Discussiones Mathematicae General Algebra and Applications 32 (2012) 115–136 doi:10.7151/dmgaa.1188

FOUR-PART SEMIGROUPS - SEMIGROUPS OF BOOLEAN OPERATIONS

PRAKIT JAMPACHON¹

Department of Mathematics KhonKaen University 40002 Thailand e-mail: prajam@kku.ac.th

YENI SUSANTI

Department of Mathematics Gadjah Mada University Yogyakarta Indonesia 55281

e-mail: inielsusan@yahoo.com

AND

KLAUS DENECKE²

Institute of Mathematics Potsdam University Potsdam Germany e-mail: kdenecke@rz.uni-potsdam.de

Abstract

Four-part semigroups form a new class of semigroups which became important when sets of Boolean operations which are closed under the binary superposition operation $f + g := f(g, \ldots, g)$, were studied. In this paper we describe the lattice of all subsemigroups of an arbitrary four-part semigroup, determine regular and idempotent elements, regular and idempotent subsemigroups, homomorphic images, Green's relations, and prove a representation theorem for four-part semigroups.

Keywords: four-part semigroup, Boolean operation.2010 Mathematics Subject Classification: 08A30, 08A40, 08A62.

¹was supported by the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission, through the Cluster of Research of Enhance the Quality of Basic Education.

²was supported by the Center of Excellence in Mathematics, the Commission on Higher Education, Ministry of Education, Thailand.

1. INTRODUCTION

Definition [2]. Let

$$\begin{split} S_1 &= \{a_{11}, a_{12}, \dots, a_{1n_r}\},\\ S_2 &= \{a_{21}, a_{22}, \dots, a_{2n_r}\},\\ S_3 &= \{a_{31}, a_{32}, \dots, a_{3n_s}\}, \quad \text{where } a^* \in S_3 \text{ is a fixed element,}\\ S_4 &= \{a_{41}, a_{42}, \dots, a_{4n_s}\}, \quad \text{where } a^{**} \in S_4 \text{ is a fixed element,} \end{split}$$

be four non-empty, finite and pairwise disjoint sets and let $S = S_1 \cup S_2 \cup S_3 \cup S_4$. We define a binary operation * on S by

$$a_{ij} * a_{lk} = \begin{cases} a_{lk} & \text{if } a_{ij} \in S_1 \\ a_{tk} & \text{if } a_{ij} \in S_2 \text{ where } t = \begin{cases} 1 & \text{if } l = 2 \\ 2 & \text{if } l = 1 \\ 3 & \text{if } l = 4 \\ 4 & \text{if } l = 3 \end{cases}$$
$$a^* \in S_3 & \text{if } a_{ij} \in S_3 \\ a^{**} \in S_4 & \text{if } a_{ij} \in S_4. \end{cases}$$

The semigroup (S; *) is said to be a four-part semigroup.

Remark 1.

1. It is easy to see that the binary operation * is well-defined and associative. Therefore (S; *) is a finite semigroup. Since the sets S_1 and S_2 have the same cardinality, as do S_3 and S_4 , any four-part semigroup has even cardinality. Four-part semigroups were introduced by R. Butkote ([1]) (see also [2]) to give an abstract description of the semigroup $(O^n(\{0,1\});+)$ of all *n*-ary Boolean operations for $n \geq 1$, where $f + g := f(g, \ldots, g), f, g \in O^n(\{0, 1\})$ is the *n*-ary Boolean operation which is defined by $(f+g)(a_1, ..., a_n) := f(g(a_1, ..., a_n), ..., g(a_1, ..., a_n)).$ The sets S_1, S_2, S_3 and S_4 are then the following collections of Boolean operations : $C_4^n := \{ f \in O^n(A) | f(0, ..., 0) = 0 \text{ and } f(1, ..., 1) = 1 \}, \neg C_4^n :=$ $\{f \in O^n(A) | f(0,...,0) = 1 \text{ and } f(1,...,1) = 0\}$ (the notation $\neg C_4^n$ means that each element of this set is the negation of an element of C_4^n , $K_0^n := \{f \in$ $O^n(A)|f(0,\ldots,0)=0$ and $f(1,\ldots,1)=0$ which contains the *n*-ary constant operation with value 0 and $K_1^n := \{ f \in O^n(A) | f(0, ..., 0) = 1 \text{ and } f(1, ..., 1) = 1 \}.$ K_1^n contains the *n*-ary constant operation with value 1. Each element of K_1^n is the negation of some element of K_0^n . Therefore, instead of K_1^n one could also write $\neg K_0^n$. Clearly, $O^n(\{0,1\}) = C_4^n \cup \neg C_4^n \cup K_0^n \cup K_1^n$ is the disjoint union of these sets and it is not difficult to see that $(O^n(\{0,1\});+)$ is a four-part semigroup

since the operation + satisfies

$$f + g = \begin{cases} g & \text{if } f \in C_4^n \\ \neg g & \text{if } f \in \neg C_4^n \\ c_0^n & \text{if } f \in K_0^n \\ c_1^n & \text{if } f \in \neg K_0^n. \end{cases}$$

Our aim is to determine the semigroup-theoretical properties of four-part semigroups. This can be applied to determine the properties of the semigroup $(O^n(\{0,1\});+).$

2. To get a semigroup not necessarily all of the sets S_1, S_2, S_3, S_4 have to be non-empty. We analyze all possible cases where at least one of our sets is empty. Clearly, $S_1 = \emptyset$ iff $S_2 = \emptyset$ and $S_3 = \emptyset$ iff $S_4 = \emptyset$. Therefore except the case that none of the sets S_1, S_2, S_3, S_4 is the empty set, we have three more cases:

- 1. $S_1 = S_2 = \emptyset, S_3 \neq \emptyset, S_4 \neq \emptyset,$
- 2. $S_3 = S_4 = \emptyset, S_1 \neq \emptyset, S_2 \neq \emptyset,$
- 3. $S_1 = S_2 = S_3 = S_4 = \emptyset$.

In the first case we have $S = S_3 \cup S_4$ with

$$a_{ij} * a_{lk} = \begin{cases} a^* & \text{if } a_{ij} \in S_3 \\ a^{**} & \text{if } a_{ij} \in S_4 \end{cases}$$

and in the second case we have $S = S_1 \cup S_2$ with

$$a_{ij} * a_{lk} = \begin{cases} a_{lk} & \text{if } a_{ij} \in S_1 \\ a_{1k} & \text{if } a_{ij} \in S_2 & \text{and } l = 2 \\ a_{2k} & \text{if } a_{ij} \in S_2 & \text{and } l = 1. \end{cases}$$

2. Subsemigroups of four-part semigroups

To study subsemigroups of four-part semigroups we define the following kinds of semigroups:

Definition. A semigroup S = (S; *) is called a constant semigroup if there is an element $b^* \in S$ such that $a * b = b^*$ for any $a, b \in S$, a right-zero constant semigroup if there are two disjoint non-empty sets S_1, S_2 such that $S = S_1 \cup S_2$ and there is a fixed element $b^* \in S_2$ such that

$$a * b = \begin{cases} b & \text{if } a \in S_1 \\ b^* & \text{if } a \in S_2, \end{cases}$$

a two-constant semigroup if there are two disjoint non-empty sets S_1, S_2 such that $S = S_1 \cup S_2$ and there are two fixed elements $b^* \in S_1$ and $b^{**} \in S_2$ such that

$$a * b = \begin{cases} b^* & \text{if } a \in S_1 \\ b^{**} & \text{if } a \in S_2, \end{cases}$$

a right-zero two-constant semigroup if there are subsets S_1, S_2, S_3 of S such that $S = \bigcup_{i=1}^{3} S_i, S_i \neq \emptyset, S_i \cap S_j = \emptyset$ for $i \neq j \in \{1, 2, 3\}$ and there are distinguished elements $b^* \in S_2$ and $b^{**} \in S_3$ such that

$$a * b = \begin{cases} b & \text{if } a \in S_1 \\ b^* & \text{if } a \in S_2 \\ b^{**} & \text{if } a \in S_3, \end{cases}$$

a right-zero φ -semigroup if there is a fixed point free bijective mapping $\varphi: S \to S$ with $\varphi \circ \varphi = id$ and there are two disjoint sets S_1, S_2 of S such that $S = S_1 \cup S_2$ and

$$a * b = \begin{cases} b & \text{if } a \in S_1 \\ \varphi(b) & \text{if } a \in S_2. \end{cases}$$

Lemma 2. Let S be a four-part semigroup. Then there is a fixed point free bijective mapping $\varphi : S \to S$ such that $\varphi \circ \varphi = id$, $\varphi(a^*) = a^{**}$, $\varphi(a^{**}) = a^*$, $\varphi(a_{1j}) = a_{2j}$, $\varphi(a_{2j}) = a_{1j}$, $\varphi(a_{3k}) = a_{4k}$ and $\varphi(a_{4k}) = a_{3k}$ for $j = 1, \ldots, n_r$ and $k = 1, \ldots, n_s$.

Proof. We can define a bijective mapping $\varphi : S \to S$ by definition $\varphi(a_{1j}) = a_{2j}$, $\varphi(a_{2j}) = a_{1j}, j = 1, \ldots, n_r$ and $\varphi(a_{3k}) = a_{4k}$ and $\varphi(a_{4k}) = a_{3k}, k = 1, \ldots, n_s$ and $\varphi(a^*) = a^{**}, \varphi(a^{**}) = a^*$. It is easy to see that φ is a fixed point free bijection satisfying $\varphi \circ \varphi = id$.

Lemma 3. Let S be a four-part semigroup and let $\mathcal{H} \subseteq S$ be a subsemigroup with $H = H_1 \cup H_2 \cup H_3 \cup H_4$, $H_i \subseteq S_i$, i = 1, 2, 3, 4. Then we have

- (i) If $H_2 \neq \emptyset$, then $H_1 \neq \emptyset$.
- (ii) If $H_2 \neq \emptyset$, then $H_3 \neq \emptyset$ if and only if $H_4 \neq \emptyset$.

Proof. (i) Let $H_2 \neq \emptyset$ and $a_{2j} \in H_2 \subseteq S_2$. Then $a_{2j} * a_{2j} = a_{1j} \in S_1 \cap H = H_1$, i.e $H_1 \neq \emptyset$.

(ii) Let $H_2 \neq \emptyset$ and $H_3 \neq \emptyset$ and let $a_{2j} \in H_2$ and $a_{3k} \in H_3$. Then $a_{2j} * a_{3k} = a_{4k} \in S_4 \cap H = H_4$, i.e., $H_4 \neq \emptyset$ and if $a_{4k} \in H_4$, then $a_{2j} * a_{4k} = a_{3k} \in S_3 \cap H = H_3$.

