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bi-BL-ALGEBRA

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Abstract

In this paper, we introduce the notion of a *bi*-BL-algebra, *bi*-filter, *bi*-deductive system and *bi*-Boolean elements of a *bi*-BL-algebra and deal with *bi*-filters in *bi*-BL-algebra. We study this structure and construct the quotient of *bi*-BL-algebra. Also present a classification for examples of proper *bi*-BL-algebras.

Keywords: *bi*-BL-algebra, *bi*-filter, *bi*-deductive system, *bi*-Boolean elements of a *bi*-BL-algebra.

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1. Introduction

*bi*structure is a tool as this answers a major problem faced by all algebraic structures - groups, semigroups, loops, groupoids etc. that is the union of two subgroups, or two subrings, or two subsemigroups etc. do not form any algebraic structure but all of them find a nice *bi*algebraic structure as *bi*groups, *bi*rings, *bi*semigroups etc. Except for this *bi*algebraic structure these would remain only as sets without any nice algebraic structure on them. Further when these *bi*algebraic structures are defined on them they enjoy not only the inherited qualities of the algebraic structure from which they are taken but also several distinct algebraic properties that are not present in algebraic structures.

The study of *bi*algebraic structures started recently. The study of *bi*groups was carried out in 1994–1996. Further research on *bi*groups and fuzzy *bi*groups was published in 1998. In the year 1999, *bi*vector spaces were introduced. In 2001, concept of free De Morgan *bi*semigroups and *bi*semilattices was studied. It is said by Zoltan Esik that these *bi*algebraic structures like *bi*groups, *bi*semigroups, *bi*near rings help in the construction of finite machines or finite automaton and semi automaton. The notion of non-associative *bi*algebraic structures was first introduced in the year 2003, [19].

BL-algebra have been invented by P. Hajek [9] in order to provide an algebraic proof of the completeness theorem of "Basic Logic" (BL, for short) arising from the continuous triangular norms, familiar in the fuzzy Logic framework. The language of propositional Hajek basic logic [9] contains the binary connectives $\odot$ and $\rightarrow$ and the constant $\overline{0}$.

Axioms of BL are:

(A_1) $(\phi \rightarrow \chi) \rightarrow ((\chi \rightarrow \psi) \rightarrow (\phi \rightarrow \psi))$
(A_2) $(\phi \odot \chi) \rightarrow \phi$
(A_3) $(\phi \odot \chi) \rightarrow (\chi \odot \phi)$
(A_4) $(\phi \odot (\phi \rightarrow \chi)) \rightarrow (\chi \odot (\chi \rightarrow \phi))$
(A_{5a}) $(\phi \rightarrow (\chi \rightarrow \psi)) \rightarrow ((\phi \odot \chi) \rightarrow \psi))$
(A_{5b}) $((\phi \odot \chi) \rightarrow \psi) \rightarrow (\phi \rightarrow (\chi \rightarrow \psi))$
(A_6) $((\phi \rightarrow \chi) \rightarrow \psi) \rightarrow (((\chi \rightarrow \phi) \rightarrow \psi) \rightarrow \psi)$
(A_7) $\overline{0} \rightarrow \omega$.

In this paper, we generalize the notion of BL-algebra and introduce notion of *bi*-BL-algebra and study it. The notions of *bi*-filter, *bi*-deductive system and *bi*-Boolean elements of a *bi*-BL-algebra are introduced and studied this structure in detail. We construct the quotient of *bi*-BL-algebra, also present classes of examples of proper *bi*-BL-algebras.

2. PRELIMINARIES

2.1. Definitions and Theorems

Definition 2.1 [9]**.** A BL-algebra is an algebra $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ with four binary operations $\wedge, \vee, \odot, \rightarrow$ and two constants $0, 1$ such that:

$(BL1)$ $(A, \wedge, \vee, \rightarrow, 0, 1)$ is a bounded lattice,

$(BL2)$ $(A, \odot, 1)$ is a commutative monoid,

$(BL3)$ $\odot$ and $\rightarrow$ form an adjoint pair i.e, $a \odot b \leq c$ if and only if $a \leq b \rightarrow c$,

$(BL4)$ $a \wedge b = a \odot (a \to b)$,

$(BL5)$ $(a \to b) \vee (b \to a) = 1$,

for all $a, b, c \in A$.

A BL-algebra is called an MV-algebra if $x^{--} = x$, for all $x \in A$, where $x^{-} = x \to 0$.

Definition 2.2 [9]**.** A filter of a BL-algebra A is a nonempty subset F of A, such that for all $x, y \in A$, we have

(1) $x, y \in F$ implies $x \odot y \in F$,

(2) $x \in F$ and $x \leq y$ imply $y \in F$.

Definition 2.3 [17]**.** A non-empty subset D of BL-algebra A is called a deductive system if

(1) $1 \in D$,

(2) If $x \in D$ and $x \to y \in D$ imply $y \in D$.

Proposition 2.4 [17]**.** *A non-empty subset F of BL-algebra is a deductive system if and only if F is a filter.*

Theorem 2.5 [9]**.** *Let F be a filter of a BL-algebra A. Define: $x \equiv_F y$ if and only if $x \to y \in F$ and $y \to x \in F$. Then $\equiv_F$ is a congruence relation on A. The set of all congruence classes is denoted by $\frac{A}{F}$, i.e., $\frac{A}{F} := \{[x] | x \in A\}$, where $[x] = \{y \in A | x \equiv_F y\}$. Define $\bullet, \rightharpoonup, \sqcap, \sqcup$ on $\frac{A}{F}$ as follows:*
$[x] \bullet [y] = [x \odot y]$, $[x] \rightharpoonup [y] = [x \to y]$, $[x] \sqcap [y] = [x \wedge y]$, $[x] \sqcup [y] = [x \vee y]$.
Therefore $(\frac{A}{F}, \sqcap, \sqcup, \bullet, \rightharpoonup, [1], [0])$ is a BL-algebra with respect to F.

Definition 2.6 [9]**.** Let L be a BL-algebra. An element $a \in L$ is called complemented if there is an $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$; If such element b exists it is called a complement of a. We will denote the set of all complement in L by $B(L)$.

For any BL-algebra A, $B(A)$ denotes the Boolean algebra of all complement elements in $L(A)$ (hence $B(A) = B(L(A))$).

Definition 2.7 [7, 9, 18]**.** Let A and B are BL-algebras. A function $f : A \to B$ is called homomorphism of BL-algebras if and only if:

(1) $f(0) = 0$,

(2) $f(x * y) = f(x) * f(y)$,

(3) $f(x \to y) = f(x) \to f(y)$,

for all $x, y \in A$.

3. *bi*-BL-algebra

3.1. Definition and some examples

Definition 3.1. A *bi*-BL-algebra is an algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ with four binary operations and two constants if $L = L_1 \cup L_2$ where L_1 and L_2 are proper subsets of L and

(i) $(L_1, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a non-trivial BL-algebra,

(ii) $(L_2, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a non-trivial BL-algebra.

Definition 3.2. If L is a *bi*-BL-algebra and also a BL-algebra, then we say that L is a super BL-algebra.

Definition 3.3. A *bi*-BL-algebra $L = L_1 \cup L_2$ is said to be finite if it has a finite number of elements and if L has infinite number of elements, then L is said to be infinite *bi*-BL-algebra.

Example 3.4. Let $L_1 = \{0, a, c, 1\}$ and $L_2 = \{0, b, c, 1\}$. Define $\odot$ and $\rightarrow$ as follow:

L_1

$\odot$	0	a	c	1
0	0	0	0	0
a	0	a	a	a
c	0	a	c	c
1	0	a	c	1

$\rightarrow$	0	a	c	1
0	1	1	1	1
a	0	1	1	1
c	0	a	1	1
1	0	a	c	1

L_2

$\odot$	0	b	c	1
0	0	0	0	0
b	0	b	b	b
c	0	b	c	c
1	0	b	c	1

$\rightarrow$	0	b	c	1
0	1	1	1	1
b	0	1	1	1
c	0	b	1	1
1	0	b	c	1

For L, whose tables are the following:

L

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	b	b
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
1	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then L_1 and L_2 are BL-algebras and $L = L_1 \cup L_2$ is a *bi*-BL-algebra but L is not a BL-algebra since $(a \to b) \vee (b \to a) = b \vee a = c \neq 1$. In this example $L_1 \cap L_2 \neq \{0, 1\}$.

Example 3.5. Let $L_1 = \{0, a, b, c, d, 1\}$ and $L_2 = \{0, d, e, 1\}$. Define $\odot$ and $\to$ as follow:

L_1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	c	c	d	a
b	0	c	b	c	d	b
c	0	c	c	c	d	c
d	0	d	d	d	0	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	b	d	1
b	0	a	1	a	d	1
c	0	1	1	1	d	1
d	d	1	1	1	1	1
1	0	a	b	c	d	1

L_2

$\odot$	0	d	e	1
0	0	0	0	0
d	0	0	d	d
e	0	d	e	e
1	0	d	e	1

$\to$	0	d	e	1
0	1	1	1	1
d	d	1	1	1
e	0	d	1	1
1	0	d	e	1

For L, whose tables are the following:

L

$\odot$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	c	c	d	e	a
b	0	c	b	c	d	b	b
c	0	c	c	c	d	e	c
d	0	d	d	d	0	d	d
e	0	e	b	e	d	e	e
1	0	a	b	c	d	e	1

$\to$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	0	1	b	b	d	e	1
b	0	a	1	a	d	d	1
c	0	1	1	1	d	e	1
d	d	1	1	1	1	1	1
e	0	d	b	d	d	1	1
1	0	a	b	c	d	e	1

Then L_1 and L_2 are BL-algebras and $L = L_1 \cup L_2$ is a *bi*-BL-algebra but L is not a BL-algebra since $(a \to e) \vee (e \to a) = e \vee d = e \neq 1$. In this case, $L_1 \cap L_2 \neq \{0, 1\}$.

