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# LEAPING CONVERGENTS OF TASOEV CONTINUED FRACTIONS

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**Abstract**

Denote the $n$-th convergent of the continued fraction by $p_n/q_n = [a_0; a_1, \dots, a_n]$. We give some explicit forms of leaping convergents of Tasoev continued fractions. For instance, $[0; ua, ua^2, ua^3, \dots]$ is one of the typical types of Tasoev continued fractions. Leaping convergents are of the form $p_{rn+i}/q_{rn+i}$ $(n = 0, 1, 2, \dots)$ for fixed integers $r \geq 2$ and $0 \leq i \leq r-1$.

**Keywords:** leaping convergents, Tasoev continued fractions.

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## 1. Hurwitz and Tasoev continued fractions

Let $\alpha = [a_0; a_1, a_2, \dots]$ denote the regular (or simple) continued fraction expansion of a real number $\alpha$, where

$$\begin{aligned} \alpha &= a_0 + 1/\alpha_1, & a_0 &= \lfloor \alpha \rfloor, \\ \alpha_n &= a_n + 1/\alpha_{n+1}, & a_n &= \lfloor \alpha_n \rfloor \quad (n \geq 1). \end{aligned}$$

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Hurwitz continued fraction expansions, quasi-periodic simple continued fractions, have the form

$$
\begin{aligned}
&[a_0; a_1, \dots, a_n, \overline{Q_1(k), \dots, Q_p(k)}]_{k=1}^{\infty} \\
&= [a_0; a_1, \dots, a_n, Q_1(1), \dots, Q_p(1), Q_1(2), \dots, Q_p(2), Q_1(3), \dots\,], \tag{1}
\end{aligned}
$$

where $a_0$ is an integer, $a_1, \dots, a_n$ are positive integers, $Q_1, \dots, Q_p$ are polynomials with rational coefficients which take positive integral values for $k = 1, 2, \dots$ and at least one of the polynomials is not constant. Well-known examples are

$$
\begin{aligned}
e &= [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, \dots\,] = [2; \overline{1, 2k, 1}]_{k=1}^{\infty}\,, \\
\tanh 1 &= \frac{e^2 - 1}{e^2 + 1} = [0; 1, 3, 5, 7, \dots\,] = [0; \overline{2k-1}]_{k=1}^{\infty}\,, \\
\tan 1 &= [1; 1, 1, 3, 1, 5, 1, \dots\,] = [1; \overline{2k-1, 1}]_{k=1}^{\infty}\,.
\end{aligned}
$$

It seems that every known example belongs to one of the three types, $e$-type, *tanh*-type and tan-type. No concrete example where the degree of any polynomial exceeds 1 has been given.

Recently, the author [5] found more general forms of Hurwitz continued fractions belonging to *tanh*-type and *tan*-type. Namely,

$$
\begin{aligned}
&[0; ua, v(a+b), u(a+2b), v(a+3b), u(a+4b), v(a+5b), \dots\,] \\
&= \frac{\sum_{n=0}^{\infty} (n!)^{-1} u^{-n-1} (vb)^{-n} \prod_{i=1}^{n} (a+bi)^{-1}}{\sum_{n=0}^{\infty} (n!)^{-1} (uvb)^{-n} \prod_{i=1}^{n-1} (a+bi)^{-1}} \tag{2}
\end{aligned}
$$

and

$$
\begin{aligned}
&[0; ua-1, 1, v(a+b)-2, 1, u(a+2b)-2, 1, v(a+3b)-2, 1, \dots\,] \\
&= \frac{\sum_{n=0}^{\infty} (-1)^n (n!)^{-1} u^{-n-1} (vb)^{-n} \prod_{i=1}^{n} (a+bi)^{-1}}{\sum_{n=0}^{\infty} (-1)^n (n!)^{-1} (uvb)^{-n} \prod_{i=1}^{n-1} (a+bi)^{-1}}\,, \tag{3}
\end{aligned}
$$

respectively. In [9], the author constituted more general forms of Hurwitz continued fractions of $e$-type, namely, the quasi-periodic continued fractions with period 3 whose partial quotients include at least one 1;

$$
\begin{aligned}
&[0;\overline{u(a+bk)-1,1,v-1}]_{k=1}^{\infty} \\
(4)\qquad &= \frac{\displaystyle\sum_{n=0}^{\infty} u^{-2n-1}v^{-2n}b^{-n}(n!)^{-1}\prod_{i=1}^{n+1}(a+bi)^{-1}}{\displaystyle\sum_{n=0}^{\infty} b^{-n}(n!)^{-1}\Bigl((uv)^{-2n}\prod_{i=1}^{n}(a+bi)^{-1}-(uv)^{-2n-1}\prod_{i=1}^{n+1}(a+bi)^{-1}\Bigr)}
\end{aligned}
$$

and

$$
\begin{aligned}
(5)\qquad &[0;\overline{v-1,1,u(a+bk)-1}]_{k=1}^{\infty} \\
&= \frac{\displaystyle\sum_{n=0}^{\infty} b^{-n}(n!)^{-1}\Bigl(u^{-2n}v^{-2n-1}\prod_{i=1}^{n}(a+bi)^{-1}+u^{-2n-1}v^{-2n-2}\prod_{i=1}^{n+1}(a+bi)^{-1}\Bigr)}{\displaystyle\sum_{n=0}^{\infty}(uv)^{-2n}b^{-n}(n!)^{-1}\prod_{i=1}^{n}(a+bi)^{-1}}\,.
\end{aligned}
$$

