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# THE RINGS WHICH ARE BOOLEAN*

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**Abstract**

We study unitary rings of characteristic 2 satisfying identity $x^p = x$ for some natural number $p$. We characterize several infinite families of these rings which are Boolean, i.e., every element is idempotent. For example, it is in the case if $p = 2^n - 2$ or $p = 2^n - 5$ or $p = 2^n + 1$ for a suitable natural number $n$. Some other (more general) cases are solved for $p$ expressed in the form $2^q + 2m + 1$ or $2^q + 2m$ where $q$ is a natural number and $m \in \{1, 2, \ldots, 2^q - 1\}$.

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A ring $\mathcal{R} = (R; +, \cdot)$ is called *Boolean* if every its element is idempotent, i.e., if $\mathcal{R}$ satisfies the identity $x^2 = x$. Boolean rings play an important role in propositional logic and in theoretical computer science as well as in lattice theory, see e.g. [2]. In particular, every unitary Boolean ring

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can be converted into a Boolean algebra and vice versa. This motivated us to classify Boolean rings among rings with restricted powers, i.e., rings satisfying the identity $x^p = x$ for a natural number $p > 2$.

A sample result is the following.

**Lemma 1.** *Let $\mathcal{R} = (R; +, \cdot)$ be a ring satisfying the identity $x^p = x$ for some integer $p \geq 2$. The following are equivalent:*

(a) *$\mathcal{R}$ is Boolean;*

(b) *$\mathcal{R}$ satisfies the identity $x^{q+1} = x^q$ for some natural number $q \leq p$.*

***Proof.*** (a) $\Rightarrow$ (b): It is evident, because $x^2 = x$ implies $x^{q+1} = x^q$ for every natural number $q$.

(b) $\Rightarrow$ (a): Then $\mathcal{R}$ satisfies also $x^{p+1} = x^p$ and hence

$$x^2 = x \cdot x = x \cdot x^p = x^{p+1} = x^p = x,$$

thus $\mathcal{R}$ is Boolean. ■

It is an easy consequence of $x^2 = x$ that every Boolean ring is of characteristic 2, i.e., it satisfies the identity $x + x = 0$. Due to this fact, we restrict our treaty only to rings of characteristic 2.

A ring $\mathcal{R} = (R; +, \cdot)$ is called *unitary* if it contains a unit, i.e., an element denoted by 1 such that $x \cdot 1 = x = 1 \cdot x$ for each $x \in R$. For further information and notation on rings, the reader is refered to basic monographs [1, 4–6].

As a motivation, we can serve with the following two particular cases.

**Lemma 2.** *Let $\mathcal{R} = (R; +, \cdot)$ be a unitary ring of characteristic 2 satisfying the identity $x^3 = x$. Then $\mathcal{R}$ is Boolean.*

***Proof.*** Every element of $\mathcal{R}$ can be written in the form $x + 1$ because $x = (x + 1) + 1$, due to the fact that $\mathcal{R}$ is unitary and of characteristic 2. Hence, we get

$$\begin{aligned} 1 + x &= (1 + x)^3 = (1 + x) \cdot (1 + x)^2 = (1 + x) \cdot (1 + x^2) \\ &= 1 + x + x^3 + x^2 = 1 + x + x + x^2 = 1 + x^2 \end{aligned}$$

whence $x = x^2$ proving that $\mathcal{R}$ is Boolean. ■

On the contrary, we can show that there exists a unitary ring of characteristic 2 satisfying the identity $x^4 = x$ which is not Boolean. In fact, we can show the whole infinite family of identities $x^p = x$, i.e., an infinite set of natural numbers $p$ such that a unitary ring of characteristic 2 satisfying the identity $x^p = x$ need not be Boolean, see the following.

**Lemma 3.** *For each natural number $k$ there exists a unitary commutative ring of characteristic 2 satisfying the identity $x^{3k+1} = x$ which is not Boolean.*

***Proof.*** Consider the four-element ring $\mathcal{R}$ whose operations $+$ and $\cdot$ are determined by the tables

| $+$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 2 | 1 | 0 |

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 0 | 2 | 3 | 1 |
| 3 | 0 | 3 | 1 | 2 |

It is an immediate reflexion that $\mathcal{R}$ is unitary, commutative and of characteristic 2. Moreover, $\mathcal{R}$ satisfies $x^{3k+1} = x$ for every natural number $k$. However, $\mathcal{R}$ is not Boolean because e.g. $2 \cdot 2 = 3 \neq 2$. ■

**Remark 4.** Let us note that if $\mathcal{R} = (R; +, \cdot)$ is a unitary ring satisfying the identity $x^p = x$ for some even $p$ then we need not suppose that $\mathcal{R}$ is of characteristic 2. In fact, in this case there exists an element $-1 \in R$ and from the identity $x^p = x$ for $x = -1$ we get $1 = (-1)^p = -1$. Then for each $x \in R$ we have $-x = (-1) \cdot x = 1 \cdot x = x$ whence $x + x = x + (-x) = 0$.