Example 3.6. Let $L_1 = \{0, a, c, 1\}$ and $L_2 = \{0, b, c, d, 1\}$. Define $\odot$ and $\to$ as follow:

L_1

$\odot$	0	a	c	1
0	0	0	0	0
a	0	a	a	a
c	0	a	a	c
1	0	a	c	1

$\to$	0	a	c	1
0	1	1	1	1
a	0	1	1	1
c	0	c	1	1
1	0	a	c	1

L_2

$\odot$	0	b	d	1
0	0	0	0	0
b	0	0	b	b
d	0	b	d	d
1	0	b	d	1

$\to$	0	b	d	1
0	1	1	1	1
b	b	1	1	1
d	0	b	1	1
1	0	b	d	1

For L, whose tables are the following:

$$L \qquad \begin{array}{c|cccccc} \odot & 0 & a & b & c & d & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & a & 0 & a & 0 & a \\ b & 0 & 0 & 0 & 0 & b & b \\ c & 0 & a & 0 & a & b & c \\ d & 0 & 0 & b & b & d & d \\ 1 & 0 & a & b & c & d & 1 \end{array} \qquad \begin{array}{c|cccccc} \to & 0 & a & b & c & d & 1 \\ \hline 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ a & d & 1 & d & 1 & d & 1 \\ b & c & c & 1 & 1 & 1 & 1 \\ c & b & c & d & 1 & d & 1 \\ d & a & a & c & c & 1 & 1 \\ 1 & 0 & a & b & c & d & 1 \end{array}$$

Then L_1 and L_2 are BL-algebras. $L = L_1 \cup L_2$ is a *bi*-BL-algebra also L is a super BL-algebra. In this case, $L_1 \cap L_2 \neq \{0, 1\}$.

Remark 3.7. Special case of *bi*-BL-algebra:
A non-empty set $(\mathrm{L}, \wedge, \vee, \odot, \to, 0, 1)$ is called a *bi*-BL-algebra if $L = L_1 \cup L_2$ where L_1 and L_2 are proper subsets of L (denote the least element by 0 and the greatest element by 1) and

(i) $(L_1, \wedge, \vee, \odot, \to, 0_1, 1_1)$ is non-trivial a BL-algebra,

(ii) $(L_2, \wedge, \vee, \odot, \to, 0_2, 1_2)$ is a non-trivial BL-algebra.

Now, we present classes of examples of proper *bi*-BL-algebras which is similar to BL-algebras [11]:

3.2. Classes of examples of *bi*-*BL*-algebras

We start details with the linearly ordered set(chain).

$$L_{n+1} = \{0, 1, 2, \dots, n\},$$

$(n \geq 1)$, organized as a lattice with $\wedge = min$ and $\vee = max$, and organized term equivalent:

$$\mathcal{L}_{n+1} = (L_{n+1}, \odot, ^-, n),$$

with:

$$x \odot y = max(0, x + y - n), x^- = x \to 0, \quad (0 = n^-),$$

hence $x \to y = max\{z | x \odot z \leq y\} = (x \odot y^-)^- = min(n, y - x + n)$. Hence, for $n = 1, \dots, 6$, we have the linearly ordered MV-algebras $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$, $\mathcal{L}_5$,$\mathcal{L}_6$, [11].

3.2.1. Classes of examples of finite, linearly ordered *bi-BL*-algebras

The examples are one of the following forms:

1. Linearly ordered MV $\cup$ linearly ordered MV,
2. Linearly ordered MV $\cup$ linearly ordered BL or linearly ordered BL $\cup$ linearly ordered MV,
3. Linearly ordered BL $\cup$ linearly ordered BL.

• (1) Examples of the form: Linearly ordered MV $\cup$ linearly ordered MV.

Denote $\mathcal{H}_{m+1,n+1} = \mathcal{L}_{m+1} \cup \mathcal{L}_{n+1}$, for $m, n \geq 1$.

1. Example of the form: $\mathcal{H}_{2,n+1} = \mathcal{L}_2 \cup \mathcal{L}_{n+1}$ for $n \geq 1$.

Denote $H_{2,n+1} = L_2 \cup L_{n+1} = \{-1, 0\} \cup \{0, 1, 2, \ldots, n\} = \{-1, 0, 1, 2, \ldots, n\}$. For $n = 1, 2, 3, 4, 5$, since elements are from integer numbers then we have the linearly ordered *bi-BL*-algebras $\mathcal{H}_{2,2} = \mathcal{L}_2 \cup \mathcal{L}_2$, $\mathcal{H}_{2,3} = \mathcal{L}_2 \cup \mathcal{L}_3$, $\mathcal{H}_{2,4} = \mathcal{L}_2 \cup \mathcal{L}_4$, $\mathcal{H}_{2,5} = \mathcal{L}_2 \cup \mathcal{L}_5$, $\mathcal{H}_{2,6} = \mathcal{L}_2 \cup \mathcal{L}_6$, whose tables are the following:

$\mathcal{H}_{2,2}$

$\odot$	-1	0	1
-1	-1	-1	-1
0	-1	0	0
1	-1	0	1

$\rightarrow$	-1	0	1
-1	1	1	1
0	-1	1	1
1	-1	0	1

$\mathcal{H}_{2,3}$

$\odot$	-1	0	1	2
-1	-1	-1	-1	-1
0	-1	0	0	0
1	-1	0	0	1
2	-1	0	1	2

$\rightarrow$	-1	0	1	2
-1	2	2	2	2
0	-1	2	2	2
1	-1	1	2	2
2	-1	0	1	2

$\mathcal{H}_{2,4}$

$\odot$	-1	0	1	2	3
-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0
1	-1	0	0	0	1
2	-1	0	0	1	2
3	-1	0	1	2	3

$\rightarrow$	-1	0	1	2	3
-1	3	3	3	3	3
0	-1	3	3	3	3
1	-1	2	3	3	3
2	-1	1	2	3	3
3	-1	0	1	2	3

$\mathcal{H}_{2,5}$

$\odot$	-1	0	1	2	3	4
-1	-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0	0
1	-1	0	0	0	0	1
2	-1	0	0	0	1	2
3	-1	0	0	1	2	3
4	-1	0	1	2	3	4

$\to$	-1	0	1	2	3	4
-1	4	4	4	4	4	4
0	-1	4	4	4	4	4
1	-1	3	4	4	4	4
2	-1	2	3	4	4	4
3	-1	1	2	3	4	4
4	-1	0	1	2	3	4

$\mathcal{H}_{2,6}$

$\odot$	-1	0	1	2	3	4	5
-1	-1	-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0	0	0
1	-1	0	0	0	0	0	1
2	-1	0	0	0	0	1	2
3	-1	0	0	0	1	2	3
4	-1	0	0	1	2	3	4
5	-1	0	1	2	3	4	5

$\to$	-1	0	1	2	3	4	5
-1	5	5	5	5	5	5	5
0	-1	5	5	5	5	5	5
1	-1	4	5	5	5	5	5
2	-1	3	4	5	5	5	5
3	-1	2	3	4	5	5	5
4	-1	1	2	3	4	5	5
5	-1	0	1	2	3	4	5

2. Example of the form: $\mathcal{H}_{3,n+1} = \mathcal{L}_3 \cup \mathcal{L}_{n+1}$ for $n \geq 1$.

Denote $H_{3,n+1} = L_3 \cup L_{n+1} = \{-2,-1,0\} \cup \{0,1,\ldots,n\} = \{-2,-1,0,1,2,\ldots,n\}$. For $n = 1,2$, since elements are from integer numbers then we have the linearly ordered *bi*-BL-algebras $\mathcal{H}_{3,2} = \mathcal{L}_3 \cup \mathcal{L}_2$, $\mathcal{H}_{3,3} = \mathcal{L}_3 \cup \mathcal{L}_3$, whose tables are:

$\mathcal{H}_{3,2}$

$\odot$	-2	-1	0	1
-2	-2	-2	-2	-2
-1	-2	-2	-1	-1
0	-2	-1	0	0
1	-2	-1	0	1

$\to$	-2	-1	0	1
-2	1	1	1	1
-1	-1	1	1	1
0	-2	-1	1	1
1	-2	-1	0	1

$\mathcal{H}_{3,3}$

$\odot$	-2	-1	0	1	2
-2	-2	-2	-2	-2	-2
-1	-2	-2	-1	-1	-1
0	-2	-1	0	0	0
1	-2	-1	0	0	1
2	-2	-1	0	1	2

$\to$	-2	-1	0	1	2
-2	2	2	2	2	2
-1	-1	2	2	2	2
0	-2	-1	2	2	2
1	-2	-1	1	2	2
2	-2	-1	0	1	2

Remark 3.8. The examples of the forms $\mathcal{H}_{m+1,n+1}$, for $m,n \geq 1$ are BL-algebras thus are super BL-algebras.