Tasoev continued fractions ([4, 5, 8, 9, 10, 18]) are also systematic but have hardly been known before. They are also quasi-periodic but at least one of $Q_j(k)$'s in (1) includes exponentials in $k$ instead of polynomials. In [5], the author found some more general Tasoev continued fractions. Namely,

$$
(6)\qquad [0;\overline{ua^k}]_{k=1}^{\infty} = \frac{\displaystyle\sum_{n=0}^{\infty} u^{-2n-1}a^{-(n+1)^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\displaystyle\sum_{n=0}^{\infty} u^{-2n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}\,,
$$

$$
(7)\qquad [0;ua-1,\overline{1,ua^{k+1}-2}]_{k=1}^{\infty} = \frac{\displaystyle\sum_{n=0}^{\infty}(-1)^n u^{-2n-1}a^{-(n+1)^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\displaystyle\sum_{n=0}^{\infty}(-1)^n u^{-2n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}\,,
$$

$$
(8)\qquad [0;\overline{ua^k,va^k}]_{k=1}^{\infty} = \frac{\displaystyle\sum_{n=0}^{\infty} u^{-n-1}v^{-n}a^{-(n+1)(n+2)/2}\prod_{i=1}^{n}(a^{i}-1)^{-1}}{\displaystyle\sum_{n=0}^{\infty} u^{-n}v^{-n}a^{-n(n+1)/2}\prod_{i=1}^{n}(a^{i}-1)^{-1}}
$$

and

$$
\begin{aligned}
&[0; ua-1, 1, va-2, \overline{1, ua^{k+1}-2, 1, va^{k+1}-2}]_{k=1}^{\infty} \\
&= \frac{\sum_{n=0}^{\infty} (-1)^n u^{-n-1} v^{-n} a^{-(n+1)(n+2)/2} \prod_{i=1}^{n} (a^i-1)^{-1}}{\sum_{n=0}^{\infty} (-1)^n u^{-n} v^{-n} a^{-n(n+1)/2} \prod_{i=1}^{n} (a^i-1)^{-1}} .
\end{aligned} \tag{9}
$$

We can safely say that Tasoev continued fractions are geometric and Hurwitz continued fractions are arithmetic ([8]). The Tasoev continued fractions corresponding to $e$-type Hurwitz continued fractions were also derived in [9];

$$
\begin{aligned}
&[0; \overline{ua^k-1, 1, v-1}]_{k=1}^{\infty} \\
&= \frac{\sum_{n=0}^{\infty} u^{-2n-1} v^{-2n} a^{-(n+1)^2} \prod_{i=1}^{n} (a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty} \left((uv)^{-2n} a^{-n^2} - (uv)^{-2n-1} a^{-(n+1)^2}\right) \prod_{i=1}^{n} (a^{2i}-1)^{-1}}
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
&[0; \overline{v-1, 1, ua^k-1}]_{k=1}^{\infty} \\
&= \frac{\sum_{n=0}^{\infty} \left(u^{-2n} v^{-2n-1} a^{-n^2} + u^{-2n-1} v^{-2n-2} a^{-(n+1)^2}\right) \prod_{i=1}^{n} (a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty} (uv)^{-2n} a^{-n^2} \prod_{i=1}^{n} (a^{2i}-1)^{-1}} .
\end{aligned} \tag{11}
$$

The different types of Tasoev continued fractions with period 3 shown in [8] are

$$
\begin{aligned}
&[0; \overline{ua^{2k-1}-1, 1, va^{2k}-1}]_{k=1}^{\infty} \\
&= \frac{\sum_{n=0}^{\infty} u^{-n-1} v^{-n} a^{-(n+1)^2} \prod_{i=1}^{n} (a^{2i}-(-1)^i)^{-1}}{\sum_{n=0}^{\infty} (-1)^n u^{-n} v^{-n} a^{-n^2} \prod_{i=1}^{n} (a^{2i}-(-1)^i)^{-1}} ,
\end{aligned} \tag{12}
$$

$$\begin{aligned}&[0;ua,\overline{va^{2k}-1,1,ua^{2k+1}-1}]_{k=1}^\infty\\ &=\frac{\sum_{n=0}^\infty(-1)^n u^{-n-1}v^{-n}a^{-(n+1)^2}\prod_{i=1}^n(a^{2i}-(-1)^i)^{-1}}{\sum_{n=0}^\infty u^{-n}v^{-n}a^{-n^2}\prod_{i=1}^n(a^{2i}-(-1)^i)^{-1}}\,,\end{aligned}\tag{13}$$

$$\begin{aligned}&[0;\overline{ua^k-1,1,va^k-1}]_{k=1}^\infty\\ &=\frac{\sum_{n=0}^\infty u^{-n-1}v^{-n}a^{-(n+1)(n+2)/2}\prod_{i=1}^n(a^i-(-1)^i)^{-1}}{\sum_{n=0}^\infty(-1)^n u^{-n}v^{-n}a^{-n(n+1)/2}\prod_{i=1}^n(a^i-(-1)^i)^{-1}}\end{aligned}\tag{14}$$

and

$$\begin{aligned}&[0;ua,\overline{va^k-1,1,ua^{k+1}-1}]_{k=1}^\infty\\ &=\frac{\sum_{n=0}^\infty(-1)^n u^{-n-1}v^{-n}a^{-(n+1)(n+2)/2}\prod_{i=1}^n(a^i-(-1)^i)^{-1}}{\sum_{n=0}^\infty u^{-n}v^{-n}a^{-n(n+1)/2}\prod_{i=1}^n(a^i-(-1)^i)^{-1}}\,.\end{aligned}\tag{15}$$