Similarly as in Lemma 2, we can determine infinite sets of natural numbers $p$ for which $x^p = x$ implies that $\mathcal{R}$ is Boolean.

**Theorem 5.** *Let $\mathcal{R} = (R; +, \cdot)$ be a unitary ring and $n$ be a natural number.*

(i) *If $\mathcal{R}$ satisfies $x^{2^n-2} = x$ for $n > 1$ then $\mathcal{R}$ is Boolean.*

(ii) *If $\mathcal{R}$ is of characteristic 2 and satisfies $x^{2^n-5} = x$ for $n > 3$ then $\mathcal{R}$ is Boolean.*

***Proof.*** (i): As mentioned above, $\mathcal{R}$ is of charactic 2. If $\mathcal{R}$ satisfies $x^{2^n-2} = x$ then it satisfies also $x^{2^n} = x^3$ thus by Lemma 3(a) from [3]

$$1 + x^{2^n} = (1+x)^{2^n} = (1+x)^3 = 1 + 3 \times x + 3 \times x^2 + x^3.$$

Since $x^{2^n} = x^3$, we conclude $3 \times (x + x^2) = x + x^2 = 0$, whence $x = x^2$.

(ii): If $\mathcal{R}$ satisfies $x^{2^n-5} = x$ for some natural number $n > 3$ then it satisfies also $x^{2^n} = x^6$ and therefore, by [3], Lemma 3(a),

$$\begin{aligned} 1 + x^6 = 1 + x^{2^n} &= (1+x)^{2^n} = (1+x)^6 = (1+x)^4 \cdot (1+x)^2 \\ &= (1+x^4) \cdot (1+x^2) = 1 + x^2 + x^4 + x^6 \end{aligned}$$

whence $x^2 = x^4$. This yields

$$x^3 = x^5 = x^7 = \cdots = x^{2^n-5} = x$$

and, applying Lemma 2, we conclude that $\mathcal{R}$ is Boolean. ■

Similarly, we can also decide the following case.

**Lemma 6.** *Let $\mathcal{R} = (R; +, \cdot)$ be a unitary ring of characteristic 2 satisfying $x^{2^q+1} = x$ for a natural number $q$. Then $\mathcal{R}$ is Boolean.*

***Proof.*** We compute

$$\begin{aligned} 1 + x = (1+x)^{2^q+1} &= (1+x) \cdot (1+x)^{2^q} = (1+x) \cdot (1+x^{2^q}) \\ &= 1 + x + x^{2^q} + x^{2^q+1} = 1 + x + x^{2^q} + x = 1 + x^{2^q}, \end{aligned}$$

i.e., for $p = 2^q + 1$ we have $x^p = x = x^{2^q} = x^{p-1}$. By Lemma 1, $\mathcal{R}$ is Boolean. ■

Another relative large set of odd natural numbers $p$, for which a unitary ring of characteristic 2 satisfying $x^p = x$ is Boolean, is discerned by the following result.

**Theorem 7.** *Let $\mathcal{R} = (R; +, \cdot)$ be a unitary ring of characteristic 2 satisfying $x^p = x$ where $p = 2^q + 2m + 1$ for some natural number $q$ and $m \in \{1, 2, \ldots, 2^{q-1} - 1\}$. If $2^q - 2m = 2^a + 2^b$ where $a$,$b$ are integers such that $q > a > b \geq 1$ then $\mathcal{R}$ is Boolean provided $2^a - 2^b$ divides $2m$.*