• (2) Examples of the form: Linearly ordered MV ∪ linearly ordered BL or linearly ordered BL ∪ linearly ordered MV.

Denote $\mathcal{H}_{m+1,n+1,p+1} = \mathcal{L}_{m+1} \cup \mathcal{H}_{n+1,p+1} = \mathcal{L}_{m+1} \cup (\mathcal{L}_{n+1} \cup \mathcal{L}_{p+1}) = (\mathcal{L}_{m+1} \cup \mathcal{L}_{n+1}) \cup \mathcal{L}_{p+1} = \mathcal{H}_{m+1,n+1} \cup \mathcal{L}_{p+1}$, by associativity of $\cup$.

Example. The set $H_{2,2,2} = L_2 \cup H_{2,2} = \{-1, 0\} \cup \{0, 1, 2\} = H_{2,2} \cup L_2 = \{-1, 0, 1\} \cup \{1, 2\} = \{-1, 0, 1, 2\}$, organized as a lattice in a obvious way and as *bi*-BL-algebra $\mathcal{H}_{2,2,2} = \mathcal{H}_{2,2} \cup \mathcal{L}_2$ with the following tables:

$\mathcal{H}_{2,2,2}$

$\odot$	-1	0	1	2
-1	-1	-1	-1	-1
0	-1	0	0	0
1	-1	0	1	1
2	-1	0	1	2

$\to$	-1	0	1	2
-1	2	2	2	2
0	-1	2	2	2
1	-1	0	2	2
2	-1	0	1	2

Remark 3.9. The examples of the forms $\mathcal{H}_{m+1,n+1,p+1}$, for $m, n, p \geq 1$ are BL-algebras thus become a super BL-algebras.

• (3) Examples of the form: Linearly ordered BL ∪ linearly ordered BL or equivalent forms.

Denote $\mathcal{H}_{m+1,n+1,p+1,q+1} = \mathcal{H}_{m+1,n+1} \cup \mathcal{H}_{p+1,q+1} = (\mathcal{L}_{m+1} \cup \mathcal{L}_{n+1}) \cup (\mathcal{L}_{p+1} \cup \mathcal{L}_{q+1}) = \mathcal{H}_{m+1,n+1,p+1} \cup \mathcal{L}_{q+1} = \mathcal{L}_{m+1} \cup \mathcal{H}_{n+1,p+1,q+1}$, by associativity of $\cup$.

Example. The set $H_{2,2,2,2} = H_{2,2} \cup H_{2,2} = H_{2,2,2} \cup L_2 = \{-1, 0, 1, 2\} \cup \{2, 3\} = \{-1, 0, 1, 2, 3\} = L_2 \cup H_{2,2,2} = \{-1, 0\} \cup \{0, 1, 2, 3\}$, organized as a lattice in a obvious way and as *bi*-BL-algebra $\mathcal{H}_{2,2,2} = \mathcal{H}_{2,2} \cup \mathcal{L}_2$ with the following tables:

$\mathcal{H}_{2,2,2,2}$

$\odot$	-1	0	1	2	3
-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0
1	-1	0	1	1	1
2	-1	0	1	2	2
3	-1	0	1	2	3

$\to$	-1	0	1	1	1
-1	3	3	3	3	3
0	-1	3	3	3	3
1	-1	0	3	3	3
2	-1	0	1	3	3
3	-1	0	1	2	3

Remark 3.10. The examples of the forms $\mathcal{H}_{m+1,n+1,p+1,q+1}$, for $m, n, p, q \geq 1$ are BL-algebras thus become a super BL-algebras.

3.3. Classes of examples of finite, non-linearly ordered *bi*-BL-algebras

The examples are one of the following forms:

1. Linearly ordered $MV \cup$ non-linearly ordered MV,
2. Linearly ordered $MV \cup$ non-linearly ordered BL or linearly ordered $BL \cup$ non-linearly ordered MV,
3. Linearly ordered $BL \cup$ non-linearly ordered BL.

• (1) Examples of the form: Linearly ordered MV $\cup$ non-linearly ordered MV.

Denote $\mathcal{H}_{p+1,(n+1)\times(m+1)} = \mathcal{L}_{p+1} \cup \mathcal{L}_{(n+1)\times(m+1)}$, for $p, m, n \geq 1$.

We present two families of examples.

1. Examples of the form: $\mathcal{H}_{2,(n+1)\times(m+1)} = \mathcal{L}_2 \cup \mathcal{L}_{(n+1)\times(m+1)}$ for $n, m \geq 1$.

Denote $H_{2,(n+1)\times(m+1)} = L_2 \cup L_{(n+1)\times(m+1)}$, with $n, m \geq 1$.

We present four examples.

Example 1. The set $H_{2,2\times 2} = L_2 \cup L_{2\times 2} = \{-1, 0\} \cup \{0, a, b, 1\} = \{-1, 0, a, b, 1\}$, organized as a lattice as and with operations $\to$ and $\odot$ in the following tables, is a non-linearly ordered *bi*-BL-algebra, denoted by $\mathcal{H}_{2,2\times 2} = \mathcal{L}_2 \cup \mathcal{L}_{2\times 2}$.

$\mathcal{H}_{2,2\times 2}$

$\odot$	-1	0	a	b	1
-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0
a	-1	0	a	0	a
b	-1	0	0	b	b
1	-1	0	a	b	1

$\to$	-1	0	a	b	1
-1	1	1	1	1	1
0	-1	1	1	1	1
a	-1	b	1	b	1
b	-1	a	a	1	1
1	-1	0	a	b	1

Example 2. The set $H_{2,3\times2} = L_2 \cup L_{3\times2} = \{-1,0\} \cup \{0,a,b,c,d,1\} = \{-1,0,a,b,c,d,1\}$, organized as a lattice as and with operations $\to$ and $\odot$ in the following tables, is a non-linearly ordered BL-algebra, denoted by $\mathcal{H}_{2,3\times2} = \mathcal{L}_2 \cup \mathcal{L}_{3\times2}$.

$\mathcal{H}_{2,3\times2}$

$\odot$	-1	0	a	b	c	d	1
-1	-1	-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0	0	0
a	-1	0	a	0	a	0	a
b	-1	0	0	0	0	b	b
c	-1	0	a	0	a	b	b
d	-1	0	0	b	b	d	d
1	-1	0	a	b	c	d	1

$\to$	-1	0	a	b	c	d	1
-1	1	1	1	1	1	1	1
0	-1	1	1	1	1	1	1
a	-1	d	1	d	1	d	1
b	-1	c	c	1	1	1	1
c	-1	b	c	d	1	d	1
d	-1	a	a	c	c	1	1
1	-1	0	a	b	c	d	1

Example 3. The set $H_{2,3\times2} = L_2 \cup L_{3\times2} = \{-1,0\} \cup \{0,a,b,c,d,e,f,g,1\}$ $= \{-1,0,a,b,c,d,e,f,g,1\}$, organized as a lattice as and with operations $\to$ and $\odot$ in the following tables, is a non-linearly ordered BL-algebra, denoted by $\mathcal{H}_{2,3\times3} = \mathcal{L}_2 \cup \mathcal{L}_{3\times3}$.

$\mathcal{H}_{2,3\times3}$

$\odot$	-1	0	a	b	c	d	e	f	g	1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0	0	0	0	0	0
a	-1	0	0	a	0	0	a	0	0	a
b	-1	0	a	b	0	a	b	0	a	b
c	-1	0	0	0	0	0	0	c	c	c
d	-1	0	0	a	0	0	a	c	c	d
e	-1	0	a	b	0	a	b	c	d	e
f	-1	0	0	0	c	c	c	f	f	f
g	-1	0	0	a	c	c	d	f	f	g
1	-1	0	a	b	c	d	e	f	g	1

$\to$	-1	0	a	b	c	d	e	f	g	1
-1	1	1	1	1	1	1	1	1	1	1
0	-1	1	1	1	1	1	1	1	1	1
a	-1	g	1	1	g	1	1	g	1	1
b	-1	f	g	1	f	g	1	f	g	1
c	-1	e	e	e	1	1	1	1	1	1
d	-1	d	e	e	g	1	1	g	1	1
e	-1	c	d	e	f	g	1	f	g	1
f	-1	b	b	b	e	e	e	1	1	1
g	-1	a	b	b	d	e	e	g	1	1
1	-1	0	a	b	c	d	e	f	g	1

Example 4. The set $H_{2,4\times2} = L_2 \cup L_{4\times2} = \{-1,0\} \cup \{0,a,b,c,d,e,f,1\} = \{-1,0,a,b,c,d,e,f,1\}$ is a *bi*-BL-algebra.

2. Examples of the form: $\mathcal{H}_{3,(n+1)\times(m+1)} = \mathcal{L}_3 \cup \mathcal{L}_{(n+1)\times(m+1)}$ for $n, m \geq 1$.

We present here only one example.