## 2. Leaping convergents

*Leaping convergents* are every $r$-th convergent of the convergents $p_n/q_n = [a_0;a_1,a_2,\dots,a_n]$. Namely, Leaping convergents are of the form $p_{rn+i}/q_{rn+i}$ ($n=0,1,2,\dots$) for fixed integers $r\ge 2$ and $0\le i\le r-1$. Some explicit forms of the leaping convergents of Hurwitz continued fractions have been known. For example, if $p_n/q_n$ and $p_n^*/q_n^*$ are the $n$-th convergent of the continued fraction $e^{1/s}=[1;\overline{(2k-1)s,1,1}\,]_{k=1}^\infty$ and $e=[2;\overline{1,2k,1}\,]_{k=1}^\infty$, respectively, then

$$p_{3n}=p_{3n-2}^*=\sum_{k=0}^n\frac{(2n-k)!}{k!(n-k)!}s^{n-k},$$
$$q_{3n}=q_{3n-2}^*=\sum_{k=0}^n(-1)^k\frac{(2n-k)!}{k!(n-k)!}s^{n-k},$$

$$p_{3n-1} = p^*_{3n-3} = n \sum_{k=0}^{n} \frac{(2n-k-1)!}{k!(n-k)!} s^{n-k},$$
$$q_{3n-1} = q^*_{3n-3} = \sum_{k=0}^{n-1} (-1)^k \frac{(2n-k-1)!}{k!(n-k-1)!} s^{n-k},$$

$$p_{3n-2} = p^*_{3n-4} = \sum_{k=0}^{n-1} \frac{(2n-k-1)!}{k!(n-k-1)!} s^{n-k},$$
$$q_{3n-2} = q^*_{3n-4} = n \sum_{k=0}^{n} (-1)^k \frac{(2n-k-1)!}{k!(n-k)!} s^{n-k}.$$

(See e.g. [11, 12, 13, 14]). The study of leaping convergents of $e^{1/s}$ has been initiated by Elsner in the case of $s = 1$ ([1]) and by the author in the case of $s \geq 2$ ([6, 7]).

Such explicit forms are useful to lead varieties of applications. In fact, explicit forms of Hurwitz continued fractions have already yielded several interesting applications. In [2, 3] we give Diophantine approximations of values of hypergeometric functions formed from Diophantine equations. In [16, 17] we give several Diophantine approximations of $tanh$, $tan$, and some linear forms of $e$ in terms of integrals. In [15] we give some new non-regular continued fraction expansions under the aspect of $N$ continued fractions. These results are obtained by using some explicit forms of Hurwitz continued fractions.

In this paper, we shall give some explicit forms of leaping convergents of Tasoev continued fractions.

## 3. Main results

Let $p_n/q_n$ be the $n$-th convergent of the Tasoev continued fraction

$$[0; \overline{ua^{2k-1}, va^{2k}}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} u^{-n-1} v^{-n} a^{-(n+1)^2} \prod_{i=1}^{n} (a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty} u^{-n} v^{-n} a^{-n^2} \prod_{i=1}^{n} (a^{2i}-1)^{-1}}.$$

**Theorem 1.** *For $n \ge 0$,*

$$p_n = \sum_{\nu=1}^{\lceil \frac{n}{2} \rceil} u^{\lceil \frac{n}{2} \rceil - \nu} v^{\lceil \frac{n+1}{2} \rceil - \nu} a^{\frac{n(n+1)}{2} - 2(\nu-1)n - \nu^2} \prod_{i=1}^{\nu-1} \frac{a^{2n} - a^{2(\nu+i-1)}}{a^{2i} - 1},$$

$$q_n = \sum_{\nu=1}^{\lfloor \frac{n}{2} \rfloor + 1} u^{\lceil \frac{n}{2} \rceil - \nu + 1} v^{\lceil \frac{n+1}{2} \rceil - \nu} a^{\frac{n(n+1)}{2} - 2(\nu-1)n - (\nu-1)^2} \prod_{i=1}^{\nu-1} \frac{a^{2n} - a^{2(\nu+i-2)}}{a^{2i} - 1}.$$

**Remark 1.** If $u = v$, then these identities become those of (6). If $u$ is replaced by $ua$ and $a$ is replaced by $a^{1/2}$, then these become those of (8).