Lemma 4. Let S be a four-part semigroup and let $\mathcal{H} \subseteq S$ be a subsemigroup of S.

- (i) If $H \cap S_2 \neq \emptyset$, then H is a four-part semigroup or a right-zero φ -semigroup.
- (ii) If $H \cap S_2 = \emptyset$, then \mathcal{H} is a right-zero, a constant, a right-zero constant, a two-constant or a right-zero two-constant semigroup.

Proof. (i) Because of $H \subseteq S_1 \cup S_2 \cup S_3 \cup S_4$ we can write $H = (S_1 \cap H) \cup (S_2 \cap H) \cup (S_3 \cap H) \cup (S_4 \cap H)$. If $H \cap S_2 \neq \emptyset$, then $H \cap S_1 \neq \emptyset$ by Lemma 3. We consider two cases:

1. $S_3 \cap H = \emptyset$. Then also $S_4 \cap H = \emptyset$ by Lemma 3 and $H = (S_1 \cap H) \cup (S_2 \cap H)$ and with a bijection $\varphi : S_1 \cup S_2 \to S_1 \cup S_2$ defined by $\varphi(a_{1j}) = a_{2j}$ and $\varphi(a_{2j}) = a_{1j}$ for all $j \in \{1, 2, ..., n_r\}$ we obtain $\varphi \circ \varphi = id$ on S and

$$a_{ij} * a_{lk} = \begin{cases} a_{lk} & \text{if } a_{ij} \in S_1 \\ \varphi(a_{lk}) & \text{if } a_{ij} \in S_2. \end{cases}$$

The restriction of φ on H satisfies $\varphi \circ \varphi = id$ on H and

$$a_{ij} * a_{lk} = \begin{cases} a_{lk} & \text{if } a_{ij} \in H_1 \\ \varphi(a_{lk}) & \text{if } a_{ij} \in H_2 \end{cases}$$

Therefore H is a right-zero φ -semigroup.

2. $S_3 \cap H \neq \emptyset$. Then by Lemma 3 also $S_4 \cap H \neq \emptyset$ and H is the union of four pairwise disjoint non-empty sets. Since \mathcal{H} is a subsemigroup by the definition of *, we get $a^*, a^{**} \in H$ and we have

$$a_{ij} * a_{lk} = \begin{cases} a_{lk} & \text{if } a_{ij} \in S_1 \cap H \\ \\ a_{tk} & \text{if } a_{ij} \in S_2 \cap H \\ \\ a^* \in S_3 & \text{if } a_{ij} \in S_3 \cap H \\ \\ a^{**} \in S_4 & \text{if } a_{ij} \in S_4 \cap H. \end{cases} \text{ where } t = \begin{cases} 1 & \text{if } l = 2 \\ 2 & \text{if } l = 1 \\ 3 & \text{if } l = 4 \\ 4 & \text{if } l = 3 \end{cases}$$

This shows that \mathcal{H} is a four-part semigroup.

(ii) If $H \cap S_2 = \emptyset$, then $H = (S_1 \cap H) \cup (S_3 \cap H) \cup (S_4 \cap H)$. Now we have the following cases:

- 1. $H \cap S_3 = H \cap S_4 = \emptyset$. Then $H = H \cap S_1$ forms a right-zero semigroup.
- 2. $H \cap S_1 = H \cap S_4 = \emptyset$ or $H \cap S_1 = H \cap S_3 = \emptyset$. Then \mathcal{H} is a constant semigroup.
- 3. $H \cap S_3 = \emptyset$ or $H \cap S_4 = \emptyset$. Then \mathcal{H} is a right-zero constant semigroup.
- 4. $H \cap S_1 = \emptyset$. Then \mathcal{H} is a two-constant semigroup.
- 5. $H \cap S_1 \neq \emptyset$, $H \cap S_3 \neq \emptyset$, $H \cap S_4 \neq \emptyset$. Then \mathcal{H} is a right-zero two-constant semigroup.

Lemma 4 answers to the question which kinds of semigroups can occur as subsemigroups of a four-part semigroup. Now we characterize the different cases.

Proposition 5. Let S be a four-part semigroup with constant elements a^* and a^{**} . Then a subset $H \subseteq S$ is the universe of a four-part semigroup with constant elements b^* and b^{**} if and only if

- (i) $H \cap S_2 \neq \emptyset, b^* = a^*, b^{**} = a^{**}$ and
- (ii) H is closed under φ , that is $\varphi(a) \in H$ for all $a \in H$.

Proof. Assume that the two conditions are satisfied. Then by Lemma 3 and condition (i), $H \cap S_1 \neq \emptyset$ and hence $H \cap S_i \neq \emptyset$ for all i = 1, 2, 3, 4 and we define $H_1 := H \cap S_1$, $H_2 := H \cap S_2$, $H_3 := H \cap S_3$, $H_4 := H \cap S_4$. Clearly, $H = H_1 \cup H_2 \cup H_3 \cup H_4$ and $H_i \cap H_j = \emptyset$ for $i \neq j$ and each of these sets is nonempty. We show that H is closed under the multiplication of S and therefore a subsemigroup. If $a_{1j} \in H_1$ and $b \in H$ is arbitrary, then $a_{1j} * b = b$ since $a_{1j} \in S_1$. If $a_{2j} \in H_2$ and $b \in H$, then $a_{2j} * b = \varphi(b) \in H$ by $\varphi(H) \subseteq H$ and if $a_{4k} \in H_4$, then $a_{2j} * a_{4k} = a_{3k} = \varphi(a_{4k}) \in H$ by $\varphi(H) \subseteq H$. If $a_{3k} \in H_3$, then $a_{3k} * b = a^* \in H$ for any $b \in H$ and if $a_{4k} \in H_4$, then $a_{4k} * b = a^{**} \in H$ for any $b \in H$. This shows that \mathcal{H} is a four-part semigroup.

Assume now conversely, that \mathcal{H} is a four-part subsemigroup of \mathcal{S} . We want to show that (i) and (ii) are satisfied. We know that $H = H_1 \cup H_2 \cup H_3 \cup H_4$, $H_i \neq \emptyset$, $H_i \cap H_j = \emptyset$ for $i \neq j$ and $b^* \in H_3$ and $b^{**} \in H_4$ are the constant elements. We need the following

Claim. $H_1 \subseteq S_1, H_2 \subseteq S_2, H_3 \subseteq S_3, H_4 \subseteq S_4.$

Proof of the Claim. Assume that $H_1 \not\subseteq S_1$. Then there is an element $a_{1j} \in H_1$ but $a_{1j} \notin S_1$ and we form $a_{1j} *_H a_{1j} = a_{1j} \in H_1$ using the multiplication $*_H$ of \mathcal{H} . But this has to be equal to $a_{1j} *_S a_{1j}$ using the multiplication of \mathcal{S} . Since $a_{1j} \notin S_1$, there are the following possibilities for a_{1j} , i.e., $a_{1j} \in H_1 \cap S_2$ or $a_{1j} \in H_1 \cap S_3$ or $a_{1j} \in H_1 \cap S_4$.

- 1. $a_{1i} \in S_2$, then $a_{1i} *_S a_{1i} = a_{2i} \in S_2$. But $a_{2i} \neq a_{1i}$, a contradiction.
- 2. $a_{1j} \in S_3$, then $a_{1j} *_H b = b$ for any $b \in H$, but $a_{1j} *_S b = a^* \in S_3$. This gives a contradiction for any $b \in H$.
- 3. $a_{1j} \in S_4$, then $a_{1j} *_H b = b$ for any $b \in H$, but $a_{1j} *_S b = a^{**} \in S_4$ for any $b \in H$, a contradiction.

These contradictions show that $H_1 \subseteq S_1$.

Assume that $H_2 \not\subseteq S_2$. Then there is an element $a_{2j} \in H_2$, but $a_{2j} \notin S_2$. We form $a_{2j} *_H a_{2j} = a_{1j}$ and this has to be equal to $a_{2j} *_S a_{2j}$. Here we have the following possibilities:

- 1. $a_{2j} \in S_1$, then $a_{2j} *_S a_{2j} = a_{2j} \neq a_{1j}$.
- 2. $a_{2j} \in S_3$, then $a_{2j} *_S a_{2j} = a^* \neq a_{1j}$.
- 3. $a_{2i} \in S_4$, then $a_{2i} *_S a_{2i} = a^{**} \neq a_{1i}$.

This contradiction shows that $H_2 \subseteq S_2$.

Assume that $H_3 \not\subseteq S_3$. Then there is an element $a_{3j} \in H_3$, but $a_{3j} \notin S_3$. If $a_{3j} \in S_1$, then $a_{3j} *_H b = b^*$ for any $b \in H$, but $a_{3j} *_S b = b$, i.e., $b = b^*$ for any $b \in H$, a contradiction. If $a_{3j} \in S_2$, then $a_{3j} *_H b = b^*$ for any $b \in H$, $b^* \in H_3$ but $a_{3j} *_S b = \varphi(b)$, i.e., $\varphi(b) = b^*$ for any b, a contradiction. If $a_{3j} \in S_4$, then $b *_H a_{3j} = a_{4j} = \varphi(a_{3j}) \in S_4$, but $b *_S a_{3j} = \varphi(a_{3j}) \notin S_4$ for any $b \in H_2 \subseteq S_2$.

 $H_4 \subseteq S_4$ can be proved in a similar way as $H_3 \subseteq S_3$. This finishes the proof of this claim.

 $H_2 \subseteq S_2$ and $H_2 \neq \emptyset$ show that $H \cap S_2 \neq \emptyset$. Moreover, since $b^* \in H_3 \subseteq S_3$ and $b^{**} \in H_4 \subseteq S_4$, we have $b^* = a_{3j} *_H b = a_{3j} *_S b = a^*$ and $b^{**} = a_{4j} *_H b = a_{4j} *_S b = a^{**}$ for any $a_{3j} \in H_3, a_{4j} \in H_4$ and $b \in H$. Since $H_2 \subseteq S_2$, then for every $a \in H$ we have $\phi(a) = a_{2j} *_H a \in H$ for $a_{2j} \in H_2$, that means (ii) is satisfied. This completes the proof.

