed equality are Σ-algebras with $A_1 \times \cdots \times A_n$ as the set of elements. To verify that their operations are the same, we consider any $m \in r(\Sigma)$, $f \in \Sigma_m$ and $\mathbf{a}_i = (a_{i1}, \ldots, a_{in}) \in A_1 \times \cdots \times A_n$ $(i = 1, \ldots, m)$. By using Lemma 5.3 and the obvious fact that $\varkappa\lambda_i = (\varkappa\lambda)_i$ for every $i \in [n]$, we get

$$\begin{aligned} f^{\varkappa(\lambda(\mathcal{A}_1,\ldots,\mathcal{A}_n))}(\mathbf{a}_1, \ldots, \mathbf{a}_m) &= \varkappa(f)^{\lambda(\mathcal{A}_1,\ldots,\mathcal{A}_n)}(\mathbf{a}_1, \ldots, \mathbf{a}_m) \\ &= (\lambda_1(\varkappa(f))^{\mathcal{A}_1}(a_{11}, \ldots, a_{m1}), \ldots, \lambda_n(\varkappa(f))^{\mathcal{A}_n}(a_{1n}, \ldots, a_{mn})) \\ &= ((\varkappa\lambda)_1(f)^{\mathcal{A}_1}(a_{11}, \ldots, a_{m1}), \ldots, (\varkappa\lambda)_n(f)^{\mathcal{A}_n}(a_{1n}, \ldots, a_{mn})) \\ &= f^{(\varkappa\lambda)(\mathcal{A}_1,\ldots,\mathcal{A}_n)}(\mathbf{a}_1, \ldots, \mathbf{a}_m). \end{aligned}$$

The converse inequality $P_{\mathcal{K}} \leq D_{\mathcal{K}} P_{\mathcal{K}}$ is obvious. ∎

Corollary 5.14. $D_{\mathcal{K}}P_f \leq D_{\mathcal{K}}P_{gf} \leq P_f D_{\mathcal{K}} \leq P_{gf}D_{\mathcal{K}}$.

Proof. $D_{\mathcal{K}}P_f \leq D_{\mathcal{K}}P_{gf} \leq D_{\mathcal{K}}P_{\mathcal{K}} = P_{\mathcal{K}} \leq P_f D_{\mathcal{K}} \leq P_{gf}D_{\mathcal{K}}$ by Lemmas 5.8 and 5.13 and Corollary 5.12. ∎

Proposition 5.15. $V_{\mathcal{K}} = H_g S_g P_{gf} D_{\mathcal{K}} = HSP_{gf}D_{\mathcal{K}}$.

Proof. Consider any class $\mathbf{U}$ of finite algebras. Since $\mathbf{U} \subseteq HSP_{gf}D_{\mathcal{K}}(\mathbf{U}) \subseteq H_g S_g P_{gf} D_{\mathcal{K}}(\mathbf{U}) \subseteq V_{\mathcal{K}}(\mathbf{U})$, it suffices to show that $\mathbf{U}' := HSP_{gf}D_{\mathcal{K}}(\mathbf{U})$ is a $\mathcal{K}$-solid GVFA. Since $\mathbf{U}'$ is, by Proposition 4.5 of [35], the GVFA generated by $D_{\mathcal{K}}(\mathbf{U})$, it remains just to verify that $\mathbf{U}'$ is closed under the $D_{\mathcal{K}}$-operator. Indeed, $D_{\mathcal{K}}(\mathbf{U}') \subseteq HD_{\mathcal{K}}SP_{gf}D_{\mathcal{K}}(\mathbf{U}) \subseteq HSD_{\mathcal{K}}P_{gf}D_{\mathcal{K}}(\mathbf{U}) \subseteq HSP_{gf}D_{\mathcal{K}}^2(\mathbf{U}) = \mathbf{U}'$ by Lemma 5.6, Corollary 5.14 and Lemma 5.2. ∎

6. The solidity of general varieties of tree languages

Among the numerous characterizations of the regular tree languages (cf. [17, 18]), the following one is particularly suitable for an algebraic treatment of the subject.

Definition 6.1. An algebra $\mathcal{A} = (A, \Sigma)$ *recognizes* a ΣX-tree language T if there exist a homomorphism $\varphi : \mathcal{T}_\Sigma(X) \to \mathcal{A}$ and a subset $F \subseteq A$ such that $T = F\varphi^{-1}$. A ΣX-tree language is *recognizable*, or *regular*, if it is recognized by a finite Σ-algebra. The set of regular ΣX-tree languages we denote by $Rec(\Sigma, X)$.

A *family of tree languages* is a mapping $\mathcal{V}$ that assigns to each pair Σ, X a set of ΣX-tree languages. We write $\mathcal{V} = \{\mathcal{V}(\Sigma, X)\}$ with the understanding that Σ and X range over all ranked alphabets and leaf alphabets, respectively. The inclusion relation, unions and intersections of these families are defined by the natural componentwise conditions. For example, for $\mathcal{U} = \{\mathcal{U}(\Sigma, X)\}$ and $\mathcal{V} = \{\mathcal{V}(\Sigma, X)\}$, $\mathcal{U} \subseteq \mathcal{V}$ means that $\mathcal{U}(\Sigma, X) \subseteq \mathcal{V}(\Sigma, X)$ for all Σ and X, and $\mathcal{U} \cap \mathcal{V} = \{\mathcal{U}(\Sigma, X) \cap \mathcal{V}(\Sigma, X)\}$. In [35] a family of tree languages $\mathcal{V} = \{\mathcal{V}(\Sigma, X)\}$ was defined to be a *general variety of tree languages (GVTL)* if the following conditions hold for all Σ, Ω, X and Y:

(T1) $\emptyset \neq \mathcal{V}(\Sigma, X) \subseteq Rec(\Sigma, X)$.

(T2) If $T \in \mathcal{V}(\Sigma, X)$, then $T_\Sigma(X) \setminus T \in \mathcal{V}(\Sigma, X)$.

(T3) If $T, U \in \mathcal{V}(\Sigma, X)$, then $T \cap U \in \mathcal{V}(\Sigma, X)$.

(T4) If $T \in \mathcal{V}(\Sigma, X)$, then $p^{-1}(T) := \{t \in T_\Sigma(X) \mid p(t) \in T\} \in \mathcal{V}(\Sigma, X)$ for every $p \in C_\Sigma(X)$.

(T5) If $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ is an $a\mathcal{S}$-morphism, then $T\varphi^{-1} \in \mathcal{V}(\Sigma, X)$ for every $T \in \mathcal{V}(\Omega, Y)$.

Since all $a\mathcal{S}$-morphisms are pure tree homomorphisms, our decision to consider pure tree homomorphisms only does not affect the definition of GVTLs.

Definition 6.2. Let $\mathcal{K} = \{\mathcal{K}(\Sigma, \Omega)\}$ be a category of substitutions. A family of tree languages $\mathcal{V} = \{\mathcal{V}(\Sigma, X)\}$ is said to be $\mathcal{K}$-*solid* if for all Σ, Ω, X and Y, and any $\mathcal{K}$-morphism $\varphi : T_\Sigma(X) \to T_\Omega(Y)$, $T\varphi^{-1} \in \mathcal{V}(\Sigma, X)$ for every $T \in \mathcal{V}(\Omega, Y)$. In particular, $\mathcal{V}$ is said to be *solid* if it is $\mathcal{S}$-solid.

The following fact is an immediate consequence of (T5) and Definition 6.2.

Proposition 6.3. *Every GVTL is $a\mathcal{S}$-solid.*

In [35] it was shown that GVFAs and GVTLs can be linked also via the usual syntactic algebras. The *syntactic congruence* of a ΣX-tree language T is the relation θ_T on $T_\Sigma(X)$ defined by

$$s\,\theta_T\,t \;\Leftrightarrow\; (\forall p \in C_\Sigma(X))(p(s) \in T \leftrightarrow p(t) \in T) \quad (s, t \in T_\Sigma(X)),$$

and the *syntactic algebra* of T is $\mathrm{SA}(T) := \mathcal{T}_\Sigma(X)/\theta_T$. The natural homomorphism $\varphi_T : \mathcal{T}_\Sigma(X) \to \mathrm{SA}(T), t \mapsto [t]_T$, where $[t]_T$ is the θ_T-class of t, is called the *syntactic homomorphism* of T. For any ΣX-tree language T, θ_T is a congruence of $\mathcal{T}_\Sigma(X)$ and it is the greatest congruence that saturates T (i.e., T is the union of some θ_T-classes), a Σ-algebra $\mathcal{A}$ recognizes T if and only if $\mathrm{SA}(T) \preceq \mathcal{A}$, and hence $T \in Rec(\Sigma, X)$ iff $\mathrm{SA}(T)$ is finite (cf. [1, 33, 34, 36]).

For any GVFA $\mathbf{U}$, let $\mathbf{U}^t(\Sigma, X) := \{T \subseteq T_\Sigma(X) \mid \mathrm{SA}(T) \in \mathbf{U}\}$ for all Σ and X. Then $\mathbf{U}^t := \{\mathbf{U}^t(\Sigma, X)\}$ is a GVTL. On the other hand, if for any GVTL $\mathcal{V} = \{\mathcal{V}(\Sigma, X)\}$, we let $\mathcal{V}^a$ be the GVFA generated by the syntactic algebras $\mathrm{SA}(T)$ where $T \in \mathcal{V}(\Sigma, X)$ for some Σ and X, we get the converse map from GVTLs to GVFAs. That is to say, if $\mathbf{U}$ is a GVFA and $\mathcal{V}$ is a GVTL, then $\mathbf{U}^{ta} = \mathbf{U}$ and $\mathcal{V}^{at} = \mathcal{V}$. For further facts about this correspondence cf. [35].