The set $H_{3,2\times 2} = L_3 \cup L_{2\times 2} = \{-2,-1,0\} \cup \{0,a,b,1\} = \{-2,-1,0,a,b,1\}$, organized as a lattice as and with operations $\rightarrow$ and $\odot$ in the following tables, is a non-linearly ordered *bi-BL*-algebra, denoted by $\mathcal{H}_{3,2\times 2} = \mathcal{L}_3 \cup \mathcal{L}_{2\times 2}$.

$\mathcal{H}_{3,2\times 2}$

$\odot$	-2	-1	0	a	b	1
-2	-2	-2	-2	-2	-2	-2
-1	-2	-1	-1	-1	-1	-1
0	-2	-1	0	0	0	0
a	-2	-1	0	a	0	a
b	-2	-1	0	0	b	b
1	-2	-1	0	a	b	1

$\rightarrow$	-2	-1	0	a	b	1
-2	1	1	1	1	1	1
-1	-2	1	1	1	1	1
0	-2	-1	1	1	1	1
a	-2	-1	b	1	b	1
b	-2	-1	a	a	1	1
1	-2	-1	0	a	b	1

Remark 3.11. The examples of forms $\mathcal{H}_{p+1,(n+1)\times(m+1)}$, for $p, n, m \geq 1$ are BL-algebras thus are super BL-algebras.

• (2) Examples of the form: Linearly ordered MV $\cup$ non-linearly ordered BL or linearly ordered BL $\cup$ non-linearly ordered MV.

Denote for $u, v, n, m \geq 1$, the *bi-BL*-algebras: $\mathcal{H}_{u+1,v+1,(n+1)\times(m+1)} = \mathcal{L}_{u+1} \cup \mathcal{L}_{v+1} \cup \mathcal{L}_{(n+1)\times(m+1)} = \mathcal{L}_{u+1} \cup \mathcal{H}_{v+1,(n+1)\times(m+1)} = \mathcal{H}_{u+1,v+1} \cup \mathcal{L}_{(n+1)\times(m+1)}$, by the associativity of $\cup$.

We present two examples.

Example 1. Consider the *bi-BL*-algebra $\mathcal{H}_{2,2,2\times 2} = \mathcal{L}_2 \cup \mathcal{H}_{2,2\times 2} = \mathcal{H}_{2,2} \cup \mathcal{L}_{2\times 2}$ the underline set, $\{-2,-1,0,a,b,1\}$ can be considered either as the union of sets:
$H_{(2,2),2\times 2)} = [\{-2,1\} \cup \{-1,0\}] \cup \{0,a,b,1\} = [L_2 \cup L_2] \cup L_{2\times 2}$
or as the union
$H_{2,(2,2\times 2)} = \{-2,-1\} \cup [\{-1,0\} \cup \{0,a,b,1\}] = L_2 \cup [L_2 \cup L_{2\times 2}] = L_2 \cup H_{2,2\times 2}$.
It has the following tables:

$\mathcal{H}_{2,2,2\times 2}$

$\odot$	-2	-1	0	a	b	1
-2	-2	-2	-2	-2	-2	-2
-1	-2	-1	-1	-1	-1	-1
0	-2	-1	0	0	0	0
a	-2	-1	0	a	0	a
b	-2	-1	0	0	b	b
1	-2	-1	0	a	b	1

$\rightarrow$	-2	-1	0	a	b	1
-2	1	1	1	1	1	1
-1	-2	1	1	1	1	1
0	-2	-1	1	1	1	1
a	-2	-1	b	1	b	1
b	-2	-1	a	a	1	1
1	-2	-1	0	a	b	1

Example 2. Consider the *bi-BL*-algebra $\mathcal{H}_{2,(2,2)\times 2} = \mathcal{L}_2 \cup \mathcal{H}_{(2,2)\times 2}$. The set $H_{2,(2,2)\times 2} = L_2 \cup H_{(2,2)\times 2} = \{-1, 0\} \cup \{0, a, b, c, d, 1\} = \{-1, 0, a, b, c, d, 1\}$, organized as a lattice and a bounded lattice with the operations $\rightarrow$ and $\odot$ form the following tables is a *bi-BL*-algebra, denoted by $\mathcal{H}_{2,(2,2)\times 2}$.

$\mathcal{H}_{2,(2,2)\times 2}$

$\odot$	-1	0	a	b	c	d	1
-1	-1	-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0	0	0
a	-1	0	a	0	a	0	a
b	-1	0	0	b	b	b	b
c	-1	0	a	b	c	b	c
d	-1	0	0	b	b	d	d
1	-1	0	a	b	c	d	1

$\rightarrow$	-1	0	a	b	c	d	1
-1	1	1	1	1	1	1	1
0	-1	1	1	1	1	1	1
a	-1	d	1	d	1	d	1
b	-1	a	a	1	1	1	1
c	-1	0	a	d	1	d	1
d	-1	a	a	c	c	1	1
1	-1	0	a	b	c	d	1

Remark 3.12. The examples of forms $\mathcal{H}_{u+1,v+1,(n+1)\times(m+1)}$, for $u, v, n, m \geq 1$ are BL-algebras thus become a super BL-algebras.

• (3) Examples of the form: Linearly ordered BL ∪ non-linearly ordered BL or equivalent forms.

Denote for $u, v, n, m, p \geq 1$, the *bi-BL*-algebras: $\mathcal{H}_{u+1,v+1,(n+1,m+1)\times(p+1)} = \mathcal{H}_{u+1,v+1} \cup \mathcal{L}_{(n+1,m+1)\times(p+1)}$.

Example. Consider the *bi-BL*-algebra $\mathcal{H}_{2,2,(2,2)\times 2} = \mathcal{H}_{2,2} \cup \mathcal{H}_{(2,2)\times 2} = (\mathcal{L}_2 \cup \mathcal{L}_2) \cup \mathcal{H}_{(2,2)\times 2} = \mathcal{L}_2 \cup \mathcal{H}_{2,(2,2)\times 2}$ with the underline set $H_{2,2,(2,2)\times 2} = H_{2,2} \cup H_{(2,2)\times 2} = \{-2, -1, 0\} \cup \{0, a, b, c, d, 1\} = \{-2, -1, 0, a, b, c, d, 1\}$, organized as a lattice, with the operations $\rightarrow$ and $\odot$ in the following tables:

$\odot$	-2	-1	0	a	b	c	d	1
-2	-2	-2	-2	-2	-2	-2	-2	-2
-1	-2	-1	-1	-1	-1	-1	-1	-1
0	-2	-1	0	0	0	0	0	0
a	-2	-1	0	a	0	a	0	a
b	-2	-1	0	0	b	b	b	b
c	-2	-1	0	a	b	c	b	c
d	-2	-1	0	0	b	b	d	d
1	-2	-1	0	a	b	c	c	1

$\rightarrow$	-2	-1	0	a	b	c	d	1
-2	1	1	1	1	1	1	1	1
-1	-1	1	1	1	1	1	1	1
0	-2	-1	1	1	1	1	1	1
a	-2	-1	d	1	d	1	d	1
b	-2	-1	a	a	1	1	1	1
c	-2	-1	0	a	d	1	d	1
d	-2	-1	a	a	c	c	1	1
1	-2	-1	0	a	b	c	d	1

Remark 3.13. The examples of forms $\mathcal{H}_{u+1,v+1,(n+1,m+1)\times(p+1)}$, for $u, v, n, m, p \geq 1$ are BL-algebras thus become a super BL-algebras.

3.3.1. Example of infinite *bi*-BL-algebras

By [11] we present example of infinite, linearly ordered *bi*-BL-algebra.

Example. The linearly ordered set(chain) $H_{P(\mathbb{Z}),2} = P(\mathbb{Z}) \cup L_2 = (\mathbb{Z}^- \cup -\infty) \cup L_2 = \{-\infty, \ldots, -3, -2, -1, 0\} \cup \{0, 1\} = \{-\infty, \ldots, -3, -2, -1, 0, 1\}$ with the operations $\rightarrow$ and $\odot$ defined by the following tables, is a linearly ordered *bi*-BL-algebra, denoted by $\mathcal{H}_{\mathcal{P}(\mathbb{Z}),2} = \mathcal{P}(\mathbb{Z}) \cup \mathcal{L}_2$.

$\mathcal{H}_{\mathcal{P}(\mathbb{Z}),2}$

$\odot$	$-\infty$	$\cdots$	-3	-2	-1	0	1
$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\vdots$	$\cdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	$\cdots$	-6	-5	-4	-3	-3
-2	$-\infty$	$\cdots$	-5	-4	-3	-2	-2
-1	$-\infty$	$\cdots$	-4	-3	-2	-1	-1
0	$-\infty$	$\cdots$	-3	-2	-1	0	0
1	$-\infty$	$\cdots$	-3	-2	-1	0	1

$\rightarrow$	$-\infty$	$\cdots$	-3	-2	-1	0	1
$-\infty$	1	$\cdots$	1	1	1	1	1
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	$\cdots$	1	1	1	1	1
-2	$-\infty$	$\cdots$	-1	1	1	1	1
-1	$-\infty$	$\cdots$	-2	-1	1	1	1
0	$-\infty$	$\cdots$	-3	-2	-1	1	1
1	$-\infty$	$\cdots$	-3	-2	-1	0	1

3.3.2. Classes of finite *bi*-BL-algebras such that are not super BL-algebras

The examples will be of the form: non-linearly ordered MV/BL-algebra $\cup$ MV/BL-algebra, more precisely of one of the following forms:

(1) non-linearly ordered $MV \bigcup$ linearly ordered MV,

(2) non-linearly ordered $MV \bigcup$ non-linearly ordered MV,

(3) non-linearly ordered $MV \bigcup$ linearly ordered BL,

(4) non-linearly ordered $MV \bigcup$ non-linearly ordered BL,

(5) non-linearly ordered $BL \bigcup$ linearly ordered MV,

(6) non-linearly ordered $BL \bigcup$ non-linearly ordered MV,

(7) non-linearly ordered $BL \bigcup$ linearly ordered BL,

(8) non-linearly ordered $BL \bigcup$ non-linearly orderedBL.