Proposition 6. let S be a four-part semigroup. Then a subset $H \subseteq S$ is the universe of a right-zero two-constant subsemigroup of S if and only if $H \subseteq S_1 \cup S_3 \cup S_4$, $H \cap S_1 \neq \emptyset$ and there are two fixed elements $b^*, b^{**}, b^* \neq b^{**}$ with $\{a^*, a^{**}\} = \{b^*, b^{**}\}.$

Proof. Let H be the universe of a right-zero two-constant subsemigroup of S. We show first that $H \cap S_2 = \emptyset$. Indeed, if $H \cap S_2 \neq \emptyset$, then there is an element $a_{2j} \in H \cap S_2$ for some $j \in \{1, \ldots, n_r\}$ and then $a_{2j} *_S b = \varphi(b)$ for any $b \in H$ with $\varphi(b) \neq b$ and b can be chosen in such a way that $\varphi(b) \neq b^* \in H_2$, $\varphi(b) \neq b^{**} \in H_3$. But by the definition of a right-zero two-constant semigroup we must have

$$a_{2j} *_S b = a_{2j} *_H b = \begin{cases} b & \text{if } a_{2j} \in H_1 \\ b^* & \text{if } a_{2j} \in H_2 \\ b^{**} & \text{if } a_{2j} \in H_3, \end{cases}$$

a contradiction. This shows that $H \subseteq S_1 \cup S_3 \cup S_4$. Now we show that $H \cap S_1 \neq \emptyset$. If $H \cap S_1 = \emptyset$, then $H \subseteq S_3 \cup S_4$ and then

$$a_{ij} *_S a_{lk} = \begin{cases} b^* & \text{if } a_{ij} \in S_3 \\ b^{**} & \text{if } a_{ij} \in S_4 \end{cases}$$

for all $a_{ij}, a_{lk} \in H$. But if $a_{ij} \in H_1 \neq \emptyset$, $a_{ij} \in S_3$, then we have $a_{ij} *_S b^{**} = b^* \neq b^{**} = a_{ij} *_H b^{**}$, a contradiction. If $a_{ij} \in H_1 \neq \emptyset$, but $a_{ij} \in S_4$ we have $a_{ij} *_S b^* = b^{**} \neq b^* = a_{ij} *_H b^*$. This shows that $H \cap S_1 \neq \emptyset$. Since $b^* \in H_2$ we have $b^* *_H b^{**} = b^*$. Since $H_2 \subseteq S_1 \cup S_3 \cup S_4$, we consider the following three possibilities for b^* :

- 1. $b^* \in S_1$, then $b^* *_S b^{**} = b^{**} \neq b^*$, a contradiction.
- 2. $b^* \in S_3$, then $b^* *_S b^{**} = a^*$ and thus $a^* = b^*$ or
- 3. $b^* \in S_4$, then $b^* *_S b^{**} = a^{**} = b^*$.

Therefore $\{a^*, a^{**}\} = \{b^*, b^{**}\}.$

Conversely, assume that $H \subseteq S$ is a subset of the universe S of a fourpart semigroup S with $H \subseteq S_1 \cup S_3 \cup S_4$, $H \cap S_1 \neq \emptyset$ and that there are two elements $b^*, b^{**} \in H$ satisfying $b^* \neq b^{**}$ and $\{a^*, a^{**}\} = \{b^*, b^{**}\}$. We show that \mathcal{H} is a right-zero two-constant subsemigroup of S. We show that H is closed under the multiplication in S. If $a \in H \cap S_1$, then for any $b \in H$ we have $a *_S b = b \in H$ and in all other cases we get elements in $\{a^*, a^{**}\}$, which are in Hsince $\{a^*, a^{**}\} = \{b^*, b^{**}\}$. Therefore H is a subsemigroup of S with $H \cap S_1 \neq \emptyset$. Further we have $H \cap S_3 \neq \emptyset$ and $H \cap S_4 \neq \emptyset$ since $\{b^*, b^{**}\} \subseteq H$. Now we set $H_1 := H \cap S_1, H_2 := H \cap S_3, H_3 := H \cap S_4$. Then $H = H_1 \cup H_2 \cup H_3$. We have to show that $b^* \in H_2$ and $b^{**} \in H_3$ or conversely and that $*_S|_H$ is the multiplication of a right-zero two-constant semigroup. If $a \in H_2$, then we have $a *_S b = a *_H b = a^* \in H \cap S_3 = H_2$ for all $b \in H$ and if $a \in H_3$, then we have $a *_S b = a *_H b = a^{**} \in H \cap S_4 = H_3$ for all $b \in H$. Now we can set $a^* = b^* \in H_2$ or $b^* = a^{**}$ or conversely. If $a \in H_1 = S_1 \cap H$, then $a *_H b = a *_S b = b$ for all $b \in H$. This shows that \mathcal{H} is a right-zero two-constant subsemigroup of S. **Proposition 7.** A subset $H \subseteq S$ of the universe of a four-part semigroup is the universe of a two-constant subsemigroup of S if and only if $H \subseteq S_3 \cup S_4$ and there are two elements $b^*, b^{**} \in H$, $b^* \neq b^{**}$ such that $\{a^*, a^{**}\} = \{b^*, b^{**}\}$.

Proof. If \mathcal{H} is a two-constant semigroup, then $H \cap S_1 = \emptyset$ and $H \cap S_2 = \emptyset$ since, if $a_{ij} \in S_1$, then $a_{ij} *_S b^* = b^*$ and $a_{ij} *_S b^{**} = b^{**}$, $a_{ij} \in H \cap S_1$ means $a_{ij} \in H_1$ or $a_{ij} \in H_2$. In the first case we have $a_{1j} *_H b^{**} = b^*$ and in the second case, $a_{1j} *_H b^* = b^{**}$. In both cases, we have a contradiction. If $a_{2j} \in H \cap S_2$, then again we have $a_{2j} \in H_1$ or $a_{2j} \in H_2$ and $a_{2j} *_S b^* = \varphi(b^*) = b^{**}$ and $a_{2j} *_S b^{**} = \varphi(b^{**}) = b^*$. If $a_{2j} \in H_1$, then $a_{2j} *_H b^* = b^*$ and if $a_{2j} \in H_2$, then $a_{2j} *_H b^{**} = b^{**}$. In both cases we have a contradiction. This shows $H \subseteq S_3 \cup S_4$. The second condition is clear. Assume now that H is a subset of $S_3 \cup S_4$. We define $H_1 := H \cap S_3$ and $H_2 := H \cap S_4$, $b^* := a^*, b^{**} := a^{**}$. If $a \in H_1$, then $a * b = b^* = a^* \in H$ and if $a \in H_2$, then $a *_S b = b^{**} \in H$ for all $b \in H$. This shows that \mathcal{H} is a two-constant semigroup.

Proposition 8. Let $H \subseteq S$ be a subset of the universe of a subsemigroup of the four-part semigroup S. Then H is the universe of a right-zero constant subsemigroup of S if and only if either $H \subseteq S_1 \cup S_3$, $H \cap S_1 \neq \emptyset$ and $a^* \in H$ or $H \subseteq S_1 \cup S_4$, $H \cap S_1 \neq \emptyset$ and $a^{**} \in H$.

Proof. Assume that \mathcal{H} is a right-zero constant subsemigroup of \mathcal{S} .

Claim.

- (i) Either $H \cap S_3 = \emptyset$ or $H \cap S_4 = \emptyset$.
- (ii) $H \not\subseteq S_1, H \not\subseteq S_2, H \not\subseteq S_3, H \not\subseteq S_4.$

Proof of Claim. (i) For the fixed element b^* and for all $a \in H$ we have $a * b^* = b^*$. If $a \in S_3$, then $a *_S b^* = a^*$ and then $a^* = b^*$ and for $a \in S_4$ we have $a *_S b^* = a^{**} = b^*$. Thus if $S_3 \cap H \neq \emptyset$, then $S_4 \cap H = \emptyset$ and if $S_4 \cap H \neq \emptyset$, then $S_3 \cap H = \emptyset$ and this proves (i).

(ii) We use that $|H| \ge 2$. Assume that $H \subseteq S_1$, then for all $a, b \in H$ we have $a *_H b = b$ and if $a \in H_2$ we get $a *_S b = b^*$, i.e., $b = b^*$ for all $b \in S$, a contradiction, which shows $H \not\subseteq S_1$. Assume that $H \subseteq S_2$, then $a *_H b = \varphi(b)$ for all $a \in H$ and $a *_S b = b$ if $a \in H_1$, i.e., $\varphi(b) = b$ for all $b \in H$, a contradiction. Assume that $H \subseteq S_3$. If $a \in H_1$, then $a *_H b = b = a^* = a *_S b$ for all $b \in H$, i.e., |H| = 1, a contradiction. In a similar way we show that $H \not\subseteq S_4$.

Now we prove that $\mathcal{H} \subseteq S$ is a right-zero constant subsemigroup if the conditions are satisfied. We show that $H \subseteq S$ is closed under $*_S$. Assume that $a \in S_1 \cap H$, then $a *_S b = b \in H$. If $a \in H \cap S_3$, then $a *_S b = a^* \in H$. In the second case we conclude in the same way. We set $H_1 = H \cap S_1 \neq \emptyset$ and $H_2 = H \cap S_3 \neq \emptyset$ and $b^* = a^*$ since $a^* \in H \cap S_3$ in the first case and $H_1 := H \cap S_1 \neq \emptyset$ and $H_2 = H \cap S_4 \neq \emptyset$ and $b^* = a^{**}$ in the second one. Then all conditions for a right-zero subsemigroup are satisfied. Now we assume that \mathcal{H} is a right-zero constant semigroup. By the claim we have $H \subseteq S_1 \cup S_2 \cup S_4$ or $H \subseteq S_1 \cup S_2 \cup S_3$. We know also that $H \not\subseteq S_1, H \not\subseteq S_2, H \not\subseteq S_3, H \not\subseteq S_4$. Note that H cannot contain an element b from S_3 together with an element $a \in S_2$ since otherwise $a *_H b = \varphi(b) \in S_4 \cap H$ which contradicts the claim. A similar argument shows that H cannot contain an element from S_4 together with an element from S_2 . Then for H we have precisely the following cases:

- 1. $H \subseteq S_3 \cup S_1, H \cap S_3 \neq \emptyset, H \cap S_1 \neq \emptyset$ or
- 2. $H \subseteq S_4 \cup S_1, H \cap S_4 \neq \emptyset, H \cap S_1 \neq \emptyset$ or
- 3. $H \subseteq S_1 \cup S_2, H \cap S_1 \neq \emptyset, H \cap S_2 \neq \emptyset$.

If $a \in S_2, b^* \in S_1$, then $a *_S b^* = \varphi(b^*) = b^* = a *_H b^*$, a contradiction and for $a \in H, b^* \in S_2$ we get $a *_S b^* = b^*$, which is also a contradiction. Therefore the third case can be excluded. In the first case from an element in $H \cap S_1 \neq \emptyset$ and an element from $H \cap S_3 \neq \emptyset$ we can produce a^* , namely by $a *_H b = a^*$ and have $a^* \in H$. In the second case we get $a^{**} \in H$. By the claim both conditions exclude each other.