Again, let $\mathcal{K} = \{\mathcal{K}(\Sigma, \Omega)\}$ be any category of substitutions.

Proposition 6.4. *If $\mathbf{U}$ is a $\mathcal{K}$-solid GVFA, then $\mathbf{U}^t$ is a $\mathcal{K}$-solid GVTL.*

Proof. Let $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ be a $\mathcal{K}$-morphism, and let $T \in \mathbf{U}^t(\Omega, Y)$. Then $\mathrm{SA}(T) \in \mathbf{U}_\Omega$, $\varphi_T : \mathcal{T}_\Omega(Y) \to \mathrm{SA}(T)$ is an epimorphism, and $T = F\varphi_T^{-1}$ for some subset F of $\mathrm{SA}(T)$. From Lemma 5.3 it follows that $\varphi : \mathcal{T}_\Sigma(X) \to \dot{\varphi}(\mathcal{T}_\Omega(Y))$ and $\varphi_T : \dot{\varphi}(\mathcal{T}_\Omega(Y)) \to \dot{\varphi}(\mathrm{SA}(T))$ are homomorphisms of Σ-algebras. Obviously $T\varphi^{-1} = F(\varphi\varphi_T)^{-1}$, and thus $T\varphi^{-1}$ is recognized by $\dot{\varphi}(\mathrm{SA}(T))$. Since $\dot{\varphi}(\mathrm{SA}(T)) \in \mathbf{U}_\Sigma$, this means that $T\varphi^{-1} \in \mathbf{U}^t(\Sigma, X)$. ■

Proposition 6.4 parallels Proposition 4 of [4] but our proof is slightly simpler. Also the following converse corresponds to a result appearing in [4].

Proposition 6.5. *If $\mathcal{V}$ is a $\mathcal{K}$-solid GVTL, then $\mathcal{V}^a$ is a $\mathcal{K}$-solid GVFA.*

Proof. Let $\mathbf{U} = \mathcal{V}^a$ and let $\mathbf{U}^*$ denote the class of all syntactic algebras in $\mathbf{U}$. Since $\mathbf{U} = HSP_{gf}(\mathbf{U}^*)$, the $\mathcal{K}$-solidity of $\mathbf{U}$ means that $D_{\mathcal{K}}HSP_{gf}(\mathbf{U}^*) \subseteq \mathbf{U}$, and as $D_{\mathcal{K}}HSP_{gf}(\mathbf{U}^*) \subseteq HSP_{gf}D_{\mathcal{K}}(\mathbf{U}^*)$, it suffices to show that $D_{\mathcal{K}}(\mathbf{U}^*) \subseteq \mathbf{U}$.

Let $\varkappa \in \mathcal{K}(\Sigma, \Omega)$ and let $\mathrm{SA}(T) = (B, \Omega)$ for some $T \in \mathcal{V}(\Omega, Y)$. If X is sufficiently large, there is a $\mathcal{K}$-morphism $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ such that $\dot{\varphi} = \varkappa$ and $T_\Sigma(X)\varphi$ contains an element of every θ_T-class. Then $\varphi : \mathcal{T}_\Sigma(X) \to \varkappa(\mathcal{T}_\Omega(Y))$ and $\varphi_T : \varkappa(\mathcal{T}_\Omega(Y)) \to \varkappa(\mathrm{SA}(T))$ are homomorphisms of Σ-algebras, and $\varphi\varphi_T : \mathcal{T}_\Sigma(X) \to \varkappa(\mathrm{SA}(T))$ is surjective. For each $b \in B$, $b\varphi_T^{-1} \in \mathcal{V}(\Omega, Y)$ (cf. Lemma 5.2 of [34]), and hence $b(\varphi\varphi_T)^{-1} = b\varphi_T^{-1}\varphi^{-1} \in \mathcal{V}(\Sigma, X)$ as $\mathcal{V}$ is $\mathcal{K}$-solid, and this means that $\mathrm{SA}(b(\varphi\varphi_T)^{-1}) \in \mathbf{U}$. It is easy to see that $\bigcap\{\theta_{a\psi^{-1}} \mid a \in A\} \subseteq \ker\psi$ for any algebra $\mathcal{A} = (A, \Sigma)$ and any epimorphism $\psi : \mathcal{T}_\Sigma(X) \to \mathcal{A}$. Since $\varkappa(\mathrm{SA}(T)) \cong \mathcal{T}_\Sigma(X)/\ker\varphi\varphi_T$, this means that $\varkappa(\mathrm{SA}(T))$ is an image of a subdirect product of the algebras $\mathrm{SA}(b(\varphi\varphi_T)^{-1})$, and therefore $\varkappa(\mathrm{SA}(T)) \in \mathbf{U}$. ∎

7. The solidity of varieties of finite g-congruences

For any Σ and X, let $\mathrm{FC}(\Sigma, X) := \{\theta \in \mathrm{Con}(\mathcal{T}_\Sigma(X)) \mid T_\Sigma(X)/\theta \text{ finite}\}$ be the set of *finite congruences* of the term algebra $\mathcal{T}_\Sigma(X)$, and let

$$\mathrm{GFC}(\Sigma, X) := \{(\sigma, \theta) \in \mathrm{GCon}(\mathcal{T}_\Sigma(X)) \mid \theta \in \mathrm{FC}(\Sigma, X)\}$$

be the set of *finite g-congruences* of $\mathcal{T}_\Sigma(X)$. Clearly, $\mathrm{FC}(\Sigma, X)$ is a filter of the congruence lattice $\mathrm{Con}(\mathcal{T}_\Sigma(X))$, and if $(\iota, \varphi) : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Omega(Y)$ is a g-morphism, then $(\iota \circ \omega \circ \iota^{-1}, \varphi \circ \theta \circ \varphi^{-1}) \in \mathrm{GFC}(\Sigma, X)$ for any $(\omega, \theta) \in \mathrm{GFC}(\Omega, Y)$. This fact will be generalized in Lemma 7.2 below.

A *family of finite g-congruences* $\mathcal{C} = \{\mathcal{C}(\Sigma, X)\}$ is a mapping that assigns to each pair Σ, X a subset $\mathcal{C}(\Sigma, X)$ of $\mathrm{GFC}(\Sigma, X)$. It is a *variety of finite g-congruences* (GVFC) if the following conditions hold for all Σ, Ω, X and Y.

(FC1) For every $\sigma \in \mathrm{Er}(\Sigma)$, the set $\mathcal{C}(\Sigma, X)_\sigma := \{\theta \in \mathrm{FC}(\Sigma, X) \mid (\sigma, \theta) \in \mathcal{C}(\Sigma, X)\}$ is a filter of $\mathrm{FC}(\Sigma, X)$.

(FC2) If $(\sigma, \theta) \in \mathcal{C}(\Sigma, X)$ and $(\tau, \theta) \in \mathrm{GFC}(\Sigma, X)$, then $(\tau, \theta) \in \mathcal{C}(\Sigma, X)$.

(FC3) If $(\iota, \varphi) : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Omega(Y)$ is any g-morphism and $(\omega, \theta) \in \mathcal{C}(\Omega, Y)$, then $(\iota \circ \omega \circ \iota^{-1}, \varphi \circ \theta \circ \varphi^{-1}) \in \mathcal{C}(\Sigma, X)$.

For any $\sigma \in \mathrm{Er}(\Sigma)$, let $\bar{\sigma}$ be the least equivalence on $T_\Sigma(\Xi)$ satisfying

(1) $\xi_i \,\bar{\sigma}\, \xi_i$ for every $i = 1, 2, 3, \ldots$, and

(2) if $m \in r(\Sigma)$, $f, g \in \Sigma_m$, and $s_1, \ldots, s_m, t_1, \ldots, t_m \in T_\Sigma(\Xi)$ are such that $f \,\sigma\, g$, and $s_i \,\bar{\sigma}\, t_i$ for every $i \in [m]$, then $f(s_1, \ldots, s_m) \,\bar{\sigma}\, g(t_1, \ldots, t_m)$.

Obviously, $s \,\bar{\sigma}\, t$ means that s and t have the same "shape" and that corresponding leaves in them are labeled by the same variable and corresponding inner nodes by σ-equivalent symbols. The following can be shown by induction on s.

Lemma 7.1. *Let $(\sigma, \theta) \in \mathrm{GCon}(\mathcal{A})$ for some algebra $\mathcal{A} = (A, \Sigma)$. If $s, t \in T_\Sigma(\Xi_n)$ $(n > 0)$ and $s \,\bar{\sigma}\, t$, then $s^{\mathcal{A}}(a_1, \ldots, a_n) \,\theta\, t^{\mathcal{A}}(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n \in A$.*

For any congruence θ of a Σ-algebra $\mathcal{A}$, there is an equivalence $M(\theta) \in \mathrm{Er}(\Sigma)$ such that for any $\sigma \in \mathrm{Er}(\Sigma)$, $(\sigma, \theta) \in \mathrm{GCon}(\mathcal{A})$ iff $\sigma \leq M(\theta)$ (cf. [35]). We define the pre-image of any $(\omega, \theta) \in \mathrm{GFC}(\Omega, Y)$ under any tree homomorphism $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ as $\varphi \circ (\omega, \theta) \circ \varphi^{-1} := (\varphi^{-1}[\omega], \varphi \circ \theta \circ \varphi^{-1})$, where $\varphi^{-1}[\omega] \in \mathrm{Er}(\Sigma)$ is defined so that for any $m \in r(\Sigma)$, $f, g \in \Sigma_m$, $f \,\varphi^{-1}[\omega]\, g$ iff $\dot{\varphi}(f) \,\bar{\omega}\, \dot{\varphi}(g)$.