***Proof.*** We shall prove the identities by induction. For simplicity, put

$$P(n) = \sum_{\nu=1}^{\lceil \frac{n}{2} \rceil} u^{\lceil \frac{n}{2} \rceil - \nu} v^{\lceil \frac{n+1}{2} \rceil - \nu} a^{\frac{n(n+1)}{2} - 2(\nu-1)n - \nu^2} \prod_{i=1}^{\nu-1} \frac{a^{2n} - a^{2(\nu+i-1)}}{a^{2i} - 1},$$

$$Q(n) = \sum_{\nu=1}^{\lfloor \frac{n}{2} \rfloor + 1} u^{\lceil \frac{n}{2} \rceil - \nu + 1} v^{\lceil \frac{n+1}{2} \rceil - \nu} a^{\frac{n(n+1)}{2} - 2(\nu-1)n - (\nu-1)^2} \prod_{i=1}^{\nu-1} \frac{a^{2n} - a^{2(\nu+i-2)}}{a^{2i} - 1}.$$

It is easy to see that $p_0 = 0 = P(0)$, $p_1 = 1 = P(1)$, $q(0) = 1 = Q(0)$ and $q(1) = ua = Q(1)$. Assume that $p_{2n-1} = P(2n-1)$ and $p_{2n-2} = P(2n-2)$. Then

$$\begin{aligned}
p_{2n} &= va^{2n} p_{2n-1} + p_{2n-2} \\
&= va^{2n} \sum_{\nu=1}^{n} u^{n-\nu} v^{n-\nu} a^{n(2n-1) - 2(\nu-1)(2n-1) - \nu^2} \prod_{i=1}^{\nu-1} \frac{a^{4n-2} - a^{2(\nu+i-1)}}{a^{2i} - 1} \\
&\quad + \sum_{\nu=1}^{n-1} u^{n-\nu-1} v^{n-\nu} a^{(n-1)(2n-1) - 2(\nu-1)(2n-2) - \nu^2} \prod_{i=1}^{\nu-1} \frac{a^{4n-4} - a^{2(\nu+i-1)}}{a^{2i} - 1} \\
&= \sum_{\nu=1}^{n} u^{n-\nu} v^{n-\nu+1} a^{n(2n+1) - 4(\nu-1)n - \nu^2} a^{2(\nu-1)} \prod_{i=1}^{\nu-1} \frac{a^{4n-2} - a^{2(\nu+i-1)}}{a^{2i} - 1} \\
&\quad + \sum_{\nu=2}^{n} u^{n-\nu} v^{n-\nu+1} a^{(n-1)(2n-1) - 2(\nu-2)(2n-2) - (\nu-1)^2} \prod_{i=1}^{\nu-2} \frac{a^{4n-4} - a^{2(\nu+i-2)}}{a^{2i} - 1}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\nu=1}^{n} u^{n-\nu} v^{n-\nu+1} a^{n(2n+1)-4(\nu-1)n-\nu^2} \big( (a^{4n} - a^{2(2\nu-1)}) + a^{2\nu}(a^{2(\nu-1)} - 1) \big) \\
&\quad \times \frac{(a^{4n} - a^{2(\nu+1)})(a^{4n} - a^{2(\nu+2)}) \dots (a^{4n} - a^{2(2\nu-2)})}{(a^2-1)(a^4-1)\dots(a^{2(\nu-1)}-1)} \\
&= \sum_{\nu=1}^{n} u^{n-\nu} v^{n-\nu+1} a^{n(2n+1)-4(\nu-1)n-\nu^2} \prod_{i=1}^{\nu-1} \frac{a^{4n} - a^{2(\nu+i-1)}}{a^{2i}-1} \\
&= P(2n)\,.
\end{aligned}
$$

Next, assume that $p_{2n} = P(2n)$ and $p_{2n-1} = P(2n-1)$. Then

$$
\begin{aligned}
p_{2n+1} &= u a^{2n+1} p_{2n} + p_{2n-1} \\
&= u a^{2n+1} \sum_{\nu=1}^{n} u^{n-\nu} v^{n-\nu+1} a^{n(2n+1)-4(\nu-1)n-\nu^2} \prod_{i=1}^{\nu-1} \frac{a^{4n} - a^{2(\nu+i-1)}}{a^{2i}-1} \\
&\quad + \sum_{\nu=1}^{n} u^{n-\nu} v^{n-\nu} a^{n(2n-1)-2(\nu-1)(2n-1)-\nu^2} \prod_{i=1}^{\nu-1} \frac{a^{4n-2} - a^{2(\nu+i-1)}}{a^{2i}-1} \\
&= \sum_{\nu=1}^{n} (uv)^{n-\nu+1} a^{(n+1)(2n+1)-2(\nu-1)(2n+1)-\nu^2} a^{2(\nu-1)} \prod_{i=1}^{\nu-1} \frac{a^{4n} - a^{2(\nu+i-1)}}{a^{2i}-1} \\
&\quad + \sum_{\nu=2}^{n+1} (uv)^{n-\nu+1} a^{n(2n-1)-2(\nu-2)(2n-1)-(\nu-1)^2} \prod_{i=1}^{\nu-2} \frac{a^{2(2n-1)} - a^{2(\nu+i-2)}}{a^{2i}-1} \\
&= \sum_{\nu=1}^{n+1} (uv)^{n-\nu+1} a^{(n+1)(2n+1)-2(\nu-1)(2n+1)-\nu^2} \\
&\quad \times \big( (a^{4n+2} - a^{2(2\nu-1)}) + a^{2\nu}(a^{2(\nu-1)} - 1) \big) \frac{\prod_{i=1}^{\nu-2} (a^{4n+2} - a^{2(\nu+i)})}{\prod_{i=1}^{\nu-1} (a^{2i}-1)} \\
&= \sum_{\nu=1}^{n+1} (uv)^{n-\nu+1} a^{(n+1)(2n+1)-2(\nu-1)(2n+1)-\nu^2} \prod_{i=1}^{\nu-1} \frac{a^{2(2n+1)} - a^{2(\nu+i-1)}}{a^{2i}-1} \\
&= P(2n+1)\,.
\end{aligned}
$$

In similar manners, we can prove that if $q_{2n-1} = Q(2n-1)$ and $q_{2n-2} = Q(2n-2)$ then $q_{2n} = Q(2n)$, and if $q_{2n} = Q(2n)$ and $q_{2n-1} = Q(2n-1)$ then $q_{2n+1} = Q(2n+1)$. ■

We can rewrite the formulas as follows.