Proposition 9. Let $H \subseteq S$ be a subset of the universe of a subsemigroup of a four-part semigroup S. Then H is the universe of a right-zero semigroup if and only if $H \subseteq S_1$ or $H = \{a^*\}$ or $H = \{a^{**}\}$.

Proof. Assume that $H \subseteq S_1$ or $\{a^*\}$ or $\{a^{**}\}$. Then H is closed under the multiplication of S and forms a right-zero semigroup. Conversely, let H be a right-zero subsemigroup of S. Assume that $H \not\subseteq S_1$, then there exists $a \in H \cap (S_2 \cup S_3 \cup S_4)$. But if $a \in S_2$ then we have $a *_H a = a \neq \varphi(a) = a *_S a$, a contradiction. Therefore $H \subseteq S_3 \cup S_4$. We show that $H \cap S_3 = \{a^*\}$ and $H \cap S_4 = \{a^{**}\}$. Let $a \in H \cap S_3, a \neq a^*$. Then $a *_H a = a^*$ which contradicts the definition of a right-zero semigroup. Therefore $H \cap S_3 = \{a^*\}$. Similarly we obtain $H \cap S_4 = \{a^{**}\}$. Hence $H \subseteq \{a^*, a^{**}\}$ and we obtain $H = \{a^*\}$ or $H = \{a^{**}\}$ or $H = \{a^*, a^{**}\}$. The latter case is impossible since otherwise $a^* *_H a^{**} = a^* *_S a^{**} = a^*$.

Proposition 10. Let $H \subseteq S$ be a subset of the universe of a subsemigroup of a four-part semigroup S. Then H is the universe of a constant semigroup if and only if $H \subseteq S_3$ and $a^* \in H$ or $H \subseteq S_4$ and $a^{**} \in H$ or $H = \{a\}, a \in S_1$.

Proof. If the condition is satisfied, then $a *_S b = a^* \in H$ if $a, b \in H \cap S_3$ or $a *_S b = a^{**}$ if $a, b \in S_4 \cap H$. Therefore the set H is closed under multiplication

and forms a constant subsemigroup of S. If $\mathcal{H} \subseteq S$ is a constant subsemigroup of S and $|H| \geq 2$, then $H \cap S_1 = \emptyset, H \cap S_2 = \emptyset, H \cap S_4 = \emptyset$ or $H \cap S_1 = \emptyset, H \cap S_2 = \emptyset, H \cap S_3 = \emptyset$. Indeed, if $a \in H \cap S_1$ and $b \neq a, b \in H$, then $a *_H b = b$, but $a *_H a = a \neq b$ and \mathcal{H} is not a constant semigroup, therefore $H \cap S_1 = \emptyset$. Let $H \cap S_2 \neq \emptyset$ and $a \in H \cap S_2$. Then $a *_S b = \varphi(b)$ and $a *_S \varphi(b) = b$. Because of $\varphi(b) \neq b$ (φ is a fixed point free mapping) is \mathcal{H} not a constant semigroup. Therefore $H \cap S_2 = \emptyset$. If $a \in H \cap S_4$ and assume that $b \in S_3$. Then $a *_S b = a^{**}$ and $b * a = a^* \in A$. Because of $a^* \neq a^{**}$, \mathcal{H} cannot be constant. In the second case we conclude in a similar way. Moreover, we cannot have elements from S_3 and from S_4 since otherwise $a *_H b = a^*$ if $a \in S_3$ and $a *_H b = a^{**}$ if $a \in S_4$ and this contradicts the assumption that \mathcal{H} is a constant semigroup. These equation show also that $a^* \in H$ if $H \subseteq S_3$ or $a^{**} \in H$ if $H \subseteq S_4$. If |H| = 1 then the only element must be idempotent, i.e. $a \in S_1$ or $a \in \{a^*, a^{**}\}$. But the second case is already included in the previous cases.

Proposition 11. Let S be a four-part semigroup. Then a non-empty subset $H \subseteq S$ is the universe of a right-zero φ -subsemigroup of S if and only if $H \subseteq S_1 \cup S_2, H \cap S_1 \neq \emptyset$ and $H \cap S_2 \neq \emptyset$ and H is closed under φ , i.e., if $a \in H$, then $\varphi(a) \in H$ for all $a \in H$.

Proof. Let H be the universe of a right-zero φ -subsemigroup of S. We prove that $H \cap S_3 = \emptyset$ and $H \cap S_4 = \emptyset$. If $H \cap S_3 \neq \emptyset$ and $a \in S_3$, then $a *_H b = b$ if $a \in H_1$ or $a *_H b = \varphi(b)$ if $a \in H_2$, but $a *_S b = a^*$ for any $b \in H$, i.e., $b = a^*$ for any $b \in H$, a contradiction or $\varphi(b) = a^*$ for any $b \in H$, which is also a contradiction. Similarly we get a contradiction if $H \cap S_4 \neq \emptyset$. Altogether, we have $H \subseteq S_1 \cup S_2$.

Suppose that $H \cap S_1 = \emptyset$. Then $H \subseteq S_2$ and since $H \neq \emptyset$, there is an element $a \in H \cap S_2$. Then $a *_S a = a *_H a = \varphi(a)$, where φ is the idempotent, fixed point free bijective mapping from S. Since $a \in S_2$, the image $\varphi(a)$ belongs to S_1 , a contradiction. If $H \cap S_2 = \emptyset$, then $H \subseteq S_1$ and with $a \in H \cap S_1$ and $b \in H_2$ we have $b *_S a = b *_H a = \varphi(a) \in H_1$, where φ is the fixed point free, bijective mapping from \mathcal{H} . The element $\varphi(a)$ belongs to S and since $a \in S_1$, we have $\varphi(a) \in S_2$, a contradiction. With $b \in H_2$ for any $a \in H$ we have $b *_H a = \varphi(a) \in H$, i.e., H is closed under φ .

Since $\mathcal{H} \subseteq S$ is a subsemigroup let conversely, $H \subseteq S$ be a subset which satisfies $H \subseteq S_1 \cup S_2, H \cap S_1 \neq \emptyset, H \cap S_2 \neq \emptyset$. Then we define $H_1 := H \cap S_1$ and $H_2 := H \cap S_2$ and use as fixed point free, bijective mapping from H the restriction of the corresponding mapping of H since H is closed under φ . Now we have

$$a *_H b = \begin{cases} b & \text{if } a \in H_1 \\ \varphi(b) & \text{if } a \in H_2 \end{cases}$$

and \mathcal{H} is a right-zero φ -semigroup.

3. Idempotent and regular subsemigroups of four-part semigroups

Proposition 12. Let S be a four-part semigroup and let $a \in S$ be arbitrary. Then a is an idempotent element of S if and only if $a \in S_1 \cup \{a^*, a^{**}\}$.

Proof. If $a \in S_1 \cup \{a^*, a^{**}\}$, then it is clear that a * a = a. Conversely, let $a \in S$ be idempotent. Assume that $a \notin S_1 \cup \{a^*, a^{**}\}$. If $a \in S_2$ then $a * a = \varphi(a) \neq a$, a contradiction. If $a \in (S_3 \cup S_4) \setminus \{a^*, a^{**}\}$, then $a * a \in \{a^*, a^{**}\}$ and thus $a * a \neq a$, a contradiction. This completes the proof.

Proposition 13. Let S be a four-part semigroup and let $H \subseteq S$. Then \mathcal{H} is an idempotent subsemigroup of S if and only if $H \subseteq S_1 \cup \{a^*, a^{**}\}$.

Proof. If $H \subseteq S_1 \cup \{a^*, a^{**}\}$, then by definition $a * b \in H$ for every $a, b \in H$ and thus \mathcal{H} is a subsemigroup of \mathcal{S} . By Proposition 12 it follows that \mathcal{H} is an idempotent subsemigroup. Conversely, if \mathcal{H} is an idempotent subsemigroup of \mathcal{S} , then by Proposition 12, $H \subseteq S_1 \cup \{a^*, a^{**}\}$.

Proposition 14. Let S be a four-part semigroup and let $a \in S$ be arbitrary. Then a is a regular element of S if and only if $a \in S_1 \cup S_2 \cup \{a^*, a^{**}\}$.

Proof. Let $a \in S_1 \cup S_2 \cup \{a^*, a^{**}\}$. If $a \in S_1$, then (a * a) * a = a * a = a, if $a_{2j} \in S_2$, then $(a_{2j} * a_{2j}) * a_{2j} = \varphi(a_{2j}) * a_{2j} = a_{1j} * a_{2j} = a_{2j}$ and if $a = a^*$ or $a = a^{**}$, then a * a * a = a. Thus any $a \in S_1 \cup S_2 \cup \{a^*, a^{**}\}$ is regular. Conversely, for arbitrary $a_{3j} \in S_3, a_{3j} \neq a^*, a_{4j} \in S_4, a_{4j} \neq a^{**}$ and for any $b \in S$ we have $(a_{3j} * b) * a_{3j} = a^* * a_{3j} = a^* \neq a_{3j}$ and $(a_{4j} * b) * a_{4j} = a^{**} * a_{4j} = a^{**} \neq a_{4j}$. Hence, $a \in (S_3 \cup S_4) \setminus \{a^*, a^{**}\}$ cannot be regular. Therefore if $a \in S$ is a regular element, then $a \in S_1 \cup S_2 \cup \{a^*, a^{**}\}$.

Proposition 15. Let S be a four-part semigroup and let $H \subseteq S$. Then \mathcal{H} is a regular subsemigroup of S if and only if $H \subseteq S_1 \cup \{a^*, a^{**}\}$ or $H \subseteq S_1 \cup S_2 \cup \{a^*, a^{**}\}$ such that $\varphi(a) \in H$ for all $a \in H$.

Proof. If $H \subseteq S_1 \cup \{a^*, a^{**}\}$, then by Proposition 13, \mathcal{H} is an idempotent subsemigroup and hence a regular subsemigroup. Now, let $H \subseteq S_1 \cup S_2 \cup \{a^*, a^{**}\}$ such that $\varphi(a) \in H$ for all $a \in H$. If $a \in S_1$, then for all $b \in H$ we have $a * b = b \in H$, if $a \in S_2$ and $b \in H$, then $a * b = \varphi(b) \in H$, if $a = a^*$ or $a = a^{**}$, then $a * b = a^* \in H$ or $a * b = a^{**}$ for all $b \in H$. Thus H is closed under multiplication and hence forms a subsemigroup and by Proposition 14, \mathcal{H} is regular. Conversely, let \mathcal{H} be a regular subsemigroup of \mathcal{S} such that $H \not\subseteq S_1 \cup \{a^*, a^{**}\}$. Then by Proposition 14 we have $H \subseteq S_1 \cup S_2 \cup \{a^*, a^{**}\}$ and $H \cap S_2 \neq \emptyset$. Since \mathcal{H} is a semigroup then for all $b \in H \cap S_2$ and $a \in H$ we have $b * a = \varphi(a) \in H$. This completes the proof.