Lemma 7.2. *Let $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ be any tree homomorphism. If $(\omega, \theta) \in \mathrm{GFC}(\Omega, Y)$, then $\varphi \circ (\omega, \theta) \circ \varphi^{-1} \in \mathrm{GFC}(\Sigma, X)$.*

Proof. Clearly, $\varphi \circ \theta \circ \varphi^{-1}$ is a finite equivalence on $T_\Sigma(X)$, and $\varphi^{-1}[\omega] \in \mathrm{Er}(\Sigma)$ by definition. To show that $\varphi \circ \theta \circ \varphi^{-1} \in \mathrm{Con}(\mathcal{T}_\Sigma(X))$, consider any $m \in r(\Sigma)$, $f \in \Sigma_m$ and $s_1, \ldots, s_m, t_1, \ldots, t_m \in T_\Sigma(X)$ such that $s_i \,\varphi \circ \theta \circ \varphi^{-1}\, t_i$ for every $i \in [m]$. Then $s_i\varphi \,\theta\, t_i\varphi$ for every $i \in [m]$, and therefore

$$
\begin{aligned}
f^{\mathcal{T}_\Sigma(X)}(s_1, \ldots, s_m)\varphi \;&=\; \varphi_m(f)[s_1\varphi, \ldots, s_m\varphi] \;=\; \dot{\varphi}(f)^{\mathcal{T}_\Omega(Y)}(s_1\varphi, \ldots, s_m\varphi) \\
&\equiv_\theta\; \dot{\varphi}(f)^{\mathcal{T}_\Omega(Y)}(t_1\varphi, \ldots, t_m\varphi) \;=\; f^{\mathcal{T}_\Sigma(X)}(t_1, \ldots, t_m)\varphi,
\end{aligned}
$$

i.e., $f^{\mathcal{T}_\Sigma(X)}(s_1, \ldots, s_m) \,\varphi \circ \theta \circ \varphi^{-1}\, f^{\mathcal{T}_\Sigma(X)}(t_1, \ldots, t_m)$. If $f \,\varphi^{-1}[\omega]\, g$ for some $m \in r(\Sigma)$, $f, g \in \Sigma_m$, and $t_1, \ldots, t_m \in T_\Sigma(X)$, then

$$
\begin{aligned}
f^{\mathcal{T}_\Sigma(X)}(t_1, \ldots, t_m)\varphi \;&=\; \varphi_m(f)[t_1\varphi, \ldots, t_m\varphi] \;=\; \dot{\varphi}(f)^{\mathcal{T}_\Omega(Y)}(t_1\varphi, \ldots, t_m\varphi) \\
&\equiv_\theta\; \dot{\varphi}(g)^{\mathcal{T}_\Omega(Y)}(t_1\varphi, \ldots, t_m\varphi) \;=\; g^{\mathcal{T}_\Sigma(X)}(t_1, \ldots, t_m)\varphi
\end{aligned}
$$

by Lemma 7.1, and hence $f^{\mathcal{T}_\Sigma(X)}(t_1, \ldots, t_m) \,\varphi \circ \theta \circ \varphi^{-1}\, g^{\mathcal{T}_\Sigma(X)}(t_1, \ldots, t_m)$. This shows that $\varphi \circ (\omega, \theta) \circ \varphi^{-1}$ is a g-congruence of $\mathcal{T}_\Sigma(X)$. ∎

In the *reduced syntactic congruence* $\rho_T := (\sigma_T, \theta_T)$ of a ΣX-tree language T, θ_T is the usual syntactic congruence of T and $\sigma_T := M(\theta_T)$. The *reduced syntactic algebra* of T is the g-quotient $\mathrm{RA}(T) := \mathcal{T}_\Sigma(X)/\rho_T$, and the *syntactic g-morphism* of T is the g-morphism $(\iota_T, \varphi_T) : \mathcal{T}_\Sigma(X) \to \mathrm{RA}(T)$, where $\iota_T : f \mapsto [f]_T$ and $\varphi_T : t \mapsto [t]_T$. Lemma 7.2 yields following fact.

Corollary 7.3. *For any tree homomorphism $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ and any ΩY-tree language $T \subseteq T_\Omega(Y)$, $\varphi^{-1}[\sigma_T] \subseteq M(\varphi \circ \theta_T \circ \varphi^{-1})$.*

We extend any tree homomorphism $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ to a tree homomorphism $\varphi_* : T_\Sigma(X \cup \{\xi\}) \to T_\Omega(Y \cup \{\xi\})$ by setting $\xi\varphi_* = \xi$. The image $p\varphi_*$ of a ΣX-context p is a unary ΩY-polynomial symbol, i.e., a member of $T_\Omega(Y \cup \{\xi\})$. If φ is non-linear, $p\varphi_*$ may contain several ξ's, and if φ is deleting, $p\varphi_*$ may be an ΩY-tree. Nevertheless, $p(t)\varphi = p\varphi_*(t\varphi)$ for any $p \in C_\Sigma(X)$ and $t \in T_\Sigma(X)$. It is also easy to see that for any ΩY-tree language T and all $s, t \in T_\Omega(Y)$,

$$s\,\theta_T\,t \;\Leftrightarrow\; (\forall q \in T_\Omega(Y \cup \{\xi\}))(q(s) \in T \leftrightarrow q(t) \in T).$$

In fact, such a definition of θ_T is used in [33] and [1], for example.

Lemma 7.4. *For any tree homomorphism $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ and any ΩY-tree language $T \subseteq T_\Omega(Y)$, $\varphi \circ \rho_T \circ \varphi^{-1} \leq \rho_{T\varphi^{-1}}$.*

Proof. We should show that (1) $\varphi^{-1}[\sigma_T] \subseteq \sigma_{T\varphi^{-1}}$ and (2) $\varphi \circ \theta_T \circ \varphi^{-1} \subseteq \theta_{T\varphi^{-1}}$.
For any $s, t \in T_\Sigma(X)$,

$$\begin{aligned} s\,\varphi \circ \theta_T \circ \varphi^{-1}\,t \;&\Leftrightarrow\; s\varphi\,\theta_T\,t\varphi \\ &\Leftrightarrow\; (\forall q \in T_\Omega(Y \cup \{\xi\}))(q(s\varphi) \in T \leftrightarrow q(t\varphi) \in T) \\ &\Rightarrow\; (\forall p \in C_\Sigma(X))(p\varphi_*(s\varphi) \in T \leftrightarrow p\varphi_*(t\varphi) \in T) \\ &\Leftrightarrow\; (\forall p \in C_\Sigma(X))(p(s)\varphi \in T \leftrightarrow p(t)\varphi \in T) \\ &\Leftrightarrow\; s\,\theta_{T\varphi^{-1}}\,t, \end{aligned}$$

from which (2) follows. Now $\varphi^{-1}[\sigma_T] \subseteq M(\varphi \circ \theta_T \circ \varphi^{-1}) \subseteq M(\theta_{T\varphi^{-1}}) = \sigma_{T\varphi^{-1}}$ by Corollary 7.3 and (2), and hence also (1) holds. ∎

Again, let $\mathcal{K}$ be any given category of substitutions.

Definition 7.5. A GVFC $\mathcal{C} = \{\mathcal{C}(\Sigma, X)\}$ is *$\mathcal{K}$-solid* if for all Σ, Ω, X, Y and $(\omega, \theta) \in \mathcal{C}(\Omega, Y)$, $\varphi \circ (\omega, \theta) \circ \varphi^{-1} \in \mathcal{C}(\Sigma, X)$ for every $\mathcal{K}$-morphism $\varphi : T_\Sigma(X) \to T_\Omega(Y)$, and $\mathcal{C}$ is *solid* if it is $\mathcal{S}$-solid.

The GVTL $\mathcal{C}^t = \{\mathcal{C}^t(\Sigma, X)\}$ that corresponds to a given GVFC $\mathcal{C} = \{\mathcal{C}(\Sigma, X)\}$ is defined [35] by the condition $\mathcal{C}^t(\Sigma, X) := \{T \subseteq T_\Sigma(X) \mid \rho_T \in \mathcal{C}(\Sigma, X)\}$.

The following result corresponds to the converse part of Proposition 6 of [4].

Proposition 7.6. *If $\mathcal{C}$ is a $\mathcal{K}$-solid GVFC, then $\mathcal{C}^t$ is a $\mathcal{K}$-solid GVTL.*

Proof. Let $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ be a $\mathcal{K}$-morphism. If $T \in \mathcal{C}^t(\Omega, Y)$, then $\rho_T \in \mathcal{C}(\Omega, Y)$. This implies $\rho_{T\varphi^{-1}} \in \mathcal{C}(\Sigma, X)$ because $\varphi \circ \rho_T \circ \varphi^{-1} \leq \rho_{T\varphi^{-1}}$ by Lemma 7.4, and hence $T\varphi^{-1} \in \mathcal{C}^t(\Sigma, X)$. ∎

As shown in [35], the GVFC $\mathbf{U}^c$ corresponding to a given GVFA $\mathbf{U}$ may be defined also by the condition $\mathbf{U}^c(\Sigma, X) := \{(\sigma,\theta) \in \mathrm{GFC}(\Sigma, X) \mid \mathcal{T}_\Sigma(X)/\theta \in \mathbf{U}\}$.