**Corollary 1.** *For $n \ge 0$,*

$$\begin{aligned} p_{2n} &= \sum_{\nu=0}^{n-1} u^\nu v^{\nu+1} a^{(\nu+2)(2\nu+1)} \prod_{i=1}^{n-\nu-1} \frac{a^{2(2\nu+i+1)}-1}{a^{2i}-1}\,, \\ p_{2n+1} &= \sum_{\nu=0}^{n} (uv)^\nu a^{\nu(2\nu+3)} \prod_{i=1}^{n-\nu} \frac{a^{2(2\nu+i)}-1}{a^{2i}-1}\,, \\ q_{2n} &= \sum_{\nu=0}^{n} (uv)^\nu a^{\nu(2\nu+1)} \prod_{i=1}^{n-\nu} \frac{a^{2(2\nu+i)}-1}{a^{2i}-1}\,, \\ q_{2n+1} &= \sum_{\nu=0}^{n} u^{\nu+1} v^\nu a^{(\nu+1)(2\nu+1)} \prod_{i=1}^{n-\nu} \frac{a^{2(2\nu+i+1)}-1}{a^{2i}-1}\,. \end{aligned}$$

Let $p_n/q_n$ be the $n$-th convergent of the Tasoev continued fraction

$$\begin{aligned} &[0; ua-1, \overline{1, va^{2k}-2, 1, ua^{2k+1}-2}\,]_{k=1}^\infty \\ &= \frac{\sum_{n=0}^\infty (-1)^n u^{-n-1} v^{-n} a^{-(n+1)^2} \prod_{i=1}^n (a^{2i}-1)^{-1}}{\sum_{n=0}^\infty (-1)^n u^{-n} v^{-n} a^{-n^2} \prod_{i=1}^n (a^{2i}-1)^{-1}}\,. \end{aligned}$$

For simplicity, put

$$P^*(n) =$$

$$= \sum_{\nu=1}^{\lceil \frac{n}{2} \rceil} (-1)^{\nu-1} u^{\lceil \frac{n}{2} \rceil - \nu} v^{\lceil \frac{n+1}{2} \rceil - \nu} a^{\frac{n(n+1)}{2} - 2(\nu-1)n - \nu^2} \prod_{i=1}^{\nu-1} \frac{a^{2n} - a^{2(\nu+i-1)}}{a^{2i}-1}\,,$$

$$Q^*(n) =$$

$$= \sum_{\nu=1}^{\lfloor \frac{n}{2} \rfloor + 1} (-1)^{\nu-1} u^{\lceil \frac{n}{2} \rceil - \nu + 1} v^{\lceil \frac{n+1}{2} \rceil - \nu} a^{\frac{n(n+1)}{2} - 2(\nu-1)n - (\nu-1)^2} \prod_{i=1}^{\nu-1} \frac{a^{2n} - a^{2(\nu+i-2)}}{a^{2i}-1}\,.$$

**Theorem 2.** *For $n \geq 0$ we have*

$$p_{2n} = P^*(n)\,, \qquad p_{2n+1} = P^*(n+1) - P^*(n)\,,$$
$$q_{2n} = Q^*(n)\,, \qquad q_{2n+1} = Q^*(n+1) - Q^*(n)\,.$$

**Remark 2.** If $u = v$, then these identities become those of (7). If $u$ is replaced by $ua$ and $a$ is replaced by $a^{1/2}$, then these become those of (9).

***Proof.*** The proof is similar to the previous one and omitted. ■

We can rewrite the formulas as follows.

**Corollary 2.** *For $n \geq 0$,*

$$P^*(2n) = \sum_{\nu=0}^{n-1} (-1)^{n-\nu+1} u^\nu v^{\nu+1} a^{(\nu+2)(2\nu+1)} \prod_{i=1}^{n-\nu-1} \frac{a^{2(2\nu+i+1)} - 1}{a^{2i} - 1}\,,$$
$$P^*(2n+1) = \sum_{\nu=0}^{n} (-1)^{n-\nu} (uv)^\nu a^{\nu(2\nu+3)} \prod_{i=1}^{n-\nu} \frac{a^{2(2\nu+i)} - 1}{a^{2i} - 1}\,,$$
$$Q^*(2n) = \sum_{\nu=0}^{n} (-1)^{n-\nu} (uv)^\nu a^{\nu(2\nu+1)} \prod_{i=1}^{n-\nu} \frac{a^{2(2\nu+i)} - 1}{a^{2i} - 1}\,,$$
$$Q^*(2n+1) = \sum_{\nu=0}^{n} (-1)^{n-\nu} u^{\nu+1} v^\nu a^{(\nu+1)(2\nu+1)} \prod_{i=1}^{n-\nu} \frac{a^{2(2\nu+i+1)} - 1}{a^{2i} - 1}\,.$$

## 4. Explicit forms of $e$-type Tasoev continued fractions

In this section, we shall show some explicit forms of the leaping convergents of $e$-type Tasoev continued fractions. Proofs are similarly done by induction and omitted.