• (1) Examples of the form: non-linearly ordered $MV \bigcup$ linearly ordered MV.

Denote, for $p, q, n \geq 1$

$$\mathcal{D}_{(p+1)\times(q+1),n+1} = \mathcal{L}_{(p+1)\times(q+1)} \cup \mathcal{L}_{n+1}.$$

We present three examples of above form.

Example 1. The *bi*-BL-algebra

$$\mathcal{D}_{2\times 2,2} = \mathcal{L}_{2\times 2} \cup \mathcal{L}_2,$$

with the underline set

$$D_{2\times 2,2} = L_{2\times 2} \cup L_2 = \{0, a, b, c\} \cup \{c, 1\} = \{0, a, b, c, 1\},$$

is organized as a lattice with the following tables:

$\mathcal{D}_{2\times 2,2}$

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

note that $\mathcal{D}_{2\times 2,2}$ is not a BL-algebra, since $(a \to b) \vee (b \to a) = b \vee a = c \neq 1$ thus $\mathcal{D}_{2\times 2,2}$ is not a super BL-algebra.

Example 2. The *bi*-BL-algebra

$$\mathcal{D}_{2\times 2,3} = \mathcal{L}_{2\times 2} \cup \mathcal{L}_3,$$

with the underline set

$$D_{2\times 2,3} = L_{2\times 2} \cup L_3 = \{0, a, b, c\} \cup \{c, d, 1\} = \{0, a, b, c, d, 1\},$$

is organized as a lattice with the following tables:

$\mathcal{D}_{2\times 2,3}$

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	a	a
b	0	0	b	b	b	b
c	0	a	b	c	c	c
d	0	a	b	c	c	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	1	1	1
b	a	a	1	1	1	1
c	0	a	b	1	1	1
d	0	a	b	d	1	1
1	0	a	b	c	d	1

note that $\mathcal{D}_{2\times 2,3}$ is not a BL-algebra, since $(a \to b) \vee (b \to a) = b \vee a = c \neq 1$ thus $\mathcal{D}_{2\times 2,3}$ is not a super BL-algebra.

Example 3. The *bi*-BL-algebra

$$\mathcal{D}_{2\times 3,2} = \mathcal{L}_{2\times 3} \cup \mathcal{L}_2,$$

with the underline set

$$D_{2\times 3,2} = L_{2\times 3} \cup L_2 = \{0, a, b, c, d, n\} \cup \{n, 1\} = \{0, a, b, c, d, n, 1\},$$

is organized as a lattice with the following tables:

$\mathcal{D}_{2\times 3,2}$

$\odot$	0	a	b	c	d	n	1
0	0	0	0	0	0	0	0
a	0	0	a	0	0	a	a
b	0	a	b	0	a	b	b
c	0	0	0	c	c	c	c
d	0	0	a	c	c	d	d
n	0	a	b	c	d	n	n
1	0	a	b	c	d	n	1

$\to$	0	a	b	c	d	n	1
0	1	1	1	1	1	1	1
a	d	1	1	d	1	1	1
b	c	d	1	c	d	1	1
c	b	b	b	1	1	1	1
d	a	b	b	d	1	1	1
n	0	a	b	c	d	1	1
1	0	a	b	c	d	n	1

note that $\mathcal{D}_{2\times 3,2}$ is not a BL-algebra, since $(b \to d) \vee (d \to b) = d \vee b = n \neq 1$ thus $\mathcal{D}_{2\times 3,2}$ is not a super BL-algebra.

• (2) Examples of the form: non-linearly ordered $MV \bigcup$ non-linearly ordered MV.

For $n, m, u, v \geq 1$, denote,

$$\mathcal{D}_{(n+1)\times(m+1),(u+1)\times(v+1)} = \mathcal{L}_{(n+1)\times(m+1)} \cup \mathcal{L}_{(u+1)\times(v+1)}.$$

Example. The *bi*-BL-algebra

$$\mathcal{D}_{2\times 2,2\times 2} = \mathcal{L}_{2\times 2} \cup \mathcal{L}_{2\times 2},$$

with the underline set

$$D_{2\times 2,2\times 2} = L_{2\times 2} \cup L_{2\times 2} = \{0, a, b, n\} \cup \{n, c, d, 1\} = \{0, a, b, n, c, d, 1\},$$

is organized as a lattice with the following tables:

$\mathcal{D}_{2\times 2,2\times 2}$

$\odot$	0	a	b	n	c	d	1
0	0	0	0	0	0	0	0
a	0	a	0	a	a	a	a
b	0	0	b	b	b	b	b
n	0	a	b	n	n	n	n
c	0	a	b	n	c	n	c
d	0	a	b	n	n	d	d
1	0	a	b	n	c	d	1

$\to$	0	a	b	n	c	d	1
0	1	1	1	1	1	1	1
a	b	1	b	1	1	1	1
b	a	a	1	1	1	1	1
n	0	a	b	1	1	1	1
c	0	a	b	d	1	d	1
d	0	a	b	c	c	1	1
1	0	a	b	n	c	d	1

note that $D_{2\times 2,2\times 2}$ is not a BL-algebra, since $(a \to b) \vee (b \to a) = b \vee a = n \neq 1$, thus $\mathcal{D}_{2\times 2,2\times 2}$ is not a super BL-algebra.

• (3) Examples of the form: non-linearly ordered $MV \bigcup$ linearly ordered BL or equivalent forms.

Denote, for $p, q, n, m \geq 1$,

$$\mathcal{D}_{(p+1)\times(q+1),n+1,m+1} = \mathcal{L}_{(p+1)\times(q+1)} \cup \mathcal{H}_{n+1,m+1}.$$

Example. The *bi*-BL-algebra

$$\mathcal{D}_{2\times 2,2,2} = \mathcal{L}_{2\times 2} \cup \mathcal{H}_{2,2} = \mathcal{L}_{2\times 2} \cup (\mathcal{L}_2 \cup \mathcal{L}_2) = \mathcal{D}_{2\times 2,2} \cup \mathcal{L}_2,$$

with the underline set

$$D_{2\times 2,2,2} = L_{2\times 2} \cup H_{2,2} = \{0, a, b, c\} \cup \{c, d, 1\} = \{0, a, b, c, d, 1\},$$

is organized as a lattice with the following tables:

$\mathcal{D}_{2\times 2,2,2}$

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	a	a
b	0	0	b	b	b	b
c	0	a	b	c	c	c
d	0	a	b	c	d	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	1	1	1
b	a	a	1	1	1	1
c	0	a	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

note that $\mathcal{D}_{2\times 2,2,2}$ is not a BL-algebra, since $(a \to b) \vee (b \to a) = b \vee a = c \neq 1$, thus $\mathcal{D}_{2\times 2,2,2}$ is not a super BL-algebra.

• (4) Examples of the form: non-linearly ordered $MV \bigcup$ non-linearly ordered BL or equivalent forms.

Denote, for $m, n, p, u, v \geq 1$,

$$\mathcal{D}_{(m+1)\times(n+1),p+1,(u+1)\times(v+1)} = \mathcal{L}_{(m+1)\times(n+1)} \cup \mathcal{H}_{p+1,(u+1)\times(v+1)}.$$

Example. The *bi*-BL-algebra:

$$\begin{aligned}\mathcal{D}_{2\times 2,2,2\times 2} &= \mathcal{L}_{2\times 2} \cup H_{2,2\times 2} = L_{2\times 2} \cup (L_2 \cup \mathcal{L}_{2\times 2}) = (\mathcal{L}_{2\times 2} \cup \mathcal{L}_2) \\ &= \mathcal{D}_{2\times 2,2} \cup \mathcal{L}_{2\times 2},\end{aligned}$$

with the underline set

$$\begin{aligned}D_{2\times 2,2,2\times 2} &= L_{2\times 2} \cup H_{2,2\times 2} = \{0,a,b,p\} \cup \{p,n\} \cup \{n,c,d,1\} \\ &= \{0,a,b,p,n,c,d,1\},\end{aligned}$$

is organized as a lattice as with the following tables:

$\mathcal{D}_{2\times 2,2,2\times 2}$

$\odot$	0	a	b	p	n	c	d	1
0	0	0	0	0	0	0	0	0
a	0	a	0	a	a	a	a	a
b	0	0	b	b	b	b	b	b
p	0	a	b	p	p	p	p	p
n	0	a	b	p	n	n	n	n
c	0	a	b	p	n	c	n	c
d	0	a	b	p	n	n	d	d
1	0	a	b	p	n	c	d	1

$\rightarrow$	0	a	b	p	n	c	d	1
0	1	1	1	1	1	1	1	1
a	b	1	b	1	1	1	1	1
b	a	a	1	1	1	1	1	1
p	0	a	b	1	1	1	1	1
n	0	a	b	p	1	1	1	1
c	0	a	b	p	d	1	d	1
d	0	a	b	p	c	c	1	1
1	0	a	b	p	n	c	d	1

note that $D_{2\times 2,2,2\times 2}$ is not a BL-algebra, since $(a \to b) \vee (b \to a) = b \vee a = p \neq 1$ thus $\mathcal{D}_{2\times 2,2,2\times 2}$ is not a super BL-algebra.