4. Homomorphisms of four-part semigroups

Lemma 16. Let S = (S; *) be a four-part semigroup with constant elements a^* and a^{**} and let S' = (S; *') be an arbitrary semigroup. Let $\phi : S \to S'$ be a homomorphism. Then the following Propositions are true for all $j, j' \in \{1, \ldots, n_r\}$ and $k, k' \in \{1, \ldots, n_s\}$.

- (i) If there are $a_{1j}, a_{2j'} \in S$ such that $(a_{1j}, a_{2j'}) \in Ker\phi$, then $(a, \varphi(b)) \in Ker\phi$ for every $(a, b) \in Ker\phi$.
- (ii) If there are $a_{1i}, a_{3k} \in S$ such that $(a_{1i}, a_{3k}) \in Ker\phi$, then ϕ is constant.
- (iii) If there are $a_{1j}, a_{4k} \in S$ such that $(a_{1j}, a_{4k}) \in Ker\phi$, then ϕ is constant.
- (iv) If there are $a_{2j}, a_{3k} \in S$ such that $(a_{2j}, a_{3k}) \in Ker\phi$, then ϕ is constant.
- (v) If there are $a_{2i}, a_{4k} \in S$ such that $(a_{2i}, a_{4k}) \in Ker\phi$, then ϕ is constant.
- (vi) If there are $a_{3k}, a_{4k'} \in S$ such that $(a_{3k}, a_{4k'}) \in Ker\phi$, then $(a^*, a^{**}) \in Ker\phi$.

Proof. Let $\phi: S \to S'$ be a homomorphism.

(i) If $(a_{1j}, a_{2j'}) \in Ker\phi$, then $(a, \varphi(b)) = (a_{1j} * a, a_{2j'} * b) \in Ker\phi$ for every $(a, b) \in Ker\phi$.

(ii) If $(a_{1j}, a_{3k}) \in Ker\phi$, then for every $b \in S$ we have $(b, a^*) = (a_{1j}*b, a_{3k}*b) \in Ker\phi$ and therefore ϕ is constant.

(iii) If $(a_{1j}, a_{4k}) \in Ker\phi$, then for every $b \in S$ we have $(b, a^{**}) = (a_{1j}*b, a_{4k}*b) \in Ker\phi$ and therefore ϕ is constant.

(iv) If $(a_{2j}, a_{3k}) \in Ker\phi$, then $(a_{1j}, a^*) = (a_{2j} * a_{2j}, a_{3k} * a_{3k}) \in Ker\phi$ and by (ii), ϕ is constant.

(v) If $(a_{2j}, a_{4k}) \in Ker\phi$, then $(a_{1j}, a^{**}) = (a_{2j} * a_{2j}, a_{4k} * a_{4k}) \in Ker\phi$ and by (iii), ϕ is constant.

(vi) If $(a_{3k}, a_{4k'}) \in Ker\phi$, then $(a^*, a^{**}) = (a_{3k} * a_{3k}, a_{4k'} * a_{4k'}) \in Ker\phi$.

Using the kernel $Ker\phi$ of a homomorphism ϕ we now give some more conditions for a homomorphism ϕ .

Theorem 17. Let S = (S; *) be a four-part semigroup with constant elements a^* and a^{**} and let S' = (S; *') be an arbitrary semigroup. If the mapping $\phi : S \to S'$ is a homomorphism then

(i) ϕ is constant and maps every element of S to an idempotent element of S' or

- (ii) ϕ satisfies $(\varphi(a), \varphi(b)) \in Ker\phi$ whenever $(a, b) \in Ker\phi$ and $(a, b) \in Ker\phi$ if and only if $a, b \in S_i$ for i = 1, 2, 3, 4 and for any $a, b \in S$ or
- (iii) ϕ satisfies $(a^*, a^{**}) \in Ker\phi$, $(\varphi(a), \varphi(b)) \in Ker\phi$ whenever $(a, b) \in Ker\phi$ and $(a, b) \in Ker\phi$ if and only if $a, b \in S_3 \cup S_4$ or $a, b \in S_1$ or $a, b \in S_2$ for any $a, b \in S$ or
- (iv) ϕ satisfies $(a, \varphi(b)) \in Ker\phi$ whenever $(a, b) \in Ker\phi$ and $(a, b) \in Ker\phi$ if and only if $a, b \in S_1 \cup S_2$ or $a, b \in S_3 \cup S_4$ for any $a, b \in S$.

Proof. Let $\phi : S \to S'$ be a homomorphism. Let $(a, b) \in Ker\phi$. We consider the following cases:

- 1. If there are $a_{1j}, a_{3k} \in S$ such that $(a_{1j}, a_{3k}) \in Ker\phi$ or there are $a_{1j}, a_{4k} \in S$ such that $(a_{1j}, a_{4k}) \in Ker\phi$ or there are $a_{2j}, a_{3k} \in S$ such that $(a_{2j}, a_{3k}) \in Ker\phi$ or there are $a_{2j}, a_{4k} \in S$ such that $(a_{2j}, a_{4k}) \in Ker\phi$, then by Lemma 16 (ii), (iii), (iv) and (v), ϕ is constant. Moreover, if ϕ maps all $a \in S$ to $c \in S'$, then $c = \phi(a * b) = \phi(a) *' \phi(b) = c *' c$, i.e., c is idempotent and we have (i).
- 2. If $(a_{1j}, a_{3k}), (a_{1j}, a_{4k}), (a_{2j}, a_{3k}), (a_{2j}, a_{4k}) \notin Ker\phi$ for all $j \in \{1, \ldots, n_r\}$ and for all $k \in \{1, \ldots, n_s\}$, then we consider the following subcases:
 - a. If $(a_{1j}, a_{2j'}), (a_{3k}, a_{4k'}) \notin Ker\phi$ for all $j, j' \in \{1, \ldots, n_r\}$ and for all $k, k' \in \{1, \ldots, n_s\}$, then $(a, b) \in Ker\phi$ if and only if a and b are in the same set S_i for all $a, b \in S$. Moreover, $(\varphi(a), \varphi(b)) = (a_{2j} * a, a_{2j} * b) \in Ker\phi$ whenever $(a, b) \in Ker\phi$ and $j \in \{1, \ldots, n_r\}$. Hence we have (ii).
 - b. If $(a_{3k}, a_{4k'}) \in Ker\phi$ for some $k, k' \in \{1, \ldots, n_s\}$ and $(a_{1j}, a_{2j'}) \notin Ker\phi$ for every $j, j' \in \{1, \ldots, n_r\}$, then $(a^*, a^{**}) = (a_{3k} * a_{3k}, a_{4k'} * a_{4k'}) \in Ker\phi$. Moreover, $(\varphi(a), \varphi(b)) = (a_{2j} * a, a_{2j} * b) \in Ker\phi$ whenever $(a, b) \in Ker\phi$ and $(a, b) \in Ker\phi$ if and only if $a, b \in S_3 \cup S_4$ or $a, b \in S_1$ or $a, b \in S_2$. Thus we have (iii).
 - c. If there is $(a_{1j}, a_{2j'}) \in Ker\phi$ for some $j, j' \in \ldots, n_r$, then by Lemma 16 (i), $(a, \varphi(b)) \in Ker\phi$ for any $(a, b) \in Ker\phi$. Moreover, $(a, b) \in Ker\phi$ if and only if $a, b \in S_1 \cup S_2$ or $a, b \in S_3 \cup S_4$ and thus we have (iv).

The opposite direction is not true. The following easy example shows that there are mappings ϕ which satisfy (ii), but are not homomorphisms. Let $\phi: S \to \mathbb{Z}_4$ with $\mathcal{Z}_4 = (\{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}; \cdot)$ be defined by $\phi(S_1) = \bar{0}, \phi(S_2) = \bar{1}, \phi(S_3) = \bar{2}, \phi(S_4) = \bar{3}$. Then ϕ satisfies (ii) but is not a homomorphism since $\phi(a^*a^*) = \phi(a^*) = \bar{2}$, but $\phi(a^*)\phi(a^*) = \bar{2} \cdot \bar{2} = \bar{0}$.

As a consequence we get the following description of congruence relations of fourpart semigroups. **Proposition 18.** Let S be a four-part semigroup with a^* and a^{**} as the constant elements. Then the following equivalence relations are congruence relations on S.

- (i) $\theta = S \times S \ or$
- (ii) $\theta = \theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4$ where θ_i is an equivalence relation on S_i for all i = 1, 2, 3, 4 such that $(\varphi(a), \varphi(b)) \in \theta$ whenever $(a, b) \in \theta$ or
- (iii) $\theta = \theta_1 \cup \theta_2 \cup \theta_3$ where θ_i is an equivalence relation on $S_3 \cup S_4$, on S_1 and on S_2 , respectively such that $(a^*, a^{**}) \in \theta$ and $(\varphi(a), \varphi(b)) \in \theta$ whenever $(a, b) \in \theta$ or
- (iv) $\theta = \theta_1 \cup \theta_2$ where θ_1, θ_2 are equivalence relations on $S_1 \cup S_2$ and on $S_3 \cup S_4$, respectively such that $(a, \varphi(a)) \in \theta$ for all $a \in S$.

Now, we consider the particular case that \mathcal{S} and \mathcal{S}' both are four-part semigroups.

Lemma 19. Let S = (S; *) and S' = (S'; *') be two four-part semigroups with the constant elements a^*, a^{**} and b^*, b^{**} respectively and let $\phi : S \to S'$ be an arbitrary homomorphism. Then the following Propositions hold:

- (i) If $\phi(S_1) \not\subseteq S'_1$, then ϕ is constant and $\phi(S) \subseteq \{b^*, b^{**}\}$.
- (ii) If $\phi(S_2) \not\subseteq S'_2$, then $(a, \varphi(a)) \in Ker\phi$ for all $a \in S$ or ϕ is constant and $\phi(S) \subseteq \{b^*, b^{**}\}.$
- (iii) If $\phi(S_3) \not\subseteq S'_3$, then $\phi(a^*) = b^{**}$ or ϕ is constant and $\phi(S) \subseteq S'_1 \cup \{b^*, b^{**}\}$.
- (iv) If $\phi(S_4) \not\subseteq S'_4$, then $\phi(a^{**}) = b^*$ or ϕ is constant and $\phi(S) \subseteq S'_1 \cup \{b^*, b^{**}\}$.

Proof. Let $\phi: S \to S'$ be a homomorphism.