Proposition 7.7. *If* $\mathbf{U}$ *is a* $\mathcal{K}$*-solid GVFA, then* $\mathbf{U}^c$ *is a* $\mathcal{K}$*-solid GVFC.*

Proof. Let $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ be a $\mathcal{K}$-morphism, and let $(\omega,\theta) \in \mathbf{U}^c(\Omega, Y)$. Then $\mathcal{T}_\Omega(Y)/\theta \in \mathbf{U}_\Omega$, and hence $\dot{\varphi}(\mathcal{T}_\Omega(Y)/\theta) \in \mathbf{U}_\Sigma$.

Let $\beta := \varphi \circ \theta \circ \varphi^{-1}$. We shall verify that

$$\psi : \mathcal{T}_\Sigma(X)/\beta \to \dot{\varphi}(\mathcal{T}_\Omega(Y)/\theta),\ [t]_\beta \mapsto [t\varphi]_\theta,$$

is a monomorphism of Σ-algebras. It is easy to see that ψ is well-defined and injective. Moreover, for any $m \in r(\Sigma)$, $f \in \Sigma_m$ and $t_1, \ldots, t_m \in T_\Sigma(X)$,

$$\begin{aligned} f^{\mathcal{T}_\Sigma(X)/\beta}([t_1]_\beta,\ldots,[t_m]_\beta)\psi &= [f(t_1,\ldots,t_m)]_\beta\psi = [f(t_1,\ldots,t_m)\varphi]_\theta \\ &= [\dot{\varphi}(f)[t_1\varphi,\ldots,t_m\varphi]]_\theta = [\dot{\varphi}(f)^{\mathcal{T}_\Omega(Y)}(t_1\varphi,\ldots,t_m\varphi)]_\theta \\ &= [f^{\dot{\varphi}(\mathcal{T}_\Omega(Y))}(t_1\varphi,\ldots,t_m\varphi)]_\theta = f^{\dot{\varphi}(\mathcal{T}_\Omega(Y))/\theta}([t_1\varphi]_\theta,\ldots,[t_m\varphi]_\theta) \\ &= f^{\dot{\varphi}(\mathcal{T}_\Omega(Y))/\theta}([t_1]_\beta\psi,\ldots,[t_m]_\beta\psi). \end{aligned}$$

Since $\dot{\varphi}(\mathcal{T}_\Omega(Y))/\theta = \dot{\varphi}(\mathcal{T}_\Omega(Y)/\theta)$ by Lemma 5.3, this means that also $\mathcal{T}_\Sigma(X)/\beta$ is in $\mathbf{U}$. It follows by Lemma 7.2 that $\varphi \circ (\omega,\theta) \circ \varphi^{-1} \in \mathbf{U}^c(\Sigma, X)$ as required. ∎

Propositions 6.5, 7.6 and 7.7 may be summed up as follows.

Theorem 7.8. *For any category of substitutions* $\mathcal{K}$*, a GVTL* $\mathcal{V}$ *is* $\mathcal{K}$*-solid iff* $\mathcal{V}^a$ *is a* $\mathcal{K}$*-solid GVFA, and also iff* $\mathcal{V}^c$ *is a* $\mathcal{K}$*-solid GVFC.*

8. The solidity of some general varieties of tree languages

We shall settle the solidity status of several GVTLs with respect to the categories of substitutions that we derived from some classes of tree homomorphisms. Their internal inclusion relations are shown by the Hasse diagram of Figure 1. If a GVTL $\mathcal{V}$ is $\mathcal{K}$-solid for some category $\mathcal{K}$, $\mathcal{V}$ is also $\mathcal{K}'$-solid for any category $\mathcal{K}'$ such that $\mathcal{K}' \subseteq \mathcal{K}$. On the other hand, if $\mathcal{K}' \subseteq \mathcal{K}$ and $\mathcal{V}$ is not $\mathcal{K}'$-solid, it cannot be $\mathcal{K}$-solid either. Thus, a complete description of the solidity of a given GVTL with respect to these categories may be presented in terms of just a couple positive and negative facts. Often a GVTL $\mathcal{V}$ is the union of an ascending chain $\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \ldots$ of sub-GVTLs. It is easy to see that if there is an $n_0 \geq 0$ such that $\mathcal{V}_n$ is $\mathcal{K}$-solid for every $n \geq n_0$, then also $\mathcal{V}$ is $\mathcal{K}$-solid. A similar remark applies to unions of (upwards) directed families of GVTLs. Most of the families of tree languages considered here were shown to be GVTLs in [35].

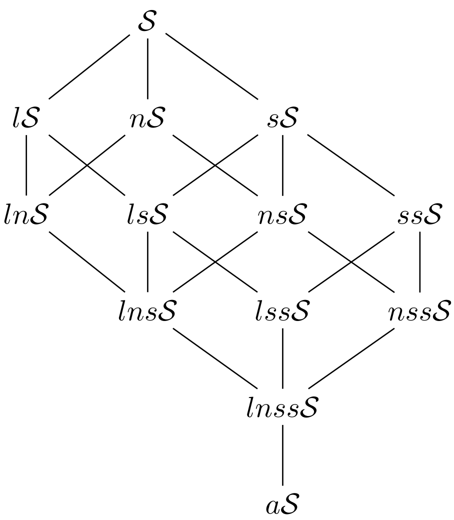

Figure 1. Our categories of substitutions.

The trivial cases. The least GVTL $Triv := \{\{\emptyset, T_\Sigma(X)\}\}$ and the greatest GVTL $Rec := \{Rec(\Sigma, X)\}$ are solid. For Rec we need the well-known fact that Rec is closed under all inverse tree homomorphisms (cf. [13, 17, 18]).

Nilpotent tree languages. For any Σ and X, let $Nil(\Sigma, X)$ consist of all finite ΣX-tree languages and their complements in $T_\Sigma(X)$, and let $Nil := \{Nil(\Sigma, X)\}$.

Proposition 8.1. *The GVTL Nil is $ns\mathcal{S}$-solid but neither $ln\mathcal{S}$- nor $lss\mathcal{S}$-solid.*

Proof. Let $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ be an $ns\mathcal{S}$-morphism, and let $T \in Nil(\Omega, Y)$. Since φ is strict and nondeleting, $\mathrm{hg}(s\varphi) \geq \mathrm{hg}(s)$ for every $s \in T_\Sigma(X)$. This implies that $t\varphi^{-1}$ is finite for every $t \in T_\Omega(Y)$. Hence, if T is finite, then so is $T\varphi^{-1}$, and if $T_\Omega(Y) \setminus T$ is finite, then $T\varphi^{-1}$ is co-finite.

To see that Nil is not $ln\mathcal{S}$-solid, let $\Sigma = \{f/1, g/1\}$, $X = \{x\}$, $T = \{f(x)\}$, and let $\varphi : T_\Sigma(X) \to T_\Sigma(X)$ be the $ln\mathcal{S}$-morphism such that $\varphi_1(f) = f(\xi_1)$, $\varphi_1(g) = \xi_1$ and $\varphi_X(x) = x$. Obviously, $T \in Nil(\Sigma, X)$ but $T\varphi^{-1}$ is neither finite nor co-finite; it consists of the ΣX-trees with exactly one f-labeled node.

To show that Nil is not $lss\mathcal{S}$-solid, let $\Sigma = \{f/2\}$, $\Omega = \{g/1\}$, $X = \{x\}$, $T = \{g(x)\}$, and let $\varphi : T_\Sigma(X) \to T_\Omega(X)$ be the $lss\mathcal{S}$-morphism such that $\varphi_2(f) = g(\xi_1)$, and $\varphi_X(x) = x$. Again, $T \in Nil(\Sigma, X)$ but $T\varphi^{-1} = \{f(x, t) \mid t \in T_\Sigma(X)\}$ is neither finite nor co-finite. ∎

A finite algebra $\mathcal{A} = (A, \Sigma)$ is *nilpotent* if there exist an $a_* \in A$ and a $k \geq 0$ such that for any $n > 0$ and $t \in T_\Sigma(\Xi_n)$, if $\mathrm{hg}(t) \geq k$, then $t^{\mathcal{A}}(a_1, \ldots, a_n) = a_*$ for all $a_1, \ldots, a_n \in A$. The class **Nil** of all nilpotent algebras is the GVFA corresponding to the GVTL Nil (cf. [34, 35]). Hence, it follows from Propositions 8.1 and 6.5 that **Nil** is $ns\mathcal{S}$-solid but neither $ln\mathcal{S}$- nor $lss\mathcal{S}$-solid.

Proposition 10 and Corollary 5 of [4] claim, for the single-type case, that Nil and **Nil** are $n\mathcal{S}$-solid. However, this holds only if we exclude unary symbols.

Definite tree languages. The *k-root* $\mathrm{rt}_k(t)$ of a ΣX-tree t is defined as follows:

(0) $\mathrm{rt}_0(t) = \varepsilon$, where ε represents the empty root segment, for every $t \in T_\Sigma(X)$;

(1) $\mathrm{rt}_1(t) = \mathrm{root}(t)$ for every $t \in T_\Sigma(X)$;

(2) for $k \geq 2$, $\mathrm{rt}_k(t) = t$ if $\mathrm{hg}(t) < k$, and $\mathrm{rt}_k(t) = f(\mathrm{rt}_{k-1}(t_1), \ldots, \mathrm{rt}_{k-1}(t_m))$ if $\mathrm{hg}(t) \geq k$ and $t = f(t_1, \ldots, t_m)$.

A ΣX-tree language T is *k-definite* (cf. [22, 34]) if for all $s, t \in T_\Sigma(X)$ such that $\mathrm{rt}_k(s) = \mathrm{rt}_k(t)$, $s \in T$ iff $t \in T$, and it is *definite* if it is k-definite for some $k \geq 0$. Let $Def_k = \{Def_k(\Sigma, X)\}$ and $Def = \{Def(\Sigma, X)\}$ be the GVTLs of k-definite ($k \geq 0$) and all definite tree languages. Clearly $Def_0 \subset Def_1 \subset Def_2 \subset \ldots$ and $Def = \bigcup_{n \geq 0} Def_k$ (cf. [35]).