• (5) Examples of the form: non-linearly ordered $BL \bigcup$ linearly ordered MV or equivalent forms.

We consider here only two examples among all possible examples.

Example 1. The *bi*-BL-algebra

$$\mathcal{D}_{2,2\times 2,2} = \mathcal{H}_{2,2\times 2} \cup \mathcal{L}_2 = (\mathcal{L}_2 \cup \mathcal{L}_{2\times 2}) \cup \mathcal{L}_2 = \mathcal{L}_2 \cup (\mathcal{L}_{2\times 2} \cup \mathcal{L}_2) = \mathcal{L}_2 \cup \mathcal{D}_{2\times 2,2},$$

with the underline set

$$D_{2,2\times 2,2} = H_{2,2\times 2} \cup L_2 = \{0,n,a,b,m\} \cup \{m,1\} = \{0,n,a,b,m,1\},$$

is organized as a lattice with the following tables:

$\mathcal{D}_{2,2\times2,2}$

$\odot$	0	n	a	b	m	1
0	0	0	0	0	0	0
n	0	n	n	n	n	n
a	0	n	a	n	a	a
b	0	n	n	b	b	b
m	0	n	a	b	m	m
1	0	n	a	b	m	1

$\to$	0	n	a	b	m	1
0	1	1	1	1	1	1
n	0	1	1	1	1	1
a	0	b	1	b	1	1
b	0	a	a	1	1	1
m	0	n	a	b	1	1
1	0	n	a	b	m	1

note that $\mathcal{D}_{2,2\times2,2}$ is not a BL-algebra, since $(a \to b) \vee (b \to a) = b \vee a = m \neq 1$ thus $\mathcal{D}_{2,2\times2,2}$ is not a super BL-algebra.

Example 2. The *bi*-BL-algebra

$$\mathcal{D}_{3,2\times2,2} = \mathcal{H}_{3,2\times2} \cup \mathcal{L}_2 = (\mathcal{L}_3 \cup \mathcal{L}_{2\times2}) \cup \mathcal{L}_2 = \mathcal{L}_3 \cup (\mathcal{L}_{2\times2} \cup \mathcal{L}_2) = \mathcal{L}_3 \cup \mathcal{D}_{2\times2,2},$$

with the underline set

$$D_{3,2\times2,2} = H_{3,2\times2} \cup L_2 = \{-2,-1,0,a,b,c\} \cup \{c,1\} = \{-2,-1,0,a,b,c,1\}$$

is organized as a lattice with the following tables:

$\mathcal{D}_{3,2\times2,2}$

$\odot$	-2	-1	0	a	b	c	1
-2	-2	-2	-2	-2	-2	-2	-2
-1	-2	-2	-1	-1	-1	-1	-1
0	-2	-1	0	0	0	0	0
a	-2	-1	0	a	0	a	a
b	-2	-1	0	0	b	b	b
c	-2	-1	0	a	b	c	c
1	-2	-1	0	a	b	c	1

$\to$	-2	-1	0	a	b	c	1
-2	1	1	1	1	1	1	1
-1	-1	1	1	1	1	1	1
0	-2	-1	1	1	1	1	1
a	-2	-1	b	1	b	1	1
b	-2	-1	a	a	1	1	1
c	-2	-1	0	a	b	1	1
1	-2	-1	0	a	b	c	1

note that $\mathcal{D}_{3,2\times2,2}$ is not a BL-algebra, since $(a \to b) \vee (b \to a) = b \vee a = c \neq 1$ thus $\mathcal{D}_{3,2\times2,2}$ is not a super BL-algebra.

• (6) Examples of the form: non-linearly ordered $BL \bigcup$ non-linearly ordered MV or equivalent forms.

Example. The *bi*-BL-algebra

$$\begin{aligned}\mathcal{D}_{2,2\times2,2\times2} &= \mathcal{H}_{2,2\times2} \cup \mathcal{L}_{2\times2} = (\mathcal{L}_2 \cup \mathcal{L}_{2\times2}) \cup \mathcal{L}_{2\times2} \\ &= \mathcal{L}_2 \cup (\mathcal{L}_{2\times2} \cup \mathcal{L}_{2\times2}) = \mathcal{L}_2 \cup \mathcal{D}_{2\times2,2\times2}\end{aligned}$$

with the support set

$$\begin{aligned} D_{2,2\times 2,2\times 2} &= H_{2,2\times 2} \cup L_{2\times 2} = \{-1, 0, a, b, n\} \cup \{n, c, d, 1\} \\ &= \{-1, 0, a, b, n, c, d, 1\}, \end{aligned}$$

is organized as a lattice with the following tables:

$\odot$	-1	0	a	b	n	c	d	1
-1	-1	-1	-1	-1	-1	-1	-1	
0	-1	0	0	0	0	0	0	0
a	-1	0	a	0	a	a	a	a
b	-1	0	0	b	b	b	b	b
n	-1	0	a	b	n	n	n	n
c	-1	0	a	b	n	c	n	c
d	-1	0	a	b	n	n	d	d
1	-1	0	a	b	n	c	d	1

$\to$	-1	0	a	b	n	c	d	1
-1	1	1	1	1	1	1	1	1
0	-1	1	1	1	1	1	1	1
a	-1	b	1	b	1	1	1	1
b	-1	a	a	1	1	1	1	1
n	-1	0	a	b	1	1	1	1
c	-1	0	a	b	d	1	d	1
d	-1	0	a	b	c	c	1	1
1	-1	0	a	b	n	c	d	1

note that $\mathcal{D}_{2,2\times 2,2\times 2}$ is not a BL-algebra, since $(a \to b) \vee (b \to a) = b \vee a = n \neq 1$ thus $\mathcal{D}_{2,2\times 2,2\times 2}$ is not super BL-algebra.

• (7) Examples of the form: non-linearly ordered $BL \bigcup$ linearly ordered BL or equivalent forms.

Example. The *bi*-BL-algebra

$$\mathcal{D}_{2,2\times 2,2,2} = \mathcal{H}_{2,2\times 2} \cup \mathcal{H}_{2,2} = (\mathcal{L}_2 \cup \mathcal{L}_{2\times 2}) \cup (\mathcal{L}_2 \cup \mathcal{L}_2) = \mathcal{L}_2 \cup \mathcal{D}_{2\times 2,2} \cup \mathcal{L}_2,$$

with the underline set

$$D_{2,2\times 2,2,2} = H_{2,2\times 2} \cup H_{2,2} = \{-1, 0, a, b, c\} \cup \{c, d, 1\} = \{-1, 0, a, b, c, d, 1\},$$

is organized as a lattice as with the following tables:

$\odot$	-1	0	a	b	c	d	1
-1	-1	-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0	0	0
a	-1	0	a	0	a	a	a
b	-1	0	0	b	b	b	b
c	-1	0	a	b	c	c	c
d	-1	0	a	b	c	c	d
1	-1	0	a	b	c	d	1

$\to$	-1	0	a	b	c	d	1
-1	1	1	1	1	1	1	1
0	-1	1	1	1	1	1	1
a	-1	b	1	b	1	1	1
b	-1	a	a	1	1	1	1
c	-1	0	a	b	1	1	1
d	-1	0	a	b	d	1	1
1	-1	0	a	b	c	d	1

note that $\mathcal{D}_{2,2\times 2,2,2}$ is not a BL-algebra, since $(a \to b) \vee (b \to a) = b \vee a = c \neq 1$ thus $\mathcal{D}_{3,2\times 2,2}$ is not a super BL-algebra.

• (8) Examples of the form: non-linearly ordered BL $\cup$ non-linearly ordered BL or equivalent forms.