(i) Let $a \in S_1$ such that $\phi(a) \notin S'_1$. Then for all $b \in S$ we have b = a * b and $\phi(b) = \phi(a * b) = \phi(a) *' \phi(b)$. If $\phi(a) \in S'_2$, then we have $\phi(b) = \phi(a) *' \phi(b) = \phi'(\phi(b))$, a contradiction. If $\phi(a) \in S'_3$ or $\phi(a) \in S'_4$, then $\phi(b) = \phi(a) *' \phi(b) = b^*$ or $\phi(b) = \phi(a) *' \phi(b) = b^{**}$, i.e., ϕ is constant and $\phi(S) \subseteq \{b^*, b^{**}\}$.

(ii) Let $a_{2j} \in S_2$ such that $\phi(a_{2j}) \notin S'_2$. Then for all $a \in S$, we have $\varphi(a) = a_{2j} * a$ and $\phi(\varphi(a)) = \phi(a_{2j}) *' \phi(a)$. If $\phi(a_{2j}) \in S'_1$, then $\phi(\varphi(a)) = \phi(a_{2j}) *' \phi(a) = \phi(a)$, i.e., $(a, \varphi(a)) \in Ker\phi$. If $\phi(a_{2j}) \in S'_3$ or $\phi(a_{2j}) \in S'_4$, then $\phi(\varphi(a)) = \phi(a_{2j}) *' \phi(a) = b^*$ or $\phi(\varphi(a)) = \phi(a_{2j}) *' \phi(a) = b^{**}$, i.e., ϕ is constant and $\phi(S) \subseteq \{b^*, b^{**}\}$.

(iii) Let $a_{3j} \in S_3$ such that $\phi(a_{3j}) \notin S'_3$. Then for all $a \in S$ we have $a_{3j} * a = a^*$ and therefore $\phi(a_{3j}) *' \phi(a) = \phi(a_{3j} * a) = \phi(a^*)$. If $\phi(a_{3j}) \in S'_1$, then $\phi(a) = \phi(a^*)$, i.e., ϕ is a constant homomorphism such that $\phi(S) \subseteq S'_1 \cup \{b^*, b^{**}\}$. If $\phi(a_{3j}) \in S'_2$, then $\phi(a^*) = \phi(a_{3j}) *' \phi(a) = \varphi'(\phi(a))$ and hence $\phi(a^*) = \varphi'(\phi(a))$.

But this is not possible for $a = a^*$ and therefore we have a contradiction. If $\phi(a_{3j}) \in S'_4$, then $b^{**} = \phi(a_{3j}) *' \phi(a) = \phi(a^*)$.

(iv) If there is $a_{4j} \in S_4$ such that $\phi(a_{4j}) \notin S'_4$, then in the same way as in (iii), we have that ϕ is a constant homomorphism such that $\phi(S) \subseteq S'_1 \cup \{b^*, b^{**}\}$ or $\phi(a^{**}) = b^*$.

Lemma 20. Let S = (S; *) and S' = (S'; *') be two four-part semigroups with the constant elements a^*, a^{**} and b^*, b^{**} respectively and let $\phi : S \to S'$ be an arbitrary homomorphism. If $(a, \varphi(a)) \in Ker\phi$ for all $a \in S$, then

- (i) ϕ is constant such that $\phi(S) \subseteq S'_1 \cup \{b^*, b^{**}\}$ or
- (ii) $\phi(S_1) \subseteq S'_1$ and $\phi(a^*) = b^*$ or
- (iii) $\phi(S_1) \subseteq S'_1$ and $\phi(a^*) = b^{**}$.

Proof. Let $\phi : S \to S'$ be a homomorphism satisfying $\phi(a) = \phi(\varphi(a))$ for all $a \in S$. Then we have $\phi(S_1) = \phi(S_2)$ and $\phi(S_3) = \phi(S_4)$. Now we will consider $H = \phi(S_1) = \phi(S_2)$ and $K = \phi(S_3) = \phi(S_4)$. If $H \not\subseteq S'_1$, then by Lemma 19 (i), ϕ is constant and $\phi(S) \subseteq \{b^*, b^{**}\}$ and we obtain (i). If $H \subseteq S'_1$, then $\phi(S_2) = H \not\subseteq S'_2$ and thus by Lemma 19 (ii), ϕ is constant such that $\phi(S) \subseteq \{b^*, b^{**}\}$, i.e., (i) or $\phi(a) = \phi(\varphi(a))$ for all $a \in S$. In the second case, if $K \subseteq S'_3$, i.e., $\phi(S_4) \not\subseteq S'_4$ then by Lemma 19 (iv), ϕ is constant such that $\phi(S) \subseteq S'_1 \cup \{b^*, b^{**}\}$ which is not possible or $\phi(a^{**}) = b^*$ implying $\phi(a^*) = \phi(\varphi(a^{**})) = \phi(a^{**}) = b^*$ and hence we obtain (ii). If $K \not\subseteq S'_3$, then by Lemma 19 (iii), ϕ is constant such that $\phi(S) \subseteq S'_1 \cup \{b^*, b^{**}\}$ or $\phi(a^*) = b^{**}$. Thus we have (i) or (iii).

Proposition 21. Let S = (S; *) and S' = (S'; *') be two four-part semigroups with the constant elements a^*, a^{**} and b^*, b^{**} respectively and let $\phi : S \to S'$ be an arbitrary mapping such that $\phi(S_i) \subseteq S'_i$ for all i = 1, 2, 3, 4. Then ϕ is a homomorphism if and only if $\phi(\varphi(a)) = \varphi'(\phi(a))$ for all $a \in S$ and $\phi(a^*) = b^*$.

Proof. Let for a mapping $\phi: S \to S'$ the conditions be satisfied. Then we have $\phi(a^{**}) = \phi(\varphi(a^*)) = \varphi'(\phi(a^*)) = \varphi'(b^*) = b^{**}$ and thus

$$\phi(a * b) = \begin{cases} \phi(b) = \phi(a) *' \phi(b) & \text{if } a \in S_1 \\ \phi(\varphi(b)) = \varphi'(\phi(b)) = \phi(a) *' \phi(b) & \text{if } a \in S_2 \\ \phi(a^*) = b^* = \phi(a) *' \phi(b) & \text{if } a \in S_3 \\ \phi(a^{**}) = b^{**} = \phi(a) *' \phi(b) & \text{if } a \in S_4. \end{cases}$$

Hence ϕ is a homomorphism. Conversely, let $\phi : S \to S'$ be a homomorphism such that $\phi(S_i) \subseteq S'_i$. Then we have $\phi(a^*) = \phi(a_{3j} * a_{3j}) = \phi(a_{3j}) *' \phi(a_{3j}) = b^*$ and for every $a \in S$ we have $\phi(\varphi(a)) = \phi(a_{2j} * a) = \phi(a_{2j}) *' \phi(a) = \varphi'(\phi(a))$.

More generally, we have

Theorem 22. Let S = (S; *) and S' = (S'; *') be two four-part semigroups with the constant elements a^*, a^{**} and b^*, b^{**} , respectively and let $\phi : S \to S'$ be an arbitrary mapping. Then ϕ is a homomorphism if and only if

- (i) ϕ is constant such that $\phi(S) \subseteq S'_1 \cup \{b^*, b^{**}\}$ or
- (ii) $\phi(S_i) \subseteq S'_i$ for i = 1, 2, 3, 4 such that $\phi(\varphi(a)) = \varphi'(\phi(a))$ for all $a \in S$ and $\phi(a^*) = b^*$ or
- (iii) $\phi(S_1) \subseteq S'_1, \ \phi(S_2) \subseteq S'_2, \ \phi(S_3) \subseteq S'_4, \ \phi(S_4) \subseteq S'_3, \ \phi(\varphi(a)) = \varphi'(\phi(a)) \ for$ all $a \in S$ and $\phi(a^*) = b^{**}$ or
- (iv) $\phi(S_1) \subseteq S'_1$, $\phi(S_3) \subseteq S'_3$, $\phi(a) = \phi(\varphi(a))$ for all $a \in S$ and $\phi(a^*) = b^*$ (or $\phi(S_1) \subseteq S'_1$, $\phi(S_3) \subseteq S'_4$, $\phi(a) = \phi(\varphi(a))$ for all $a \in S$ and $\phi(a^*) = b^{**}6$).

Proof. Let S, S' be two four-part semigroups with a^*, a^{**} and b^*, b^{**} being constant elements of S and S', respectively. Let $\phi : S \to S'$ be a homomorphism. We will consider the different cases from Theorem 17:

- 1. If ϕ is constant and maps every element of S to an idempotent element of S', then by Proposition 12, $\phi(S) \subseteq S'_1 \cup \{b^*, b^{**}\}$. Thus we have (i)
- 2. Let ϕ satisfy $(\varphi(a), \varphi(b)) \in Ker\phi$ whenever $(a, b) \in Ker\phi$ and $(a, b) \in Ker\phi$ only if $a, b \in S_i$ for i = 1, 2, 3, 4 and for every $a, b \in S$. It is clear that ϕ is not constant and $(a, \varphi(a)) \notin Ker\phi$ for all $a \in S$. Then by Lemma 19 (i) and Lemma 19 (ii), $\phi(S_1) \subseteq S'_1$ and $\phi(S_2) \subseteq S'_2$. Now we consider the following cases:
 - a. If $\phi(S_1) \subseteq S'_1$, $\phi(S_2) \subseteq S'_2$ and $\phi(S_3) \not\subseteq S'_3$, then by Lemma 19 (iii), $\phi(a^*) = b^{**}$. In this case, $a_{3j} * a_{3j} = a^*$ implies $\phi(a_{3j}) *' \phi(a_{3j}) = \phi(a^*) = b^{**}$, i.e., $\phi(a_{3j}) \in S'_4$ and hence $\phi(S_3) \subseteq S'_4$. For any $a_{4j} \in S_4$ and for $a_{2j} \in S_2$ we obtain $\phi(a_{4j}) = \phi(a_{2j} * a_{3j}) = \phi(a_{2j}) *' \phi(a_{3j}) = \varphi'(\phi(a_{3j})) \in \varphi'(S'_4) = S'_3$, i.e., $\phi(S_4) \subseteq S'_3$. Moreover, for every $a \in S$ and for $a_{2j} \in S_2$ we get $\varphi(a) = a_{2j} * a$ and hence $\phi(\varphi(a)) = \phi(a_{2j}) *' \phi(a) = \varphi'(\phi(a))$. Therefore we have (iii).
 - b. If $\phi(S_1) \subseteq S'_1$, $\phi(S_2) \subseteq S'_2$, $\phi(S_3) \subseteq S'_3$ and $\phi(S_4) \not\subseteq S'_4$, then by Lemma 19 (iv) we get $\phi(a^{**}) = b^*$. In this case, we have $\phi(a^{**}) = \phi(a_{4j} * a_{4j}) = \phi(a_{4j}) *' \phi(a_{4j}) = b^*$ for every $a_{4j} \in S_4$. This is possible iff $\phi(a_{4j}) \in S'_3$ and hence $\phi(S_4) \subseteq S'_3$. Therefore for every $a_{2j} \in S_2$ and $a_{3j} \in S_3$ we obtain $\phi(a_{3j}) = \phi(a_{2j} * a_{4j}) = \phi(a_{2j}) *' \phi(a_{4j}) = \varphi'(\phi(a_{4j})) \in \varphi'(S'_3) = S'_4$, i.e., $\phi(S_3) \subseteq S'_4$, a contradiction.