***Proof.*** Assume $p = 2^q + 2m + 1$ with $2^q - 2m = 2^a + 2^b$ for some integers $q, m, a, b$ such that $q > a > b \geq 1$ and $m \in \{1, 2, \ldots, 2^{q-1} - 1\}$. If $2^a - 2^b$ divides $2m$ then $2m = r(2^a - 2^b)$ for some natural number $r$. Since $x^{2^q+2m+1} = x$, we have

$$x^{2^{q+1}} = x^{2^q+2m+1} \cdot x^{2^q-2m-1} = x \cdot x^{2^q-2m-1} = x^{2^q-2m}.$$

Hence, using [3], Lemma 3(a),

$$\begin{aligned} 1 + x^{2^q-2m} &= 1 + x^{2^{q+1}} = (1+x)^{2^{q+1}} = (1+x)^{2^q-2m} = (1+x)^{2^a+2^b} \\ &= (1+x)^{2^a} \cdot (1+x)^{2^b} = (1+x^{2^a}) \cdot (1+x^{2^b}) \\ &= 1 + x^{2^a} + x^{2^b} + x^{2^a+2^b} = 1 + x^{2^a} + x^{2^b} + x^{2^q-2m} \end{aligned}$$

This yields $0 = x^{2^a} + x^{2^b}$, thus $x^{2^a} = x^{2^b}$. Since $2m = r \cdot (2^a - 2^b)$ and $2^q + 2m + 1 > 2^q + 1 > 2^a$, we conclude

$$\begin{aligned} x^{2^q+1} &= x^{2^q+(2^a-2^b)+1} = \cdots = x^{2^q+r\cdot(2^a-2^b)+1} \\ &= x^{2^q+2m+1} = x^p = x. \end{aligned}$$

By Lemma 6, $\mathcal{R}$ is Boolean. ■

**Corollary 8.** *Let $\mathcal{R} = (R; +, \cdot)$ be a unitary ring of characteristic 2 satisfying $x^p = x$ where $p = 2^q + 2m + 1$ for some natural number $q$ and $m \in \{1, 2, \ldots, 2^{q-1} - 1\}$. If $2^q - 2m = 2^{a+1} + 2^a$ for some integer $a$ such that $q - 1 > a \geq 1$ then $\mathcal{R}$ is Boolean.*

***Proof.*** If $2^q - 2m = 2^{a+1} + 2^a$ for some integers $q, m, a$ such that $q - 1 > a \geq 1$, $m \in \{1, 2, \ldots, 2^{q-1} - 1\}$ then $2^{a+1} - 2^a = 2^a$. Thus it divides

$$2m = 2^q - 2^{a+1} - 2^a = 2^a \cdot (2^{q-a} - 3),$$

which, by Theorem 7, means that $\mathcal{R}$ is Boolean. ■

Hence, we get the sequence of numbers

$$p = 3, 5, 9, 17, 33, \ldots, \tag{S}$$

by Lemma 6, for which a unitary ring of characteristic 2 satisfying the identity $x^p = x$ is Boolean. In what follows, we will detect other natural numbers $p$ of this property.

**Remark 9.** We can recognize that (ii) of Theorem 5 can be included in the cases treated in Theorem 7. Namely, if $p = 2^n - 5$ for some integer $n > 3$ then we can compute

$$p = 2^{n-1} + (2^{n-1} - 5) = 2^{n-1} + (2^{n-1} - 6) + 1.$$

Using the notation from Theorem 7 we have

$$2^q - 2m = 2^{n-1} - (2^{n-1} - 6) = 6 = 2^2 + 2^1.$$

Thus, applying Corollary 8, we obtain that $\mathcal{R}$ is Boolean. Hence, we can extend our sequence (S) with numbers

$$p = 11, 27, 59, 123, \dots,$$

Moreover, Corollary 8 enables us to insert also numbers of the form $2^n - 11$ ($n > 4$), i.e.,

$$p = 21, 53, 117, 245, \dots,$$

further numbers of the form $2^n - 23$ ($n > 5$), i.e.,

$$p = 41, 105, 233, 489, \dots,$$

etc. We can generalize this approach in the following result.