The single-type version of the following lemma appears in [4]. Also here it can easily be verified by induction on k.

Lemma 8.2. *Let $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ be an $s\mathcal{S}$-morphism. For any $s, t \in T_\Sigma(X)$ and $k \geq 0$, if $\mathrm{rt}_k(s) = \mathrm{rt}_k(t)$, then $\mathrm{rt}_k(s\varphi) = \mathrm{rt}_k(t\varphi)$.*

Counterparts to the positive statements of the following proposition appear, in their respective forms, in [15] and [4].

Proposition 8.3. *The GVTL Def is $s\mathcal{S}$-solid, but not $ln\mathcal{S}$-solid. Def_k is $s\mathcal{S}$-solid for every $k \geq 0$, but for no $k \geq 1$ is Def_k $ln\mathcal{S}$-solid. Def_0 is solid.*

Proof. That Def_k ($k \geq 0$) and Def are $s\mathcal{S}$-solid follows from Lemma 8.2.

To prove the second claim, let $\Sigma = \{f/1, g/1\}$ and $X = \{x\}$, and define the $ln\mathcal{S}$-morphism $\varphi : T_\Sigma(X) \to T_\Sigma(X)$ by $\varphi_1(f) = f(\xi_1)$, $\varphi_1(g) = \xi_1$ and $\varphi_X(x) = x$. Now $T = \{x\}$ is k-definite for every $k \geq 1$ but $T\varphi^{-1} = \{g^n(x) \mid n \geq 0\}$ is not k-definite for any $k \geq 0$. This shows that neither Def nor Def_k for any $k \geq 1$ is $ln\mathcal{S}$-solid. For the last assertion, it suffices to note that $Def_0 = Triv$. ■

For any $k \geq 0$, let $\mathbf{Def}_k$ be the class of finite k-definite algebras, i.e., the GVFA Def_k^a corresponding to Def_k. Similarly, let $\mathbf{Def} := Def^a$ be the GVFA of all finite definite algebras. In [14] Ésik showed that for any Σ, the Σ-VFA of all k-definite finite Σ-algebras is defined by the identities of the form $u \approx v$ such that $u, v \in T_\Sigma(\Xi_n)$ for some $n \geq 1$ and $\mathrm{rt}_k(u) = \mathrm{rt}_k(v)$. From Propositions 6.5 and 8.3 it follows that the GVFAs $\mathbf{Def}_k$ and $\mathbf{Def}$ are $s\mathcal{S}$-solid, but it is easy to show this also directly when we note that if $\varkappa : \Sigma \to \Omega$ is a strict $\Sigma\Omega$-substitution, then $\mathrm{rt}_k(u) = \mathrm{rt}_k(v)$ $(u, v \in T_\Sigma(\Xi_n))$ implies $\mathrm{rt}_k(\varkappa(u)) = \mathrm{rt}_k(\varkappa(v))$.

Reverse definite tree languages. For each $k \geq 0$ and any $t \in T_\Sigma(X)$, let $S_k(t) := \{s \in \mathrm{sub}(t) \mid \mathrm{hg}(s) < k\}$ be the set of subtrees of t of height $< k$. In particular, $S_0(t) = \emptyset$. A ΣX-tree language T is *reverse k-definite* if for all $s, t \in T_\Sigma(X)$ such that $S_k(s) = S_k(t)$, $s \in T$ iff $t \in T$, and it is *reverse definite* if it is reverse k-definite for some $k \geq 0$. Let $RDef_k = \{RDef_k(\Sigma, X)\}$ and $RDef = \{RDef(\Sigma, X)\}$ be the GVTLs of reverse k-definite ($k \geq 0$) and reverse definite tree languages, respectively. Clearly, $RDef_0 \subset RDef_1 \subset RDef_2 \subset \ldots$ and $RDef = \bigcup_{k \geq 0} RDef_k$ (cf. [35]).

Lemma 8.4. *Let $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ be an nss$\mathcal{S}$-morphism. If $s \in T_\Sigma(X)$ and $t \in \mathrm{sub}(s\varphi)$, then $t \in \mathrm{sub}(s'\varphi)$ for some $s' \in \mathrm{sub}(s)$ such that $\mathrm{hg}(s') \leq \mathrm{hg}(t)$. Hence, $S_k(s\varphi) = \bigcup\{S_k(s'\varphi) \mid s' \in S_k(s)\}$ for all $k \geq 0$ and $s \in T_\Sigma(X)$.*

Proof. Clearly, $\mathrm{hg}(s\varphi) \geq \mathrm{hg}(s)$ for every $s \in T_\Sigma(X)$. We prove the first assertion by induction on s. If $s \in X$, we may choose $s' = s$ for every $t \in \mathrm{sub}(s\varphi)$. Now, let $s = f(s_1, \ldots, s_m)$ for some $m > 0$, $f \in \Sigma_m$ and $s_1, \ldots, s_m \in T_\Sigma(X)$, and assume that the claim holds for all ΣX-trees of height $< \mathrm{hg}(s)$. Since φ is nondeleting and symbol-to-symbol, we have $\varphi_m(f) = g(\xi_{i_1}, \ldots, \xi_{i_n})$ for some $n > 0$, $g \in \Omega_n$ and $i_1, \ldots, i_n$ such that $\{i_1, \ldots, i_n\} = [m]$. Hence, $s\varphi = g(s_{i_1}\varphi, \ldots, s_{i_n}\varphi)$. There are two cases to consider. If $t = s\varphi$, and we may choose $s' = s$. Otherwise, $t \in \mathrm{sub}(s_{i_j}\varphi)$ for some $j \in [n]$, we may apply the inductive assumption to find an $s' \in \mathrm{sub}(s_{i_j})$ such that $\mathrm{hg}(s') \leq \mathrm{hg}(t)$ and $t \in \mathrm{sub}(s'\varphi)$. Then $s' \in \mathrm{sub}(s)$, too, and we are done. The second claim follows immediately from the first one. ∎

Proposition 8.5.

(a) *$RDef$ is nss$\mathcal{S}$-solid, but neither lns$\mathcal{S}$- nor lss$\mathcal{S}$-solid.*

(b) *For each $k \geq 2$, $RDef_k$ is nss$\mathcal{S}$-solid, but neither lns$\mathcal{S}$- nor lss$\mathcal{S}$-solid.*

(c) *$RDef_1$ is n$\mathcal{S}$-solid, but not lss$\mathcal{S}$-solid.*

(d) *$RDef_0$ is solid.*

Proof. Let $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ be an $nss\mathcal{S}$-morphism, and let $T \in RDef_k(\Omega, Y)$ for some $k \geq 0$. Consider any ΣX-trees s and t such that $S_k(s) = S_k(t)$ and $s \in T\varphi^{-1}$. Then $s\varphi \in T$, and $S_k(s\varphi) = S_k(t\varphi)$ by Lemma 8.4, and hence also $t \in T\varphi^{-1}$. This shows that $RDef_k$ and $RDef$ are $nss\mathcal{S}$-solid.

The negative parts of (a) and (b) are proved by the following two examples. First, let $\Sigma = \{f_1/2, f_2/2, g/1\}$, $\Omega = \{f/2, g/1, h/1\}$. $X = \{x\}$, and let $\varphi : T_\Sigma(X) \to T_\Omega(X)$ be defined by $\varphi_2(f_1) = f(g(\xi_1), \xi_2)$, $\varphi_2(f_2) = f(h(\xi_1), \xi_2)$,

$\varphi_1(g) = g(\xi_1)$ and $\varphi_X(x) = x$. Then φ is linear, nondeleting and strict. The ΩX-tree language $T := \{t \in T_\Omega(Y) \mid S_2(t) = \{x, g(x)\}\}$ is reverse k-definite for every $k \geq 2$. For each $n \geq 1$, let $s_n := f_1(x, g^n(x))$ and $t_n = f_2(x, g^n(x))$. Then $S_k(s_n) = S_k(t_n)$ for every $k \leq n+1$, but $s_n\varphi = f(g(x), g^n(x)) \in T$ while $t_n\varphi = f(h(x), g^n(x)) \notin T$. Hence, $T\varphi^{-1}$ is not reverse k-definite for any $k \geq 0$.

Next, let $\Sigma = \{f/2, g/1\}$, $\Omega = \{h/1\}$, $X = \{x, x'\}$ and define the $lss\mathcal{S}$-morphism $\varphi : T_\Sigma(X) \to T_\Omega(X)$ by $\varphi_2(f) = \varphi_1(g) = h(\xi_1)$, $\varphi_X(x) = x$ and $\varphi_X(x') = x'$. The ΩX-tree language $T := \{x, h(x), h^2(x), \ldots\}$ is reverse k-definite for every $k \geq 1$, but $T\varphi^{-1}$ is not reverse k-definite for any $k \geq 0$. Indeed, if $s_k := f(g^k(x), g^k(x'))$ and $t_k = f(g^k(x'), g^k(x))$ $(k \geq 1)$, then $S_k(s_k) = S_k(t_k)$, but $s_k\varphi = h^{k+1}(x) \in T$ while $t_k\varphi = h^{k+1}(x') \notin T$.