Example. The *bi-BL*-algebra

$$\begin{aligned}\mathcal{D}_{2,2\times 2,2,2\times 2} &= \mathcal{H}_{2,2\times 2} \cup \mathcal{H}_{2,2\times 2} = (\mathcal{L}_2 \cup \mathcal{L}_{2\times 2}) \cup (\mathcal{L}_2 \cup \mathcal{L}_{2\times 2}) \\ &= \mathcal{L}_2 \cup \mathcal{D}_{2\times 2,2} \cup \mathcal{L}_{2\times 2} = \mathcal{D}_{2,2\times 2,2} \cup \mathcal{L}_{2\times 2},\end{aligned}$$

with the underline set

$$\begin{aligned}D_{2,2\times 2,2,2\times 2} &= H_{2,2\times 2} \cup H_{2,2\times 2} = \{0, m, a, b, p\} \cup \{p, n, c, d, 1\} \\ &= \{0, m, a, b, p, n, c, d, 1\},\end{aligned}$$

is organized as a lattice as with the following tables:

$\odot$	0	m	a	b	p	n	c	d	1
0	0	0	0	0	0	0	0	0	0
m	0	m	m	m	m	m	m	m	m
a	0	m	a	m	a	a	a	a	a
b	0	m	m	b	b	b	b	b	b
p	0	m	a	b	p	p	p	p	p
n	0	m	a	b	p	n	n	n	n
c	0	m	a	b	p	n	c	n	c
n	0	m	a	b	p	n	n	d	d
1	0	m	a	b	p	n	c	d	1

$\to$	0	m	a	b	p	n	c	d	1
0	1	1	1	1	1	1	1	1	1
m	0	1	1	1	1	1	1	1	1
a	0	b	1	b	1	1	1	1	1
b	0	a	a	1	1	1	1	1	1
p	0	m	a	b	1	1	1	1	1
n	0	m	a	b	p	1	1	1	1
c	0	m	a	b	p	d	1	1	1
n	0	m	a	b	p	c	c	1	1
1	0	m	a	b	p	n	c	d	1

note that $\mathcal{D}_{2,2\times2,2,2\times2}$ is not a BL-algebra, since $(a \to b) \vee (b \to a) = b \vee a = p \neq 1$ thus $\mathcal{D}_{2,2\times2,2,2\times2}$ is not a super BL-algebra.

3.4. *bi*-Homomorphisms, *bi*-Filters and *bi*-Boolean center

Definition 3.14. Let $L = (L_1 \cup L_2, \wedge, \vee, \odot, \to, 0, 1)$ and $K = (K_1 \cup K_2, \wedge, \vee, \odot, \to, 0, 1)$ be two *bi*-BL-algebras. We say a map ϕ from L to K is a *bi*-homomorphism of *bi*-BL-algebras. If $\phi = \phi_1 \cup \phi_2$ where $\phi_1 = \phi\mid_{L_1}$ from L_1 to K_1 and $\phi_2 = \phi\mid_{L_2}$ from L_2 to K_2 are BL-homomorphisms.

Definition 3.15. Let $\phi : L \to K$ be a *bi*-homomorphism, where $L = L_1 \cup L_2$ and $K = K_1 \cup K_2$ are *bi*-BL-algebras the kernel of the *bi*- homomorphism ϕ as $Ker(\phi) = Ker(\phi_1) \cup Ker(\phi_2)$; here $Ker(\phi_1) = \{a_1 \in L_1 \mid \phi_1(a_1) = 1\}$ and $Ker(\phi_2) = \{a_2 \in L_2 \mid \phi_2(a_2) = 1\}$, i.e., $Ker(\phi) = \{a_1 \in L_1, a_2 \in L_2 \mid \phi_1(a_1) = 1, \phi_2(a_2) = 1\}$.

Example 3.16. Let $L = \mathcal{D}_{2\times2,2}$ and $K = \mathcal{D}_{2\times2,3}$. Define $\phi = \phi_1 \cup \phi_2$ as follow: $\phi_1 : \mathcal{L}_{2\times2} \to \mathcal{L}_{2\times2}$ where ϕ_1 is a identity map and $\phi_2 : \mathcal{L}_2 \to \mathcal{L}_3$ where $\phi_2(c) = c$ and $\phi_2(1) = 1$, then ϕ is a *bi*-homomorphism from L to K and $Ker(\phi_1) = \{c\}$ and $Ker(\phi_2) = \{1\}$, so $Ker(\phi) = \{c, 1\}$.

Definition 3.17. Let $L = L_1 \cup L_2$ be a *bi*-BL-algebra. We say that subset $S = S_1 \cup S_2$ of L is a sub *bi*-BL-algebra of L if $L_1 \cap S = S_1$ and $L_2 \cap S = S_2$ are subalgebra of L_1 and L_2 respectively.

Example 3.18. In the Example 3.2, consider $S_1 = \{0, a, c, 1\}$ and $S_2 = \{0, e, 1\}$, then $S = S_1 \cup S_2 = \{0, a, c, e, 1\}$ is a sub *bi*-BL-algebra of L, since $S \cap L_1 = S_1$ and $S \cap L_2 = S_2$ are subalgebras of L_1 and L_2 respectively.

Definition 3.19. Let $L = L_1 \cup L_2$ be a *bi*-BL-algebras. We say the subset $F = F_1 \cup F_2$ of L is a *bi*-filter of L if F_i is a filter of L_i, where $i = 1, 2$ respectively.

Theorem 3.20. *Let $L = L_1 \cup L_2$ and $K = K_1 \cup K_2$ are bi-BL-algebras and $\phi : L \to K$ is a bi-BL-algebra homomorphism. Then $Ker(\phi)$ is a bi-filter of L.*

Example 3.21. In Example 3.2, consider $F_1 = \{a, 1\}$ and $F_2 = \{e, 1\}$. Then $F = F_1 \cup F_2 = \{a, e, 1\}$ is a *bi*-filter of L.

Theorem 3.22. *Let $F = F_1 \cup F_2$ be a bi-filter of a bi-BL-algebra $L = L_1 \cup L_2$ such that F_i is a filter of L_i where $i = 1, 2$. Then $\frac{\mathcal{L}}{\mathcal{F}} := \frac{L_1}{F_1} \cup \frac{L_2}{F_2}$ is a bi-BL-algebra where $\frac{L_i}{F_i} = \{[x]_{F_i} | x \in L_i\}$ and $[x]_{F_i} = \{y \in L_i | x \to y \in F_i,\ y \to x \in F_i\}$, where $x \in L_i$ and $i = 1, 2$.*

Definition 3.23. Let $F = F_1 \cup F_2$ be a *bi*-filter of a super BL-algebra $L = L_1 \cup L_2$. If F is a filter of L, then we say that F is a super filter of L.

Example 3.24. Let $L_1 = \{0, a, c, 1\}$ and $L_2 = \{0, b, c, d, 1\}$. Define $\odot$ and $\to$ as follow:

L_1

$\odot$	0	a	c	1
0	0	0	0	0
a	0	a	c	a
c	0	c	c	c
1	0	a	c	1

$\to$	0	a	c	1
0	1	1	1	1
a	0	1	c	1
c	0	1	1	1
1	0	a	c	1

L_2

$\odot$	0	b	c	d	1
0	0	0	0	0	0
b	0	b	c	d	b
c	0	c	c	d	c
d	0	d	d	0	d
1	0	b	c	d	1

$\to$	0	b	c	d	1
0	1	1	1	1	1
b	0	1	c	d	1
c	0	1	1	d	1
d	d	1	1	1	1
1	0	b	c	d	1

For L, whose tables are the following:

L

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	c	c	d	a
b	0	c	b	c	d	b
c	0	c	c	c	d	c
d	0	d	d	d	0	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	b	d	1
b	0	a	1	a	d	1
c	0	1	1	1	d	1
d	d	1	1	1	1	1
1	0	a	b	c	d	1

Then $L1$ and L_2 are BL-algebras and $L = L_1 \cup L_2$ is a *bi*-BL-algebra and also L is a super BL-algebra, consider $F_1 = \{a, 1\}$ and $F_2 = \{b, c, 1\}$ are filters of L_1 and L_2, respectively. Also $F = F_1 \cup F_2 = \{a, b, c, 1\}$ is super filter of L.

Corollary 3.25. *In a super BL-algebra, any super filter is a filter.*

In the following example we show that converse of above corollary is not true.

Example 3.26. In super BL-algebra $\mathcal{H}_{2,2\times 2}$, we consider $F = \{a, 1\}$. Then F is a filter of $\mathcal{H}_{2,2\times 2}$ but is not a super filter of $\mathcal{H}_{2,2\times 2}$, since F is not union each pair filters such that $F = F_1 \cup F_2$ and F_i is filter of L_i, respectively.

Proposition 3.27. *Let $L = L_1 \cup L_2$ be a super BL-algebra and $F = F_1 \cup F_2$ be a super filter of L, then $\frac{L}{F}$ is a BL-algebra.*

If $L = L_1 \cup L_2$ is a super BL-algebra and $F = F_1 \cup F_2$ a super filter of L, then $\frac{L}{F} \neq \frac{\mathcal{L}}{\mathcal{F}}$.

Example 3.28. In Example 3.2 we have:

$[0]_{F_1} = \frac{0}{F_1} = \{x \in L_1 | x \to 0 \in F_1, 0 \to x \in F_1\} = \{0\}$,
$[a]_{F_1} = \frac{a}{F_1} = \{x \in L_1 | x \to a \in F_1, a \to x \in F_1\} = \{a, 1\}$,
$[c]_{F_1} = \frac{c}{F_1} = \{x \in L_1 | x \to c \in F_1, c \to x \in F_1\} = \{c\}$,
$[1]_{F_1} = \frac{1}{F_1} = \{x \in L_1 | x \to 1 \in F_1, 1 \to x \in F_1\} = \{a, 1\}$,

Thus $[a]_{F_1} = [1]_{F_1}$, therefore $\frac{L_1}{F_1} = \{[0]_{F_1}, [c]_{F_1}, [1]_{F_1}\}$.