**Theorem 10.** *Let $\mathcal{R} = (R; +, \cdot)$ be a unitary ring of characteristic 2 satisfying $x^p = x$ for some natural number $p$ of the form $2^n - (3 \cdot 2^l - 1)$ where $n, l$ are arbitrary natural numbers such that $n - 3 \geq l$. Then $\mathcal{R}$ is Boolean.*

***Proof.*** If $p = 2^n - (3 \cdot 2^l - 1)$ for some natural numbers satisfying $n - 3 \geq l$ then

$$3 \cdot 2^l \leq 3 \cdot 2^{n-3} < 4 \cdot 2^{n-3} = 2^{n-1}$$

and, therefore, $p = 2^{n-1} + (2^{n-1} - 3 \cdot 2^l) + 1$. We put $q = n - 1$, $2m = 2^{n-1} - 3 \cdot 2^l$ and then obtain

$$2^q - 2m = 2^{n-1} - (2^{n-1} - 3 \cdot 2^l) = 3 \cdot 2^l = 2 \cdot 2^l + 2^l = 2^{l+1} + 2^l$$

which, due to Corollary 8, means that $\mathcal{R}$ is Boolean. ■

In the next theorem, we will analyse the case of Theorem 7 in more details to obtain a general method how to produce sequences of $p$'s for which $\mathcal{R}$ is Boolean.

**Theorem 11.** *Let $\mathcal{R} = (R; +, \cdot)$ be a unitary ring of characteristic 2 satisfying $x^p = x$ where $p = 2^q + 2m + 1$ for some natural number $q$ and $m \in \{1, 2, \ldots, 2^{q-1} - 1\}$ such that $2^q - 2m = 2^a + 2^b$ where $a, b$ are integers satisfying $a > b \geq 1$ and, moreover, $q = (a+1) + k \cdot (a-b)$ for some nonnegative integer $k$. Then $\mathcal{R}$ is Boolean.*

***Proof.*** Consider a unitary ring $\mathcal{R}$ of characteristic 2 satisfying the identity $x^p = x$ for a number $p$ possessing the assumption. Then

$$\begin{aligned}
2m &= 2^q - 2^a - 2^b = 2^{(a+1)+k\cdot(a-b)} - 2^a - 2^b \\
&= 2^b \cdot \left(2^{(a+1)+k\cdot(a-b)-b} - 2^{a-b} - 1\right) \\
&= 2^b \cdot \left(2^{(k+1)\cdot(a-b)+1} - 2^{a-b} - 1\right) \\
&= 2^b \cdot \left(2 \cdot 2^{(k+1)\cdot(a-b)} - 2^{a-b} - 1\right) \\
&= 2^b \cdot \left[\left(\left(2^{a-b}\right)^{k+1} - 2^{a-b}\right) + \left(\left(2^{a-b}\right)^{k+1} - 1\right)\right] \\
&= 2^b \cdot \left[2^{a-b} \cdot \left(\left(2^{a-b}\right)^{k} - 1\right) + \left(\left(2^{a-b}\right)^{k+1} - 1\right)\right] \\
&= 2^b \cdot \Big[2^{a-b} \cdot \left(2^{a-b} - 1\right) \cdot \left(\left(2^{a-b}\right)^{k-1} + \ldots + 2^{a-b} + 1\right) \\
&\qquad + \left(2^{a-b} - 1\right) \cdot \left(\left(2^{a-b}\right)^{k} + \ldots + 2^{a-b} + 1\right)\Big] \\
&= 2^b \cdot \left(2^{a-b} - 1\right) \cdot \left(2 \cdot \left(2^{a-b}\right)^{k} + \ldots + 2 \cdot 2^{a-b} + 1\right) \\
&= \left(2^a - 2^b\right) \cdot \left(2 \cdot \left(2^{a-b}\right)^{k} + \ldots + 2 \cdot 2^{a-b} + 1\right).
\end{aligned}$$

Hence, $2^a - 2^b$ divides $2m$ and, by Theorem 7, $\mathcal{R}$ is Boolean. ■

**Remark 12.** Theorem 11 shows us how to construct numbers $p$ for which the unitary ring of characteristic 2 satisfying $x^p = x$ is Boolean.

It is enough to choose arbitrary integers $a, b$ such that $a > b \geq 1$ then to take $q = (a+1) + k \cdot (a-b)$ for some nonnegative integer $k$ and to compute $2m = 2^q - 2^a - 2^b$. Then $p = 2^q + 2m + 1$ is the number which we look for.

**Example 13.** If we take $a = 8$, $b = 3$ and $k = 1$, we have $q = (8+1) + 1 \cdot (8-3) = 14$ and, consequently, $2m = 2^{14} - 2^8 - 2^3 = 16120$. In fact, we have proved that the unitary ring of characteristic 2 satisfying the identity $x^{32505} = x$ is Boolean, because $32505 = 2^{14} + 16120 + 1$.