- c. If $\phi(S_1) \subseteq S'_1$, $\phi(S_2) \subseteq S'_2$, $\phi(S_3) \subseteq S'_3$ and $\phi(S_4) \subseteq S'_4$, then by Proposition 21, we have (ii).
- 3. Let ϕ satisfy $(a^*, a^{**}) \in Ker\phi$, $(\varphi(a), \varphi(b)) \in Ker\phi$ whenever $(a, b) \in Ker\phi$ and $(a, b) \in Ker\phi$ only if $a, b \in S_3 \cup S_4$ or $a, b \in S_1$ or $a, b \in S_2$ for every $a, b \in S$. It is clear that ϕ is not constant and $(a_{1j}, \varphi(a_{1j})) \notin Ker\phi$ for $a_{1j} \in S_1$. Then by Lemma 19 (i) and Lemma 19 (ii), $\phi(S_1) \subseteq S'_1$ and $\phi(S_2) \subseteq S'_2$. Now we will consider all possible cases:
 - a. If $\phi(S_1) \subseteq S'_1$, $\phi(S_2) \subseteq S'_2$ and $\phi(S_3) \not\subseteq S'_3$, then by Lemma 19 (iii), $\phi(a^*) = b^{**}$ and we have $b^{**} = \phi(a^*) = \phi(a^{**})$. Then for every $a_{3j} \in S_3$ and for every $a_{4j} \in S_4$ we have $\phi(a_{3j}) *' \phi(a_{3j}) = \phi(a_{3j} * a_{3j}) = \phi(a^*) = b^{**}$ and $\phi(a_{4j}) *' \phi(a_{4j}) = \phi(a_{4j} * a_{4j}) = \phi(a^{**}) = b^{**}$ i.e., $\phi(a_{3j}), \phi(a_{4j}) \in S'_4$. Hence we obtain $\phi(a_{2j} * a_{3j}) = \phi(a_{4j}) \in S'_4$ and $\phi(a_{2j} * a_{3j}) = \phi(a_{2j}) *' \phi(a_{3j}) \in S'_3$, a contradiction.
 - b. If $\phi(S_1) \subseteq S'_1$, $\phi(S_2) \subseteq S'_2$, $\phi(S_3) \subseteq S'_3$ and $\phi(S_4) \not\subseteq S'_4$, then by Lemma 19 (iv), $\phi(a^{**}) = b^*$. Thus we have $b^* = \phi(a^*) = \phi(a^{**})$. Hence for every $a_{3j} \in S_3$ and for every $a_{4j} \in S_4$ we have $\phi(a_{3j}) *' \phi(a_{3j}) = \phi(a_{3j} * a_{3j}) = \phi(a^*) = b^*$ and $\phi(a_{4j}) *' \phi(a_{4j}) = \phi(a_{4j} * a_{4j}) = \phi(a^{**}) = b^*$ i.e., $\phi(a_{3j}), \phi(a_{4j}) \in S'_3$. Therefore we obtain $\phi(a_{2j} * a_{3j}) = \phi(a_{4j}) \in S'_3$ and $\phi(a_{2j} * a_{3j}) = \phi(a_{2j}) *' \phi(a_{3j}) \in S'_4$, a contradiction.
 - c. If $\phi(S_1) \subseteq S'_1$, $\phi(S_2) \subseteq S'_2$, $\phi(S_3) \subseteq S'_3$ and $\phi(S_4) \subseteq S'_4$, then we have a contradiction to $(a^*, a^{**}) \in Ker\phi$.
- 4. Let ϕ satisfy $(a, \varphi(b)) \in Ker\phi$ whenever $(a, b) \in Ker\phi$ and $(a, b) \in Ker\phi$ only if $a, b \in S_1 \cup S_2$ or $a, b \in S_3 \cup S_4$ for every $a, b \in S$. It is obvious that $(a, \varphi(a)) \in Ker\phi$ for all $a \in S$. Thus by Lemma 20, we have two possible cases $\phi(S_1) \subseteq S'_1$ and $\phi(a^*) = b^*$ or $\phi(S_1) \subseteq S'_1$ and $\phi(a^*) = b^{**}$. For every $a_{3j} \in S_3$, in the first case we obtain $\phi(a_{3j})*'\phi(a_{3j}) = \phi(a_{3j}*a_{3j}) = \phi(a^*) = b^*$, i.e., $\phi(a_{3j}) \in S'_3$ and hence $\phi(S_3) \subseteq S'_3$ and in the second case we have $\phi(a_{3j}) *'\phi(a_{3j}) = \phi(a_{3j}*a_{3j}) = \phi(a^*) = b^{**}$, i.e., $\phi(a_{3j}) \in S'_4$ and hence $\phi(S_3) \subseteq S'_4$. Therefore we have (iv).

Conversely, let $\phi : S \to S'$ be a mapping. If ϕ satisfies (i) and $\phi(a) = c$ for all $a \in S$ with $c \in S'_1 \cup \{b^*, b^{**}\}$, then we get $\phi(a*b) = c = c*c = \phi(a)*\phi(b)$ and hence ϕ is a homomorphism. If ϕ satisfies (ii), then ϕ is a homomorphism by Proposition 21. If ϕ satisfies (iii), then $\phi(a^{**}) = \phi(\varphi(a^*)) = \varphi'(\phi(a^*)) = \varphi'(b^{**}) = b^*$ and we obtain

$$\phi(a * b) = \begin{cases} \phi(b) = \phi(a) *' \phi(b) & \text{if } a \in S_1 \\ \phi(\varphi(b)) = \varphi'(\phi(b)) = \phi(a) *' \phi(b) & \text{if } a \in S_2 \\ \phi(a^*) = b^{**} = \phi(a) *' \phi(b) & \text{if } a \in S_3 \\ \phi(a^{**}) = b^* = \phi(a) *' \phi(b) & \text{if } a \in S_4, \end{cases}$$

i.e., ϕ is a homomorphism. If ϕ satisfies (iv), i.e., $\phi(a) = \phi(\varphi(a))$ for all $a \in S$, $\phi(S_1) \subseteq S'_1, \phi(S_3) \subseteq S'_3$ and $\phi(a^*) = b^*$, then we have $\phi(S_2) = \phi(\varphi(S_1)) = \phi(S_1) \subseteq S'_1, \phi(S_4) = \phi(\varphi(S_3)) = \phi(S_3) \subseteq S'_3$ and $\phi(a^{**}) = \phi(\varphi(a^*)) = \phi(a^*) = b^*$. Therefore we have

$$\phi(a * b) = \begin{cases} \phi(b) = \phi(a) *' \phi(b) & \text{if } a \in S_1 \\ \phi(\varphi(b)) = \phi(b) = \phi(a) *' \phi(b) & \text{if } a \in S_2 \\ \phi(a^*) = b^* = \phi(a) *' \phi(b) & \text{if } a \in S_3 \\ \phi(a^{**}) = b^* = \phi(a) *' \phi(b) & \text{if } a \in S_4. \end{cases}$$

Hence ϕ is a homomorphism. Similarly, ϕ is a homomorphism if $\phi(a) = \phi(\varphi(a))$ for all $a \in S$, $\phi(S_1) \subseteq S'_1$, $\phi(S_3) \subseteq S'_4$ and $\phi(a^*) = b^{**}$. This completes the proof.

Proposition 23. Let S = (S; *) and S' = (S'; *') be two four-part semigroups with the constant elements a^*, a^{**} and b^*, b^{**} respectively and let $\phi : S \to S'$ be a homomorphism. Then the following Propositions are true.

- (i) If φ is a homomorphism of the first type of Theorem 22, then Imφ forms a constant subsemigroup of S'.
- (ii) If φ is a homomorphism of the second type or of the third type of Theorem 22, then Imφ forms a four-part subsemigroup of S'.
- (iii) If ϕ is a homomorphism of the fourth type of Theorem 22, then $Im\phi$ forms a right-zero constant subsemigroup of S'.

Proof. (i) is obvious.

(ii) Let $\phi: S \to S'$ be a homomorphism of the third type of Theorem 22, i.e., $\phi(S_i) \subseteq S'_i$ for $i = 1, 2, \phi(S_3) \subseteq S'_4, \phi(S_4) \subseteq S'_3, \phi(\varphi(a)) = \varphi'(\phi(a))$ for all $a \in S$, $\phi(a^*) = b^{**}$ and $\phi(a^{**}) = b^*$. Then it is clear that $Im\phi \cap S'_2 \neq \emptyset$ and $b^*, b^{**} \in Im\phi$. Moreover, if $b \in Im\phi$, then there is $a \in S$ such that $b = \phi(a)$. Thus, by assumption, we obtain $b = \phi(a) = \phi(a_{2j} * \varphi(a)) = \phi(a_{2j}) *' \phi(\varphi(a)) = \varphi'(\phi(\varphi(a)))$ for $a_{2j} \in S_2$ and hence $\varphi'(b) = \phi(\varphi(a)) \in Im\phi$. Therefore $Im\phi$ satisfies the two conditions in Proposition 5 and hence forms a four-part subsemigroup of \mathcal{S}' . By the same argumentation, if ϕ is a homomorphism of the second type of Theorem 22, then $Im\phi$ forms a four-part subsemigroup of \mathcal{S}' .

(iii) Let $\phi : S \to S'$ be a homomorphism of the fourth type of Theorem 22, i.e., $\phi(a) = \phi(\varphi(a))$ for all $a \in S$ such that $\phi(S_1) \subseteq S'_1$, $\phi(S_3) \subseteq S'_3$ and $\phi(a^*) = b^*$ (or $\phi(a) = \phi(\varphi(a))$ for all $a \in S$ such that $\phi(S_1) \subseteq S'_1$, $\phi(S_3) \subseteq S'_3$ and $\phi(a^*) = b^*$). Then $Im\phi \subseteq S'_1 \cup S'_3$, $Im\phi \cap S'_1 \neq \emptyset$ and $b^* \in Im\phi$ (or $Im\phi \subseteq S'_1 \cup S'_4$, $Im\phi \cap S'_1 \neq \emptyset$ and $b^{**} \in Im\phi$). Thus by Proposition 8, $Im\phi$ forms a right-zero constant subsemigroup of S'.