$[0]_{F_2} = \frac{0}{F_2} = \{x \in L_2 | x \to 0 \in F_2, 0 \to x \in F_2\} = \{0\}$,
$[b]_{F_2} = \frac{b}{F_2} = \{x \in L_2 | x \to b \in F_2, b \to x \in F_2\} = \{b, c, 1\}$,
$[c]_{F_2} = \frac{c}{F_2} = \{x \in L_2 | x \to c \in F_2, c \to x \in F_2\} = \{b, c, 1\}$,
$[d]_{F_2} = \frac{d}{F_2} = \{x \in L_2 | x \to d \in F_2, d \to x \in F_2\} = \{d\}$,
$[1]_{F_2} = \frac{1}{F_2} = \{x \in L_2 | x \to 1 \in F_2, 1 \to x \in F_2\} = \{b, c, 1\}$,

Thus $[b]_{F_2} = [c]_{F_2} = [1]_{F_2}$, therefore $\frac{L_2}{F_2} = \{[0]_{F_2}, [d]_{F_2}, [1]_{F_2}\}$.

So $\frac{\mathcal{L}}{\mathcal{F}} = \frac{L_1}{F_1} \cup \frac{L_2}{F_2} = \{[0]_{F_1}, [c]_{F_1}, [1]_{F_1}, [0]_{F_2}, [d]_{F_2}, [1]_{F_2}\}$ is a *bi*-BL-algebra.

In $\frac{L}{F}$ we have:

$[0]_F = \frac{0}{F} = \{x \in L | x \to 0 \in F, 0 \to x \in F\} = \{0\}$,
$[a]_F = \frac{a}{F} = \{x \in L | x \to a \in F, a \to x \in F\} = \{a, b, c, 1\}$,
$[b]_F = \frac{b}{F} = \{x \in L | x \to b \in F, b \to x \in F\} = \{a, b, c, 1\}$,
$[c]_F = \frac{c}{F} = \{x \in L | x \to c \in F, c \to x \in F\} = \{a, b, c, 1\}$,
$[d]_F = \frac{d}{F} = \{x \in L | x \to d \in F, d \to x \in F\} = \{d\}$.

So $\frac{L}{F} = \{[0]_F, [d]_F, [1]_F\}$, we show that $\frac{L}{F} \neq \frac{\mathcal{L}}{\mathcal{F}}$, since $[1]_{F_1} \in \frac{\mathcal{L}}{\mathcal{F}}$, but $[1]_{F_1} \notin \frac{L}{F}$.

Definition 3.29. Let $A = A_1 \cup A_2$ be a *bi*-BL-algebra and$\mathcal{B}(A)$ be the Boolean *bi*-algebra associated with the bounded distributive lattice $L(A)$. Elements of $\mathcal{B}(A)$ are called the *bi*-Boolean elements of A and $\mathcal{B}(A) := B(A_1) \cup B(A_2)$, where $B(A_1)$ and $B(A_2)$ are Boolean elements of A_1 and A_2, respectively.

Example 3.30. In Example 3.1, we have $B(L_1) = B(L_2) = \{0, 1\}$, then $\mathcal{B}(A) = B(L_1) \cup B(L_2) = \{0, 1\}$.

Example 3.31. In Example 3.3, we have $\mathcal{B}(A) = B(L_1) = B(L_2) = \{0, 1\}$ but the Boolean elements of L are $B(L) = \{0, a, d, 1\}$. Thus the *bi*-Boolean elements of a super BL-algebra are not equal with Boolean elements of L.

Remark 3.32. Suppose $L = L_1 \cup L_2$ be a chain super BL-algebra and also $1_{L_1} = 1_{L_2}$ and $0_{L_1} = 0_{L_2}$, then the *bi*-Boolean elements of L is equal the Boolean elements of L, i.e., $\mathcal{B}(L_1) = B(L_1) = B(L_2) = B(L)$.

Example 3.33. Let $L_1 = \{0, 1, 2, 4\}$ and $L_2 = \{0, 2, 3, 4\}$. Then $L_1 \cup L_2 = \{0, 1, 2, 3, 4\} = \mathcal{L}_5$. Define $\odot$ and $\to$ as follow:

$\mathcal{L}_1$

$\odot$	0	1	2	4
0	0	0	0	0
1	0	0	0	1
2	0	0	1	2
4	0	1	2	4

$\to$	0	1	2	4
0	4	4	4	4
1	2	4	4	4
2	1	2	4	4
4	0	1	2	4

$\mathcal{L}_2$

$\odot$	0	2	3	4
0	0	0	0	0
2	0	0	0	2
3	0	0	2	3
4	0	2	3	4

$\to$	0	2	3	4
0	4	4	4	4
2	3	4	4	4
3	2	3	4	4
4	0	2	3	4

L_1 and L_2 are BL-algebras, thus $\mathcal{L}_5 = L_1 \cup L_2$ is a super BL-algebra and we have $B(L_1) = B(L_2) = B(\mathcal{L}_5) = \mathcal{B}(\mathcal{L}_5) = \{0, 1\}$.

Otherwise suppose $L_1 = \{0, 1, 2, 3\}$ and $L_2 = \{0, 2, 3, 4\}$, then $L_1 \cup L_2 = \{0, 1, 2, 3, 4\} = \mathcal{L}_5$. Define $\odot$ and $\rightarrow$ as follow:

$\mathcal{L}_1$

$\odot$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	1	2
3	0	1	2	3

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	2	3	3	3
2	1	2	3	3
3	0	1	2	3

$\mathcal{L}_2$

$\odot$	0	2	3	4
0	0	0	0	0
2	0	0	0	2
3	0	0	2	3
4	0	2	3	4

$\rightarrow$	0	2	3	4
0	4	4	4	4
2	3	4	4	4
3	2	3	4	4
4	0	2	3	4

L_1 and L_2 are BL-algebras, thus $\mathcal{L}_5 = L_1 \cup L_2$ is a super BL-algebra and we have $B(L_1) = \{0, 3\}$ and $B(L_2) = \{0, 4\}$, then $\mathcal{B}(\mathcal{L}_5) = \{0, 3, 4\}$ but $B(\mathcal{L}_5) = \{0, 4\}$. So $\mathcal{B}(\mathcal{L}_5) \neq B(\mathcal{L}_5)$.

Definition 3.34. Let $L = L_1 \cup L_2$ be a *bi*-BL-algebra. We denote by $\mathcal{D}_s(L) = D_s(L_1) \cup D_s(L_2)$ the set of all *bi*-deductive systems of L where $D_s(L_1)$ and $D_s(L_1)$ are deductive systems of L_1 and L_2 respectively. If L is a super BL-algebra, then $D_s(L)$ is the set of all deductive systems of L.

Example 3.35. Consider $\mathcal{H}_{2,2\times2}$. Then $D_s(\mathcal{L}_2) = \{\{0\}, \mathcal{L}_2\}$ and $D_s(\mathcal{L}_{2\times2}) = \{\{1\}, \{a, 1\}, \{b, 1\}, \mathcal{L}_{2\times2}\}$, thus $\mathcal{D}_s(\mathcal{H}_{2,2\times2}) = D_s(\mathcal{L}_2) \cup D_s(\mathcal{L}_{2\times2}) = \{\{0\}, \{1\}, \{a, 1\}, \{b, 1\}, \mathcal{L}_2, \mathcal{L}_{2\times2}\}$. But $D_s(\mathcal{H}_{2,2\times2}) = \{\{1\}, \{a, 1\}, \{b, 1\}, \mathcal{L}_{2\times2}, \mathcal{H}_{2,2\times2}\}$, thus $\mathcal{D}_s(\mathcal{H}_{2,2\times2}) \neq D_s(\mathcal{H}_{2,2\times2})$, since $\mathcal{L}_2 \notin D_s(\mathcal{H}_{2,2\times2})$.

Remark 3.36. Let L be a *bi*-BL-algebra. Then $D_s(L)$ together with inclusion relation is not a lattice.

In above example $(\mathcal{D}_s(\mathcal{H}_{2,2\times2}), \subseteq)$ is not a lattice, since $\mathcal{L}_2 \cap \{b, 1\} = \{-1, 0\} \cap \{b, 1\} = \emptyset$ and $\emptyset \notin \mathcal{D}_s(\mathcal{H}_{2,2\times2})$.

4. Conclusion

The union of two subgroups, or two subrings, or two subsemigroups etc. do not form any algebraic structure but all of them find a nice *bi*algebraic structure as *bi*groups, *bi*rings, *bi*semigroups etc. Except for this *bi*algebraic structure these would remain only as sets without any nice algebraic structure on them. Further when these *bi*algebraic structures are defined on them they enjoy not only the inherited qualities of the algebraic structure from which they are taken but also several distinct algebraic properties that are not present in algebraic structures.

We introduced the notion of a *bi*-BL-algebra and study it in detail. After that the notions of a *bi*-filter, *bi*-deductive system and *bi*-Boolean center of a *bi*-BL-algebra are introduced. We have also presented classes of *bi*-BL-algebras and we stated relation between *bi*-filters and quotient *bi*-BL-algebra. Finally we show that the set of all deductive systems of a *bi*-BL-algebra together with inclusion relation is not a lattice.

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