Until now, except Lemma 5(i) and partially also Lemma 3, we have dealed with unitary rings of characteristic 2 satisfying the identity $x^p = x$ only for odd natural numbers $p$. Further, we will discuss some cases when $p$ is even.

It is worth noticing that we have already solved the case of unitary ring satisfying $x^{2^r} = x$ for $r$ even. As mentioned in Remark 4, such a ring is of characteristic 2 and we can write here $2^r = 3k + 1$ for some odd natural number $k$. Hence, by Lemma 3, such a ring need not be Boolean.

If we consider a unitary ring satisfying $x^{2^r} = x$ for $r$ odd then this ring is of characteristic 2 and we can express $2^r$ in the form $3k + 2$ for some even $k$. Such a ring also need not be Boolean in general, see the following example for $r = 3$.

**Example 14.** The eight-element ring $\mathcal{R}$ whose operations $+$ and $\cdot$ are determined by the tables

| $+$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1 | 1 | 0 | 3 | 2 | 6 | 7 | 4 | 5 |
| 2 | 2 | 3 | 0 | 1 | 5 | 4 | 7 | 6 |
| 3 | 3 | 2 | 1 | 0 | 7 | 6 | 5 | 4 |
| 4 | 4 | 6 | 5 | 7 | 0 | 2 | 1 | 3 |
| 5 | 5 | 7 | 4 | 6 | 2 | 0 | 3 | 1 |
| 6 | 6 | 4 | 7 | 5 | 1 | 3 | 0 | 2 |
| 7 | 7 | 5 | 6 | 4 | 3 | 1 | 2 | 0 |

| $\cdot$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 0 | 2 | 4 | 5 | 3 | 7 | 1 | 6 |
| 3 | 0 | 3 | 5 | 6 | 7 | 1 | 4 | 2 |
| 4 | 0 | 4 | 3 | 7 | 5 | 6 | 2 | 1 |
| 5 | 0 | 5 | 7 | 1 | 6 | 2 | 3 | 4 |
| 6 | 0 | 6 | 1 | 4 | 2 | 3 | 7 | 5 |
| 7 | 0 | 7 | 6 | 2 | 1 | 4 | 5 | 3 |

is unitary, of characteristic 2, but it is evidently not Boolean.

We finish with the result which solves the problem for even natural numbers $p$ which are sum of two consequently standing powers of two, i.e. for numbers

$$p = 6, 12, 24, 48, \dots \ .$$

**Theorem 15.** *Let $\mathcal{R} = (R; +, \cdot)$ be a unitary ring satisfying the identity $x^p = x$ where $p = 2^{a+1} + 2^a$ for some natural number $a$. Then $\mathcal{R}$ is Boolean.*

***Proof.*** Consider a unitary ring $\mathcal{R}$ satisfying $x^{2^{a+1}+2^a} = x$ for some natural number $a$. By Remark 4, this ring is of characteristic 2 and, by [3], Lemma 3(a), we have

$$\begin{aligned} 1 + x &= (1+x)^{2^{a+1}+2^a} = (1+x)^{2^{a+1}} \cdot (1+x)^{2^a} \\ &= (1 + x^{2^{a+1}}) \cdot (1 + x^{2^a}) = 1 + x^{2^a} + x^{2^{a+1}} + x^{2^{a+1}+2^a}. \end{aligned}$$

Hence, $x^{2^{a+1}} = x^{2^a}$, and further

$$\begin{aligned} x^{2^{a+2}} &= x^{2^{a+1}+2^{a+1}} = x^{2^{a+1}} \cdot x^{2^{a+1}} = x^{2^{a+1}} \cdot x^{2^a} \\ &= x^{2^{a+1}+2^a} = x^p = x. \end{aligned}$$

From the identity $x^p = x^{2^{a+1}+2^a} = x$ we can also obtain

$$x^{2^{a+2}} = x^{(2^{a+1}+2^a)+(2^{a+1}-2^a)} = x^{1+(2^{a+1}-2^a)} = x^{1+2^a}.$$

Altogether we have $x = x^{2^{a+2}} = x^{2^a+1}$, and, by Lemma 6, $\mathcal{R}$ is Boolean. ■

**Remark 16.** It is easily seen that all the numbers $p$ which are determined by Theorem 15 are just the numbers of the form $p = 6 \cdot 2^{k-1}$ where $k$ is an arbitrary natural number.

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