The previous example shows that $RDef_1$ is not $lss\mathcal{S}$-solid. For any $t \in T_\Sigma(X)$, $S_1(t)$ is the set of leaf symbols appearing in t, and it is easy to see that if $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ is an $n\mathcal{S}$-morphism, then $S_1(t\varphi) = \bigcup\{S_1(x\varphi) \mid x \in S_1(t)\}$. Hence, if $s, t \in T_\Sigma(X)$ and $S_1(s) = S_1(t)$, then $S_1(s\varphi) = S_1(t\varphi)$. From this it immediately follows that $T\varphi^{-1} \in RDef_1(\Sigma, X)$ for every $T \in RDef_1(\Omega, Y)$.

Finally, it is clear that $RDef_0$ equals $Triv$, and is therefore solid. ■

Generalized definite tree languages. Generalized definite tree languages, introduced by Heuter [23], combine conditions on a root-segment and on the subtrees up to a given height. For any $j, k \geq 0$, a ΣX-tree language T is called *j,k-definite* if $S_j(s) = S_j(t)$ and $\mathrm{rt}_k(s) = \mathrm{rt}_k(t)$ imply that $s \in T$ iff $t \in T$ $(s, t \in T_\Sigma(X))$, and it is *generalized definite* if it is j,k-definite for some $j, k \geq 0$. Let $GDef_{j,k} = \{GDef_{j,k}(\Sigma, X)\}$ and $GDef = \{GDef(\Sigma, X)\}$ be the GVTLs of j,k-definite and all generalized definite ΣX-tree languages, respectively (cf. [35]). Clearly, $GDef_{j,k} \subseteq GDef_{j',k'}$ whenever $j \leq j'$ and $k \leq k'$, and the inclusion is proper if $j < j'$ or $k < k'$. Moreover, $GDef = \bigcup_{j,k\geq 0} GDef_{j,k}$. Since $GDef_{0,k} = Def_k$ for every $k \geq 0$ and $GDef_{j,0} = RDef_j$ for every $j \geq 0$, these cases are not treated separately in the following proposition.

Proposition 8.6.

(a) *$GDef$ is $nss\mathcal{S}$-solid, but neither $lns\mathcal{S}$- nor $lss\mathcal{S}$-solid.*

(b) *For all $j, k \geq 0$, $GDef_{j,k}$ is $nss\mathcal{S}$-solid.*

(c) *For all $j \geq 2$ and $k \geq 0$, $GDef_{j,k}$ is neither $lns\mathcal{S}$- nor $lss\mathcal{S}$-solid.*

(d) *For every $k \geq 1$, $GDef_{1,k}$ is $ns\mathcal{S}$-solid but neither $ln\mathcal{S}$- nor $lss\mathcal{S}$-solid.*

Proof. Let $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ be an $nss\mathcal{S}$-morphism and let $T \in GDef_{j,k}(\Omega, Y)$ for some $j, k \geq 0$. If $s, t \in T_\Sigma(X)$ are such that $S_j(s) = S_j(t)$ and $\mathrm{rt}_k(s) = \mathrm{rt}_k(t)$, then $S_j(s\varphi) = S_j(t\varphi)$ by Lemma 8.4, and $\mathrm{rt}_k(s\varphi) = \mathrm{rt}_k(t\varphi)$ by Lemma 8.2, and hence $s\varphi \in T$ iff $t\varphi \in T$, which shows that $T\varphi^{-1} \in GDef_{j,k}(\Sigma, X)$. Hence, $GDef_{j,k}$ and $GDef$ are $nss\mathcal{S}$-solid. This proves (b) and the first part of (a).

For the first part of (d), let $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ be an $ns\mathcal{S}$-morphism and let $T \in GDef_{1,k}(\Omega, Y)$ with $k \geq 0$. Let $s, t \in T_\Sigma(X)$ satisfy $S_1(s) = S_1(t)$ and $\mathrm{rt}_k(s) = \mathrm{rt}_k(t)$. Since φ is nondeleting, $S_1(s) = S_1(t)$ implies $S_1(s\varphi) = S_1(t\varphi)$ (cf. the proof of Proposition 8.5). Since φ is strict, we get $\mathrm{rt}_k(s\varphi) = \mathrm{rt}_k(t\varphi)$ by Lemma 8.2. This means that $s\varphi \in T$ iff $t\varphi \in T$, and hence $T\varphi^{-1} \in GDef_{1,k}(\Sigma, X)$.

The negative statements are again proved by counter-examples. First, let $\Sigma = \{f_1/2, f_2/2, g/1\}$, $\Omega = \{f/2, g/1, h/1\}$ and $X = \{x\}$. Define the $lns\mathcal{S}$-morphism $\varphi : T_\Sigma(X) \to T_\Omega(X)$ by $\varphi_2(f_1) = f(g(\xi_1), \xi_2)$, $\varphi_2(f_2) = f(h(\xi_1), \xi_2)$, $\varphi_1(g) = g(\xi_1)$ and $\varphi_X(x) = x$. Clearly, $T := \{t \in T_\Omega(X) \mid S_2(t) = \{x, g(x)\}, \mathrm{rt}_0(t) = \varepsilon\}$ is 2,0-definite. For any $j, k \geq 0$, let $s_{j,k} := g^k(f_1(x, g^j(x))$ and $t_{j,k} := g^k(f_2(x, g^j(x)))$. Then $S_j(s_{j,k}) = S_j(t_{j,k})$ and $\mathrm{rt}_k(s_{j,k}) = \mathrm{rt}_k(t_{j,k})$, but $s_{j,k}\varphi = g^k(f(g(x), g^j(x))) \in T$ while $t_{j,k}\varphi = g^k(f(h(x), g^j(x))) \notin T$. Therefore $T\varphi^{-1}$ is not j,k-definite for any $j, k \geq 0$, and hence not generalized definite. This proves the first part of (c) and the second claim of (a).

Next, let $\Sigma = \{f/2, g/1\}$, $\Omega = \{h/1\}$, $X = \{x, x'\}$, and define the $lss\mathcal{S}$-morphism $\varphi : T_\Sigma(X) \to T_\Omega(X)$ by $\varphi_2(f) = \varphi_1(g) = h(\xi_1)$, $\varphi_X(x) = x$ and $\varphi_X(x') = x'$. Now $T := \{t \in T_\Omega(X) \mid S_1(t) = \{x\}, \mathrm{rt}_0(t) = \varepsilon\}$ is 1,0-definite, but $T\varphi^{-1}$ is not j,k-definite for any $j, k \geq 0$, which can be seen by considering the trees $s = f(g^{j+k}(x), g^{j+k}(x'))$ and $t = f(g^{j+k}(x'), g^{j+k}(x))$. This example proves the last statements of (a), (c) and (d).

That $GDef_{1,k}$ is not $ln\mathcal{S}$-solid for any $k \geq 1$ is shown by the example used in the proof of Proposition 8.3. Indeed, the tree language $T = \{x\}$ is $1,k$-definite for every $k \geq 1$, but $T\varphi^{-1} = \{g^n(x) \mid n \geq 0\}$ is $1,k$-definite for no $k \geq 1$. ∎

Locally testable tree languages. The set $\mathrm{fork}(t)$ of *forks* of a ΣX-tree t is defined by setting $\mathrm{fork}(x) = \emptyset$ for $x \in X$, and $\mathrm{fork}(t) = \mathrm{fork}(t_1) \cup \ldots \cup \mathrm{fork}(t_m) \cup \{f\langle \mathrm{root}(t_1), \ldots, \mathrm{root}(t_m)\rangle\}$ for $t = f(t_1, \ldots, t_m)$. A ΣX-tree language T is *local* if for all $s, t \in T_\Sigma(X)$, if $\mathrm{root}(s) = \mathrm{root}(t)$ and $\mathrm{fork}(s) = \mathrm{fork}(t)$, then $s \in T$ iff $t \in T$ (cf. [17, 18, 35]). Let $Loc = \{Loc(\Sigma, X)\}$ be the GVTL local tree languages.

Proposition 8.7. *Loc is $ns\mathcal{S}$-solid but neither $lss\mathcal{S}$- nor $ln\mathcal{S}$-solid.*

Proof. Let $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ be an $ns\mathcal{S}$-morphism, $T \in Loc(\Omega, Y)$, and let $\mathrm{root}(s) = \mathrm{root}(t)$ and $\mathrm{fork}(s) = \mathrm{fork}(t)$ for some $s, t \in T_\Sigma(X)$. Since φ is strict, $\mathrm{root}(s\varphi) = \mathrm{root}(t\varphi)$. For the same reason, each fork in $s\varphi$ or $t\varphi$ can be ascribed to a fork in s or t, respectively. If $f\langle d_1, \ldots, d_m\rangle \in \mathrm{fork}(s)(= \mathrm{fork}(t))$, then there are subtrees $s' = f(s_1, \ldots, s_m) \in \mathrm{sub}(s)$ and $t' = f(t_1, \ldots, t_m) \in \mathrm{sub}(t)$ such that $\mathrm{root}(s_i) = d_i = \mathrm{root}(t_i)$ for every $i \in [m]$. As φ is nondeleting, $s'\varphi \in \mathrm{sub}(s\varphi)$ and $t'\varphi \in \mathrm{sub}(t\varphi)$. All forks in $s'\varphi$ and $t'\varphi$ that arise from $f(d_1, \ldots, d_m)$ appear in $\varphi_m(f)[\mathrm{root}(s_1\varphi), \ldots, \mathrm{root}(s_m\varphi)]$ and $\varphi_m(f)[\mathrm{root}(t_1\varphi), \ldots, \mathrm{root}(t_m\varphi)]$, respectively. Since $\mathrm{root}(s_i\varphi) = \mathrm{root}(t_i\varphi)$ for every $i \in [m]$, this implies $\mathrm{fork}(s\varphi) = \mathrm{fork}(t\varphi)$, and hence $s\varphi \in T$ iff $t\varphi \in T$. This shows that $T\varphi^{-1} \in Loc(\Sigma, X)$.