Let $p_n/q_n$ be the $n$-th convergent of the Tasoev continued fraction (10):

$$[0; \overline{ua^k - 1, 1, v - 1}]_{k=1}^\infty$$
$$= \frac{\sum_{n=0}^\infty u^{-2n-1} v^{-2n} a^{-(n+1)^2} \prod_{i=1}^n (a^{2i} - 1)^{-1}}{\sum_{n=0}^\infty \big((uv)^{-2n} a^{-n^2} - (uv)^{-2n-1} a^{-(n+1)^2}\big) \prod_{i=1}^n (a^{2i} - 1)^{-1}}\,.$$

**Theorem 3.** *For $n \geq 0$ we have*

$$
\begin{aligned}
p_{3n} &= \sum_{\nu=1}^{\lceil \frac{n}{2} \rceil} u^{n-2\nu+1} v^{n-2\nu+2} a^{\frac{n(n+1)}{2}-2(\nu-1)n-\nu^2} \prod_{i=1}^{\nu-1} \frac{a^{2n}-a^{2(\nu+i-1)}}{a^{2i}-1}, \\
p_{3n+2} &= \sum_{\nu=1}^{n+1} (uv)^{n-\nu+1} a^{\frac{(n-\nu+2)(n-\nu+3)}{2}-1} \prod_{i=1}^{\lceil \nu/2 \rceil -1} \frac{a^{2(n-\nu+i+1)}-1}{a^{2i}-1} \\
&= \sum_{\nu=1}^{n+1} (uv)^{n-\nu+1} a^{\frac{n(n+1)}{2}-(\nu-2)n-\lfloor \frac{\nu+1}{2} \rfloor^2+\frac{1-(-1)^\nu}{2}} \prod_{i=1}^{\lfloor \frac{\nu-1}{2} \rfloor} \frac{a^{2n}-a^{2(\lfloor \frac{\nu}{2} \rfloor+i-1)}}{a^{2i}-1}, \\
q_{3n} &= \sum_{\kappa=0}^{\lfloor \frac{n}{2} \rfloor} (uv)^{n-2\kappa} a^{\frac{n(n+1)}{2}-2\kappa n-\kappa^2} \prod_{i=1}^{\kappa} \frac{a^{2n}-a^{2(\kappa+i-1)}}{a^{2i}-1} \\
&\quad - \sum_{\kappa=0}^{\lfloor \frac{n-1}{2} \rfloor} (uv)^{n-2\kappa-1} a^{\frac{n(n+1)}{2}-2\kappa n-(\kappa+1)^2} \prod_{i=1}^{\kappa} \frac{a^{2n}-a^{2(\kappa+i)}}{a^{2i}-1} \\
&= \sum_{\nu=1}^{n+1} (-1)^{\nu-1} (uv)^{n-\nu+1} a^{\frac{n(n+1)}{2}-2\lfloor \frac{\nu-1}{2} \rfloor n-\lfloor \frac{\nu}{2} \rfloor^2} \prod_{i=1}^{\lfloor \frac{\nu-1}{2} \rfloor} \frac{a^{2n}-a^{2(\lfloor \frac{\nu}{2} \rfloor+i-1)}}{a^{2i}-1}, \\
q_{3n+2} &= \sum_{\kappa=0}^{\lfloor \frac{n}{2} \rfloor} u^{n-2\kappa+1} v^{n-2\kappa} a^{\frac{(n+1)(n+2)}{2}-2\kappa n-\kappa^2} \prod_{i=1}^{\kappa} \frac{a^{2n}-a^{2(\kappa+i-2)}}{a^{2i}-1} \\
&\quad + \sum_{\kappa=0}^{\lfloor \frac{n-1}{2} \rfloor} u^{n-2\kappa} v^{n-2\kappa-1} a^{\frac{n(n+1)}{2}-2\kappa n-\kappa^2} \prod_{i=1}^{\kappa} \frac{a^{2n}-a^{2(\kappa+i-1)}}{a^{2i}-1} \\
&\quad - \sum_{\kappa=0}^{\lfloor \frac{n-1}{2} \rfloor} u^{n-2\kappa} v^{n-2\kappa-1} a^{\frac{(n+1)(n+2)}{2}-2\kappa n-(\kappa+1)^2} \prod_{i=1}^{\kappa} \frac{a^{2n}-a^{2(\kappa+i-1)}}{a^{2i}-1} \\
&\quad - \sum_{\kappa=0}^{\lfloor \frac{n-2}{2} \rfloor} u^{n-2\kappa-1} v^{n-2\kappa-2} a^{\frac{n(n+1)}{2}-2\kappa n-(\kappa+1)^2} \prod_{i=1}^{\kappa} \frac{a^{2n}-a^{2(\kappa+i)}}{a^{2i}-1}.
\end{aligned}
$$

**Remark 3.** The forms of $p_{3n+1}$ and $q_{3n+1}$ are not simpler, and are obtained by using the recurrence relations: $p_{3n+1} = p_{3n+2} - p_{3n}$ and $q_{3n+1} = q_{3n+2} - q_{3n}$.