5. Green's relations on four-part semigroups

Let a and b be two elements in the semigroup S = (S; *). Recall that Green's relations are defined in the following way: $a\mathcal{L}b$ iff a = b or there exist $c, d \in S$ such that c * a = b and d * b = a, $a\mathcal{R}b$ iff a = b or there exist $c, d \in S$ such that a * c = b and b * d = a, $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$. It is well-known that for a finite semigroup \mathcal{D} and \mathcal{J} are the same.

Proposition 24. Let S = (S; *) be a four-part semigroup with a^* and a^{**} as constant elements. Then $\mathcal{L}_a = \{a, \varphi(a)\}$ for all $a \in S$.

Proof. Let $a, b \in S$ such that $a \neq b$ satisfy $a\mathcal{L}b$. Thus there are $c, d \in S$ such that c * a = b and d * b = a. Assume that $b \neq \varphi(a)$. If $a = a_{1j} \in S_1$, then $a_{1j} \neq b \neq \varphi(a_{1j}) = a_{2j} \in S_2$. Thus we have

$$c * a = c * a_{1j} = \begin{cases} a_{1j} \neq b & \text{if } c \in S_1 \\ \varphi(a_{1j}) = a_{2j} \neq b & \text{if } c \in S_2 \\ a^* & \text{if } c \in S_3 \\ a^{**} & \text{if } c \in S_4. \end{cases}$$

Therefore c * a = b is only possible for $b = a^*$ or $b = a^{**}$. But if $b = a^*$, then we have

$$d * b = d * a^* = \begin{cases} a^* \neq a_{1j} = a & \text{if } d \in S_1 \\ \varphi(a^*) = a^{**} \neq a_{1j} = a & \text{if } d \in S_2 \\ a^* \neq a_{1j} = a & \text{if } d \in S_3 \\ a^{**} \neq a_{1j} = a & \text{if } d \in S_4, \end{cases}$$

a contradiction. Similarly, we have a contradiction when $b = a^{**}$. If $a = a_{2j} \in S_2$, then in the same way we also obtain a contradiction.

Now, if $a = a_{3j} \in S_3$, then $a_{3j} \neq b \neq \varphi(a_{3j}) = a_{4j}$. Thus we have

$$c * a = c * a_{3j} = \begin{cases} a_{3j} \neq b & \text{if } c \in S_1 \\ \varphi(a_{3j}) = a_{4j} \neq b & \text{if } c \in S_2 \\ a^* & \text{if } c \in S_3 \\ a^{**} & \text{if } c \in S_4 \end{cases}$$

Thus c * a = b is only possible for $b = a^*$ or $b = a^{**}$. But if $b = a^*$, then we have

$$d * b = d * a^* = \begin{cases} a^* & \text{if } d \in S_1 \\ \varphi(a^*) = a^{**} \neq a_{3j} = a & \text{if } d \in S_2 \\ a^* & \text{if } d \in S_3 \\ a^{**} \neq a_{3j} = a & \text{if } d \in S_4, \end{cases}$$

and therefore d * b = a is possible only when $a = a_{3j} = a^*$ and we have $a = a^* = b$, a contradiction. If $b = a^{**}$, then we obtain

$$d * b = d * a^{**} = \begin{cases} a^{**} \neq a_{3j} = a & \text{if } d \in S_1 \\ \varphi(a^{**}) = a^* & \text{if } d \in S_2 \\ a^* & \text{if } d \in S_3 \\ a^{**} \neq a_{3j} = a & \text{if } d \in S_4, \end{cases}$$

and therefore d * b = a is possible only when $a = a_{3j} = a^*$ and thus $b = a^{**} = \varphi(a^*) = \varphi(a)$, a contradiction. Similarly, we also have a contradiction for the case $a = a_{4j} \in S_4$. Therefore $b = \varphi(a)$ and hence $\mathcal{L} = \{a, \varphi(a)\}$.

Proposition 25. Let S = (S; *) be a four-part semigroup with a^* and a^{**} as constant elements and let $a \in S$. Then $\mathcal{R}_a = \{a\}$ or $\mathcal{R}_a = S_1 \cup S_2$.

Proof. First we show that $a\mathcal{R}b$ for every $a, b \in S_1 \cup S_2$. Let $a \neq b$. If $a, b \in S_1$, then clearly $a\mathcal{R}b$ with c = d, b = a. If $a, b \in S_2$, then with $c = \varphi(b)$ and $d = \varphi(a)$ we have $a * c = a * \varphi(b) = \varphi(\varphi(b)) = b$ and $b * d = b * \varphi(a) = \varphi(\varphi(a)) = a$ and hence $a\mathcal{R}b$. If $a \in S_1$ and $b \in S_2$, then a * c = b and b * d = a for c = b and $d = \varphi(a)$ and thus $a\mathcal{R}b$. Now, we show that $\mathcal{R}_a = \{a\}$ if $a \in S_3 \cup S_4$. Let $a \in S_3$ and assume that $\mathcal{R}_a \neq \{a\}$, i.e., there is $b \in \mathcal{R}_a$ such that $b \neq a$. Hence for every $c, d \in S$ satisfying a * c = b and b * d = a, we obtain $a^* = a * c = b$ and therefore $a = b * d = a^* * d = a^* = b$, a contradiction. Thus there is no $a \neq b \in S$ such that $b \in \mathcal{R}_a$. Similarly, there is no $b \neq a$ such that $a\mathcal{R}b$ for $a \in S_4$. Hence $\mathcal{R}_a = \{a\}$ for $a \in S_3 \cup S_4$. This completes the proof.

Proposition 26. Let S = (S : *) be a four-part semigroup and let $a \in S$. Then $\mathcal{H}_a = \{a, \varphi(a)\}$ if $a \in S_1 \cup S_2$ and $\mathcal{H}_a = \{a\}$ if $a \in S_3 \cup S_4$.

Proof. Since $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, by Proposition 24 and Proposition 25, we have that $\mathcal{H}_a = \{a, \varphi(a)\}$ if $a \in S_1 \cup S_2$ and $\mathcal{H}_a = \{a\}$ for every $a \in S_3 \cup S_4$.

Proposition 27. Let S = (S; *) be a four-part semigroup. Then $D_a = J_a = \{a, \gamma a\}$ or $D_a = J_a = S_1 \cup S_2$.

Proof. We show that $a\mathcal{D}b$ ($a\mathcal{J}b$) for all $a, b \in S_1 \cup S_2$. If $a, b \in S_1 \cup S_2$, then by taking c = a we have $a\mathcal{L}c$ and $c\mathcal{R}b$ by Proposition 24 and Proposition 25, i.e., $a\mathcal{D}b$. If $a \notin S_1 \cup S_2$, by taking $c = \gamma a$ we have $a\mathcal{L}c$ and $c\mathcal{R}c$ by Proposition 24 and Proposition 25, i.e., $a\mathcal{D}\gamma a$. Now, let $a, b \notin S_1 \cup S_2$ and $a\mathcal{D}b$. Then there exists $c \in S$ such that $a\mathcal{L}c$ and $c\mathcal{R}b$. By Proposition 24, we have c = a or $c = \gamma a$ and by Proposition 25, we have c = b since $b \notin S_1 \cup S_2$. Therefore we have two possibilities a = c = b or $b = c = \gamma a$. Thus $\mathcal{D}_a = \{a, \gamma a\}$. By the finiteness of S, we have $\mathcal{D} = \mathcal{J}$. This completes the proof.

6. Representation of four-part semigroups

Theorem 28. Let S = (S; *) be an arbitrary four-part semigroup with the constant elements a^* and a^{**} . Then there is a natural number $n \ge 1$ such that S is isomorphic to a four-part subsemigroup of $(O^n(\{0,1\});+)$.

Proof. Let S be a four-part semigroup with $|S_1| = |S_2| = n_r$ and $|S_3| = |S_4| = n_s$. We choose a natural number n such that $max(n_r, n_s) \leq 2^{2^n-2}$ and consider $O^n(\{0,1\})$. Now, define a one-to-one mappings $\phi_1 : S_1 \to C_4^n \subseteq O^n(\{0,1\})$ and $\phi_3 : S_3 \to K_0^n \subseteq O^n(\{0,1\})$ such that $\phi_3(a^*) = c_0^n$ and define mappings $\phi_2 : S_2 \to \neg C_4^n \subseteq O^n(\{0,1\})$ and $\phi_4 : S_4 \to K_1^n \subseteq O^n(\{0,1\})$ by $\phi_2(a_{2j}) = \neg \phi_1(a_{1j})$ and $\phi_4(a_{4j}) = \neg \phi_3(a_{3j})$. It is clear that $\phi : S \to O^n(\{0,1\})$ defined by $\phi(a_{ij}) = \phi_i(a_{ij})$ is a one-to-one mapping satisfying $\phi(\varphi(a)) = \neg \phi(a)$ for all $a \in S$. Therefore, $\neg \phi(a) \in \phi(S)$ for every $a \in S$. Moreover, $S_1' := \phi(S_1) \subseteq C_4^n$, $S_2' := \phi(S_2) = \neg \phi(S_1) \subseteq \neg C_4^n$, $S_3' := \phi(S_3) \subseteq K_0^n$ and $S_4' := \phi(S_4) = \neg \phi(S_3) \subseteq K_1^n$ and for $a, b \in \phi(S) = S_1' \cup S_2' \cup S_3' \cup S_4'$ we have

$$a+b = \begin{cases} b \in \phi(S) & \text{if } a \in S'_1 \\ \neg b \in \phi(S) & \text{if } a \in S'_2 \\ c_0^n \in \phi(S) & \text{if } a \in S'_3 \\ c_1^n \in \phi(S) & \text{if } a \in S'_4, \end{cases}$$

i.e., $\phi(S) = S'_1 \cup S'_2 \cup S'_3 \cup S'_4$ forms a four-part subsemigroup of $(O^n(\{0,1\});+)$. Furthermore, considering the unary operation \neg as φ' in $O^n(\{0,1\})$, then by Theorem 22 (ii), $\phi: S \to O^n(\{0,1\})$ is a homomorphism. Therefore $S \cong \phi(S) \subseteq \mathcal{O}^n(\{0,1\})$.

References

- R. Butkote and K. Denecke, Semigroup Properties of Boolean Operations, Asian-Eur. J. Math. 1 (2008) 157–176.
- R. Butkote, Universal-algebraic and Semigroup-theoretical Properties of Boolean Operations (Dissertation Universität Potsdam, 2009).

Received 24 November 2012 Revised 28 November 2012