For showing that Loc is not $lss\mathcal{S}$-solid, let $\Sigma = \{f/2\}$, $\Omega = \{g/1\}$, $X = \{x, x'\}$ and $Y = \{y, y'\}$, and let $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ be defined by $\varphi_2(f) = g(\xi_1)$, $\varphi_X(x) = y$ and $\varphi_X(x') = y'$. Consider the local ΩY-tree language

$$T := \{t \in T_\Omega(Y) \mid \mathrm{root}(t) = g, \mathrm{fork}(t) = \{g\langle g\rangle, g\langle y\rangle\}$$

and the ΣX-trees $s := f(f(x,x), f(x',x'))$ and $t := f(f(x',x'), f(x,x))$. Now, $\mathrm{root}(s) = f = \mathrm{root}(t)$ and $\mathrm{fork}(s) = \{f\langle f,f\rangle, f\langle x,x\rangle, f\langle x',x'\rangle\} = \mathrm{fork}(t)$, but $s\varphi = g(g(y)) \in T$ while $t\varphi = g(g(y')) \notin T$.

To show that Loc is not $ln\mathcal{S}$-solid, let $\Sigma = \{f/1, g/1, h/1\}$ and $X = \{x\}$, and define $\varphi : T_\Sigma(X) \to T_\Sigma(X)$ by $\varphi_1(f) = \xi_1$, $\varphi_1(g) = g(\xi_1)$, $\varphi_1(h) = h(\xi_1)$ and $\varphi_X(x) = x$. Clearly, φ is linear and nondeleting. Consider the local ΣX-tree language $T = \{t \in T_\Sigma(X) \mid \mathrm{root}(t) = g\}$. If $s = f(g(f(h(g(x)))))$ and $t = f(h(g(f(g(x)))))$, then $\mathrm{root}(s) = \mathrm{root}(t)$ and $\mathrm{fork}(s) = \mathrm{fork}(t)$, but $s\varphi = g(h(g(x))) \in T$ while $t\varphi = h(g(g(x))) \notin T$, and hence $T\varphi^{-1}$ is not local. ∎

Aperiodic tree languages. Aperiodic tree languages were defined by Thomas [38] who characterized them by syntactic monoids. A regular ΣX-tree language T is *aperiodic* if for some $n \geq 0$ and all $p, q \in C_\Sigma(X)$ and $t \in T_\Sigma(X)$, $t \cdot p^{n+1} \cdot q \in T$ iff $t \cdot p^n \cdot q \in T$. For T aperiodic, let $\mathrm{ia}(T)$ be the least n satisfying the above condition. Let $Ap = \{Ap(\Sigma, X)\}$ be the GVTL of aperiodic tree languages ([35]).

In Section 7 we extended any tree homomorphism $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ to a tree homomorphism $\varphi_* : T_\Sigma(X \cup \{\xi\}) \to T_\Omega(Y \cup \{\xi\})$ by setting $\xi\varphi_* = \xi$. If φ is linear and $p \in C_\Sigma(X)$, then $p\varphi_*$ is either an ΩY-context or an ΩY-tree. Let us extend the products $p \cdot q$ of two contexts and the product $t \cdot p$ of a tree and a context to a product $u \cdot v$ where $u, v \in T_\Omega(Y) \cup C_\Omega(Y)$ by putting $u \cdot v = v$ whenever $v \in T_\Omega(Y)$. The following facts can be proved by induction on q.

Lemma 8.8. *If $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ is an $l\mathcal{S}$-morphism, then $(t \cdot q)\varphi = t\varphi \cdot q\varphi_*$ and $(p \cdot q)\varphi_* = p\varphi_* \cdot q\varphi_*$ for all $p, q \in C_\Sigma(X)$ and $t \in T_\Sigma(X)$.*

Proposition 8.9. *The GVTL Ap is $l\mathcal{S}$-solid, but it is not $nss\mathcal{S}$-solid.*

Proof. Let $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ be linear and consider any $T \in Ap(\Omega, Y)$. Let $n := \max(\mathrm{ia}(T), 1)$. To prove $T\varphi^{-1} \in Ap(\Sigma, X)$, consider any $t \in T_\Sigma(X)$ and $p, q \in C_\Sigma(X)$. By Lemma 8.8, we should show that

$$(\mathrm{Ap}^*) \qquad t\varphi \cdot (p\varphi_*)^{n+1} \cdot q\varphi_* \in T \;\Leftrightarrow\; t\varphi \cdot (p\varphi_*)^n \cdot q\varphi_* \in T.$$

If $q\varphi_* \in T_\Omega(Y)$, then $t\varphi \cdot (p\varphi_*)^{n+1} \cdot q\varphi_* = q\varphi_* = t\varphi \cdot (p\varphi_*)^n \cdot q\varphi_*$ and (Ap^*) trivially holds. Secondly, if $q\varphi_* \in C_\Omega(Y)$ but $p\varphi_* \in T_\Omega(Y)$, then (Ap^*) follows from $t\varphi\cdot(p\varphi_*)^{n+1}\cdot q\varphi_* = p\varphi_*\cdot q\varphi_* = t\varphi\cdot(p\varphi_*)^n\cdot q\varphi_*$. Finally, if $p\varphi_*, q\varphi_* \in C_\Omega(Y)$, then (Ap*) follows from $T \in Ap(\Omega, Y)$ and $\mathrm{ia}(T) \leq n$.

That Ap is not $nss\mathcal{S}$-solid will follow from Proposition 8.10 below. ∎

The *syntactic monoid congruence* μ_T of a ΣX-language T is defined by

$$p\,\mu_T\,q \;:\Leftrightarrow\; (\forall t \in T_\Sigma(X))(\forall r \in C_\Sigma(X))(t\cdot p\cdot r \in T \;\leftrightarrow\; t\cdot q\cdot r \in T) \quad (p, q \in C_\Sigma(X)),$$

and the *syntactic monoid* of T is the quotient monoid $\mathrm{SM}(T) := C_\Sigma(X)/\mu_T$. In [31] it was shown that for any recognizable tree language, $\mathrm{SM}(T)$ is isomorphic to the monoid of translations $\mathrm{Tr}(\mathrm{SA}(T))$ of the syntactic algebra of T.

Recall that a *variety of finite monoids* (VFM) is a class of finite monoids closed under submonoids, epimorphic images and finite direct products. A monoid is *aperiodic* if it has no non-trivial subgroups. The finite aperiodic monoids form the VFM **Ap** (cf. [12, 29]). With any VFM **M**, we associate the family of tree languages $\mathbf{M}^t = \{\mathbf{M}^t(\Sigma, X)\}$ where $\mathbf{M}^t(\Sigma, X) := \{T \in Rec(\Sigma, X) \mid \mathrm{SM}(T) \in \mathbf{M}\}$. Furthermore, let $\mathbf{M}^a$ be the class of the finite algebras $\mathcal{A}$ such that $\mathrm{Tr}(\mathcal{A}) \in \mathbf{M}$. Then $\mathbf{M}^t$ is a GVTL, $\mathbf{M}^a$ is a GVFA, $\mathbf{M}^{at} = \mathbf{M}^t$ and $\mathbf{M}^{ta} = \mathbf{M}^a$ (cf. [35]). Thomas' [38] characterization of the aperiodic tree languages says that $\mathbf{Ap}^t = Ap$. (The GVTLs definable this way by syntactic monoids were characterized in [30].) Note that $Ap^a = \mathbf{Ap}^a$.

Proposition 8.10. *The GVFA Ap^a is not $nss\mathcal{S}$-solid.*

Proof. Let $\Omega = \{g/2\}$. The Ω-algebra $\mathcal{B} = (B, \Omega)$ in which $B = \{0, a, b\}$, $g^{\mathcal{B}}(a, a) = b$, $g^{\mathcal{B}}(b, b) = a$ and $g^{\mathcal{B}}(a_1, a_2) = 0$ for all other $(a_1, a_2) \in B^2$, has three elementary translations:

$$\alpha := g^{\mathcal{B}}(\_, a) = g^{\mathcal{B}}(a, \_) : 0 \mapsto 0, a \mapsto b, b \mapsto 0;$$

$$\beta := g^{\mathcal{B}}(\_, b) = g^{\mathcal{B}}(b, \_) : 0 \mapsto 0, a \mapsto 0, b \mapsto a;$$

$$o := g^{\mathcal{B}}(\_, 0) = g^{\mathcal{B}}(0, \_) : 0 \mapsto 0, a \mapsto 0, b \mapsto 0.$$

From these we get as compositions two more translations, namely

$$\gamma := \alpha\beta : 0 \mapsto 0, a \mapsto a, b \mapsto 0, \text{ and } \delta := \beta\alpha : 0 \mapsto 0, a \mapsto 0, b \mapsto b.$$

Hence, $\mathrm{Tr}(\mathcal{B}) = \{1, \alpha, \beta, \gamma, \delta, o\}$, where $1 = 1_B$ is the identity and o is a zero element. The monoid $\mathrm{Tr}(\mathcal{B})$ has the four idempotents 1, o, γ and δ, but none of them belongs to a nontrivial group. Hence, $\mathrm{Tr}(\mathcal{B}) \in \mathbf{Ap}$ and $\mathcal{B} \in \mathbf{Ap}^a (= Ap^a)$.