Let $p_n/q_n$ be the $n$-th convergent of the Tasoev continued fraction (11):

$$
\begin{aligned}
&[0;\overline{v-1,1,ua^k-1}]_{k=1}^{\infty}\\
&=\frac{\sum_{n=0}^{\infty}\left(u^{-2n}v^{-2n-1}a^{-n^2}+u^{-2n-1}v^{-2n-2}a^{-(n+1)^2}\right)\prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}(uv)^{-2n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}\,.
\end{aligned}
$$

**Theorem 4.** *For $n\geq 0$ we have*

$$
\begin{aligned}
p_{3n}=&\sum_{\kappa=0}^{\lfloor\frac{n-1}{2}\rfloor}u^{n-2\kappa}v^{n-2\kappa-1}a^{\frac{n(n+1)}{2}-2\kappa n-\kappa^2}\prod_{i=1}^{\kappa}\frac{a^{2n}-a^{2(\kappa+i-1)}}{a^{2i}-1}\\
&+\sum_{\kappa=0}^{\lfloor\frac{n-2}{2}\rfloor}u^{n-2\kappa-1}v^{n-2\kappa-2}a^{\frac{n(n+1)}{2}-2\kappa n-(\kappa+1)^2}\prod_{i=1}^{\kappa}\frac{a^{2n}-a^{2(\kappa+i)}}{a^{2i}-1}\\
&-\sum_{\kappa=0}^{\lfloor\frac{n-2}{2}\rfloor}u^{n-2\kappa-1}v^{n-2\kappa-2}a^{\frac{n(n-1)}{2}-2\kappa n-\kappa^2}\prod_{i=1}^{\kappa}\frac{a^{2n}-a^{2(\kappa+i)}}{a^{2i}-1}\\
&-\sum_{\kappa=0}^{\lfloor\frac{n-3}{2}\rfloor}u^{n-2\kappa-2}v^{n-2\kappa-3}a^{\frac{n(n-1)}{2}-2\kappa n-(\kappa+1)^2}\prod_{i=1}^{\kappa}\frac{a^{2n}-a^{2(\kappa+i+1)}}{a^{2i}-1}\,,\\
p_{3n+2}=&\sum_{\kappa=0}^{\lfloor\frac{n}{2}\rfloor}(uv)^{n-2\kappa}a^{\frac{n(n+1)}{2}-2\kappa n-\kappa^2}\prod_{i=1}^{\kappa}\frac{a^{2n}-a^{2(\kappa+i-1)}}{a^{2i}-1}\\
&+\sum_{\kappa=0}^{\lfloor\frac{n-1}{2}\rfloor}(uv)^{n-2\kappa-1}a^{\frac{n(n+1)}{2}-2\kappa n-(\kappa+1)^2}\prod_{i=1}^{\kappa}\frac{a^{2n}-a^{2(\kappa+i)}}{a^{2i}-1}\\
=&\sum_{\nu=1}^{n+1}(uv)^{n-\nu+1}a^{\frac{n(n+1)}{2}-2\lfloor\frac{\nu-1}{2}\rfloor n-\lfloor\frac{\nu}{2}\rfloor^2}\prod_{i=1}^{\lfloor\frac{\nu-1}{2}\rfloor}\frac{a^{2n}-a^{2(\lfloor\frac{\nu}{2}\rfloor+i-1)}}{a^{2i}-1}\,,\\
q_{3n}=&\sum_{\kappa=0}^{\lfloor\frac{n}{2}\rfloor}(uv)^{n-2\kappa}a^{\frac{n(n+1)}{2}-2\kappa n-\kappa^2}\prod_{i=1}^{\kappa}\frac{a^{2n}-a^{2(\kappa+i-1)}}{a^{2i}-1}
\end{aligned}
$$

$$
- \sum_{\kappa=0}^{\lfloor \frac{n-1}{2} \rfloor} (uv)^{n-2\kappa-1} a^{\frac{n(n-1)}{2}-2\kappa n-\kappa^2} \prod_{i=1}^{\kappa} \frac{a^{2n} - a^{2(\kappa+i)}}{a^{2i}-1}
$$

$$
= \sum_{\nu=1}^{n+1} (-1)^{\nu-1} (uv)^{n-\nu+1} a^{\frac{n(n+1)}{2}-(\nu-1)n-\lfloor \frac{\nu-1}{2} \rfloor^2} \prod_{i=1}^{\lfloor \frac{\nu-1}{2} \rfloor} \frac{a^{2n} - a^{2(\lfloor \frac{\nu}{2} \rfloor+i-1)}}{a^{2i}-1}\,,
$$

$$
q_{3n+2} = \sum_{\nu=1}^{\lceil \frac{n+1}{2} \rceil} u^{n-2\nu+2} v^{n-2\nu+3} a^{\frac{n(n+5)}{2}-2\nu n-(\nu-1)^2} \prod_{i=1}^{\nu-1} \frac{a^{2n} - a^{2(\nu+i-2)}}{a^{2i}-1}\,.
$$

**Remark 4.** The forms of $p_{3n+1}$ and $q_{3n+1}$ are not simpler, and are obtained by using the recurrence relations: $p_{3n+1} = p_{3n+2} - p_{3n}$ and $q_{3n+1} = q_{3n+2} - q_{3n}$.