Now, let $\Sigma = \{f/1\}$ and define $\varkappa \in nss\mathcal{S}(\Sigma, \Omega)$ by $\varkappa(f) = g(\xi_1, \xi_1)$. Then $\varkappa(\mathcal{B}) = (\{0, a, b\}, \Sigma)$ is the Σ-algebra such that $f^{\varkappa(\mathcal{B})} : 0 \mapsto 0, a \mapsto b, b \mapsto a$. Clearly, $\mathrm{Tr}(\varkappa(\mathcal{B})) = \{1_B, f^{\varkappa(\mathcal{B})}\}$ is a 2-element group, and hence $\varkappa(\mathcal{B}) \notin Ap^a$. ∎

Piecewise testable tree languages. As the last case we consider the piecewise testable tree languages studied by Piirainen [28]. Their definition is based on the *homeomorphic embedding order* of trees used for proving the termination of term rewriting systems (cf. [3], for example). For any Σ and X, this relation $\trianglelefteq$ on $T_\Sigma(X)$ is defined by stipulating that for any $s, t \in T_\Sigma(X)$, $s \trianglelefteq t$ iff (1) $s = t$, or (2) $s = f(s_1, \ldots, s_m)$, $t = f(t_1, \ldots, t_m)$ and $s_1 \trianglelefteq t_1, \ldots, s_m \trianglelefteq t_m$, or (3) $t = f(t_1, \ldots, t_m)$ and $s \trianglelefteq t_i$ for some $i \in [m]$.

If $s \trianglelefteq t$, we call s a *piecewise subtree* of t. For any $k \geq 0$ and $t \in T_\Sigma(X)$, let $P_k(t) := \{s \in T_\Sigma(X) \mid s \trianglelefteq t, \mathrm{hg}(s) < k\}$. Now, the relation

$$\pi^k(\Sigma, X) := \{(s,t) \mid s, t \in T_\Sigma(X), P_k(s) = P_k(t)\}$$

is in $\mathrm{FC}(\Sigma, X)$. A ΣX-tree language is *piecewise k-testable* if it is saturated by $\pi^k(\Sigma, X)$, and it is *piecewise testable* if it is piecewise k-testable for some $k \geq 0$.

Our setting differs from [28] at two points. Firstly, for any $k \geq 0$, our "piecewise $(k+1)$-testable" is the "k-piecewise testable" of [28] while our "0-testable" means no testing at all. Secondly, since we consider pure tree homomorphisms only, some of the results of [28] appear here in a slightly stronger form.

Let $Pwt_k = \{Pwt_k(\Sigma, X)\}$ and $Pwt = \{Pwt(\Sigma, X)\}$ be the GVTLs of piecewise k-testable and piecewise testable tree languages, respectively (cf. [28]). Piirainen also showed that Pwt is closed under inverse non-deleting tree homomorphisms, i.e., that Pwt is $n\mathcal{S}$-solid. We can extend this result to the GVTLs Pwt_k. For this, we need the following versions of Lemmas 6.1 and 6.2 of [28].

Lemma 8.11. *Let $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ be an $n\mathcal{S}$-morphism. For any $k \geq 0$, $s \in T_\Sigma(X)$ and $t \in P_k(s\varphi)$, there exists an $s' \in P_k(s)$ such that $t \in P_k(s'\varphi)$.*

Proof. We proceed by induction on k. The case $k = 0$ is trivial since $P_0(s\varphi) = \emptyset$. If $t \in P_1(s\varphi)$, then $t \in Y$ and since φ is pure, there must be an $x \in P_1(s)$ such that $t \in P_1(x\varphi)$. Assuming now that $k \geq 2$, we proceed by induction on s.

If $s \in X$, then $\mathrm{hg}(s) < k$ and we may choose $s' = s$ for every $t \in P_k(s\varphi)$. Now, let $s = f(s_1, \ldots, s_m)$ and assume that the claim holds for all smaller trees. Then $s\varphi = \varphi_m(f)[s_1\varphi, \ldots, s_m\varphi]$, and we distinguish two cases.

If $t \trianglelefteq s_i\varphi$ for some $i \in [m]$, then $t \in P_k(s'\varphi)$ for some $s' \in P_k(s_i) \subseteq P_k(s)$. Otherwise, $t = t'[t_1, \ldots, t_m]$ for some $t' \in T_\Omega(\Xi_m)$ and $t_1, \ldots, t_m \in T_\Omega(Y)$ such that $t' \trianglelefteq \varphi_m(f)$ ($\trianglelefteq$-relation of $T_\Omega(\Xi_m)$) and $t_1 \in P_{k-1}(s_1\varphi), \ldots, t_m \in P_{k-1}(s_m\varphi)$. By the main inductive assumption, there are trees $s'_1 \in P_{k-1}(s_1), \ldots, s'_m \in P_{k-1}(s_m)$ such that $t_1 \in P_{k-1}(s'_1\varphi), \ldots, t_m \in P_{k-1}(s'_m\varphi)$. Obviously, $s' := f(s'_1, \ldots, s'_m)$ is in $P_k(s)$, and since φ is nondeleting, $t = t'[t_1, \ldots, t_m] \trianglelefteq \varphi_m(f)[s'_1\varphi, \ldots, s'_m\varphi] = s'\varphi$, which means that $t \in P_k(s'\varphi)$. ∎

Lemma 8.12. *If $\varphi : T_\Sigma(X) \to T_\Omega(Y)$ is an $n\mathcal{S}$-morphism, then $s\,\pi^k(\Sigma, X)\,t$ implies that $s\varphi\,\pi^k(\Omega, Y)\,t\varphi$.*

Proof. Let $s\,\pi^k(\Sigma, X)\,t$ for some $s, t \in T_\Sigma(X)$, and consider any $u \in P_k(s\varphi)$. By Lemma 8.11, $u \in P_k(s'\varphi)$ for some $s' \in P_k(s) = P_k(t)$. Then also $s' \trianglelefteq t$. By Lemma 5.2 of [28], this implies $s'\varphi \trianglelefteq t\varphi$, and hence $u \in P_k(s'\varphi) \subseteq P_k(t\varphi)$. This proves $P_k(s\varphi) \subseteq P_k(t\varphi)$, and the converse inclusion is shown the same way. ∎

Proposition 8.13.

(a) *Pwt is $n\mathcal{S}$-solid but not $lss\mathcal{S}$-solid.*

(b) *For each $k \geq 1$, Pwt_k is $n\mathcal{S}$-solid but not $lss\mathcal{S}$-solid.*

(c) *$Pwt_0 = Triv$ is solid.*

Proof. The GVTL Pwt is $n\mathcal{S}$-solid by Theorem 6.3 of [28], but here our stronger Lemma 8.12 implies that also every Pwt_k is $n\mathcal{S}$-solid. Of course, $Pwt_0 = Triv$ is solid. It remains to show that Pwt and Pwt_k ($k \geq 1$) are not $lss\mathcal{S}$-solid.

Let $\Sigma = \{f/2\}$, $X = \{x, x'\}$ and $\Omega = \{g/1\}$, and let $\varphi : T_\Sigma(X) \to T_\Omega(X)$ be the $lss\mathcal{S}$-morphism defined by $\varphi_2(f) = g(\xi_1)$, $\varphi_X(x) = x$ and $\varphi_X(x') = x'$. The ΩX-tree language $T = \{g^n(x) \mid n \geq 0\} = \{x, g(x), g(g(x)), \ldots\}$ is clearly piecewise k-testable for any $k \geq 1$. Let us now define (1) $s_0 := x$ and $t_0 := x'$, and (2) $s_n := f(s_{n-1}, t_{n-1})$ and $t_n := f(t_{n-1}, s_{n-1})$ for any $n \geq 1$.

It is easy to see that for any $k \geq 1$, $u \trianglelefteq s_k$ and $u \trianglelefteq t_k$ for every $u \in T_\Sigma(X)$ such that $\mathrm{hg}(u) < k$. Hence, $s_k\,\pi^k(\Sigma, X)\,t_k$ for every $k \geq 1$. On the other hand, $s_k\varphi = g^{k-1}(x) \in T$ while $t_k\varphi = g^{k-1}(x') \notin T$. Hence $T\varphi^{-1} \notin Pwt_k(\Sigma, X)$ for all $k \geq 1$, and thus also $T\varphi^{-1} \notin Pwt(\Sigma, X)$. ∎

Note that Example 6.4 of [28], which was used for showing that Pwt_1 is not $ln\mathcal{S}$-solid, is not valid here as we consider pure tree homomorphisms only.

9. Some concluding remarks

We have presented a framework for the study of the solidity of general varieties of tree languages, that contain tree languages over all alphabets, as well as the corresponding general varieties of finite algebras and general varieties of finite congruences. Secondly, we established the solidity properties of several known families of regular tree languages with respect to certain categories of substitutions that were derived from some important classes of tree homomorphisms. These GVTLs turned out to have quite different solidity properties, and largest of the categories with respect to which all of them are solid, is that of linear nondeleting symbol-to-symbol substitutions. However, at least GVTLs like the one considered in Example 10.4 of [35] are not even $lnss\mathcal{S}$-solid. As noted also by Baltazar [4], the tree language varieties of Ésik [15] are solid and his more general $+$-varieties are $s\mathcal{S}$-solid (when applying our terminology to the single-type case). Hence none of the nontrivial GVTLs considered in Section 8 is a variety in the sense of [15] and just the definite tree languages form a $+$-variety.

Of course, many more questions related to our topic remain to be studied. For example, some important GVTLs were not yet studied, and there could also be some further interesting categories of substitutions to consider.

It was indicated by the Referee that the bi-algebras of Yu. M. Movsisjan (cf. [10, 26] for references) resemble our g-algebras, and hence some of the results of Sections 4 and 5 and may have counterparts in Movsisjan's work. So far, I have not got access to that work and cannot settle the matter. In any case, in view of the main focus of this paper, it was natural to stick to the formalism of [35].

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