Let $p_n/q_n$ be the $n$-th convergent of the Tasoev continued fraction (12):

$$
[0; \overline{ua^{2k-1}-1, 1, va^{2k}-1}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} u^{-n-1} v^{-n} a^{-(n+1)^2} \prod_{i=1}^{n} (a^{2i} - (-1)^i)^{-1}}{\sum_{n=0}^{\infty} (-1)^n u^{-n} v^{-n} a^{-n^2} \prod_{i=1}^{n} (a^{2i} - (-1)^i)^{-1}}\,.
$$

**Theorem 5.** *For $n \ge 0$ we have*

$$
p_{3n} = \\
= \sum_{\nu=1}^{n} u^{n-\nu} v^{n-\nu+1} a^{n(2n+1)-4(\nu-1)n-\nu^2} \prod_{i=1}^{\nu-1} \frac{a^{4n} - (-1)^{\nu+i-1} a^{2(\nu+i-1)}}{a^{2i} - (-1)^i}\,,
$$

$$
p_{3n+2} = \\
= \sum_{\nu=1}^{n+1} (uv)^{n-\nu+1} a^{(2n+1)(n+1)-2(\nu-1)(2n+1)-\nu^2} \prod_{i=1}^{\nu-1} \frac{a^{4n+2} - (-1)^{\nu+i} a^{2(\nu+i-1)}}{a^{2i} - (-1)^i}\,,
$$

$$
q_{3n} = \\
= \sum_{\nu=1}^{n+1} (-1)^{\nu-1} (uv)^{n-\nu+1} a^{n(2n+1)-4(\nu-1)n-(\nu-1)^2} \prod_{i=1}^{\nu-1} \frac{a^{4n} - (-1)^{\nu+i} a^{2(\nu+i-2)}}{a^{2i} - (-1)^i}\,,
$$

$$q_{3n+2}=\sum_{\nu=1}^{n+1}(-1)^{\nu-1}u^{n-\nu+2}v^{n-\nu+1}a^{(2n+1)(n+1)-2(\nu-1)(2n+1)-(\nu-1)^2}\times\prod_{i=1}^{\nu-1}\frac{a^{4n+2}-(-1)^{\nu+i-1}a^{2(\nu+i-2)}}{a^{2i}-(-1)^i}.$$

**Remark 5.** The forms of $p_{3n+1}$ and $q_{3n+1}$ are not simpler, and are obtained by using the recurrence relations: $p_{3n+1}=p_{3n+2}-p_{3n}$ and $q_{3n+1}=q_{3n+2}-q_{3n}$.

If $u$ is replaced by $ua$ and $a$ is replaced by $a^{1/2}$, then these become those of (14).

Let $p_n/q_n$ be the $n$-th convergent of the Tasoev continued fraction (13):

$$[0;ua,\overline{va^{2k}-1,1,ua^{2k+1}-1}]_{k=1}^{\infty}=\frac{\sum_{n=0}^{\infty}(-1)^n u^{-n-1}v^{-n}a^{-(n+1)^2}\prod_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}}{\sum_{n=0}^{\infty}u^{-n}v^{-n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}}.$$

**Theorem 6.** *For $n\ge 0$ we have*

$$p_{3n}=\sum_{\nu=1}^{n}(-1)^{\nu-1}u^{n-\nu}v^{n-\nu+1}a^{n(2n+1)-4(\nu-1)n-\nu^2}\times\prod_{i=1}^{\nu-1}\frac{a^{4n}-(-1)^{\nu+i-1}a^{2(\nu+i-1)}}{a^{2i}-(-1)^i},$$

$$p_{3n+1}=\sum_{\nu=1}^{n+1}(-1)^{\nu-1}(uv)^{n-\nu+1}a^{(2n+1)(n+1)-2(\nu-1)(2n+1)-\nu^2}\times\prod_{i=1}^{\nu-1}\frac{a^{4n+2}-(-1)^{\nu+i}a^{2(\nu+i-1)}}{a^{2i}-(-1)^i},$$

$$q_{3n}=\sum_{\nu=1}^{n+1}(uv)^{n-\nu+1}a^{n(2n+1)-4(\nu-1)n-(\nu-1)^2}\times\prod_{i=1}^{\nu-1}\frac{a^{4n}-(-1)^{\nu+i}a^{2(\nu+i-2)}}{a^{2i}-(-1)^i},$$

$$q_{3n+1}=\sum_{\nu=1}^{n+1}u^{n-\nu+2}v^{n-\nu+1}a^{(2n+1)(n+1)-2(\nu-1)(2n+1)-(\nu-1)^2}\\ \times\prod_{i=1}^{\nu-1}\frac{a^{4n+2}-(-1)^{\nu+i-1}a^{2(\nu+i-2)}}{a^{2i}-(-1)^i}\,.$$

**Remark 6.** The forms of $p_{3n+2}$ and $q_{3n+2}$ are not simpler, and are obtained by using the recurrence relations: $p_{3n+2}=p_{3n+3}-p_{3n+1}$ and $q_{3n+2}=q_{3n+3}-q_{3n+1}$.

If $u$ is replaced by $ua$ and $a$ is replaced by $a^{1/2}$, then these become those of (15).

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