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L-ZERO-DIVISOR GRAPHS OF DIRECT PRODUCTS OF L-COMMUTATIVE RINGS

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Abstract

L-zero-divisor graphs of L-commutative rings have been introduced and studied in [5]. Here we consider L-zero-divisor graphs of a finite direct product of L-commutative rings. Specifically, we look at the preservation, or lack thereof, of the diameter and girth of the L-ziro-divisor graph of a L-ring when extending to a finite direct product of L-commutative rings.

Keywords: μ-zero-divisor, L-zero-divisor graph, μ-diameter, μ-girth, finite direct products.

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1. Introduction

In [14], Zadeh introduced the concept of fuzzy set, which is a very useful tool to describe the situation in which the data is imprecise or vague. Fuzzy sets handle such situations by attributing a degree to which a certain object belongs to a set. Many researchers used this concept to generalize some notions of algebra. Goguen in [6] generalized the notion of fuzzy subset of X to that of an L-subset, namely a function from X to a lattice L. In [11], Rosenfeld considered the fuzzification of algebraic structures. Liu [7], introduced and examined the notion of a fuzzy ideal of a ring. Since then several authors have obtained interesting results on L-ideals of a ring R and L-modules (see [8, 9, 1]). Rosenfeld in [12] considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975. During the same time, Yeh and Bang in [13] also introduced various connectedness concepts in fuzzy graphs. After the pioneering work of Rosenfeld and Yeh and Bang in 1975, when some basic fuzzy graph theoretic concepts and applications we are indicated, several authors have been finding deeper results and fuzzy analogues of many other graph-theoretic concepts. See [9] for a comprehensive survey of the literature on these developments.

Among the most interesting graphs are the zero-divisor graphs, because these involve both ring theory and graph theory. By studying these graphs, we can gain a broader insight into the concepts and properties that involve both graphs and rings. It was Beck (see [3]) who first introduced the notion of a zero-divisor graph for commutative rings. This notion was later redefined by D.F. Anderson and P.S. Livingston in [1]. Since then, there has been a lot of interest in this subject and various papers were published establishing different properties of these graphs as well as relations between graphs of various extensions (see [1, 2, 3, 9]). The zero-divisor graph of a direct product of commutative rings have been studied by Axtell, Stickles and Warfel in [2]. In the present paper, we characterize the diameter and girth of the L-zero-divisor graph of a direct product of L-commutative rings not necessarily with identity.

2. Preliminaries

Throughout this paper, R is a commutative ring, not necessarily with identity, and L stands for a complete lattice with least element 0 and greatest element 1. In order to make this paper easier to follow, we recall in this section various notions from graph theory and fuzzy commutative algebra theory which will be used in the sequel. For a graph Γ, by $E(\Gamma)$ and $V(\Gamma)$, we denote the set of all edges and vertices, respectively. We recall that a graph is connected if there exists a path connecting any two distinct vertices. The distance between two distinct vertices a and b, denoted by $\mathrm{d}(a, b)$, is the length of a shortest path connecting them (if such a path does not exist, then $d(a, a) = 0$ and $d(a, b) = \infty$). The diameter of a graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is equal to $\sup\{d(a, b) : a, b \in V(\Gamma)\}$. A graph is complete if it is connected with diameter less than or equal to one. The girth of a graph Γ, denoted $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise; $\mathrm{gr}(\Gamma) = \infty$.

If R is a commutative ring, let $Z(R)$ denote the set of zero-divisors of R and let $Z(R)^*$ denote the set of non-zero zero-divisors of R. We consider the undirected graph $\Gamma(R)$ with vertices in the set $V(\Gamma(R)) = Z(R)^*$, such that for distinct vertices a and b there is an edge connecting them if and only if $ab = 0$. Then $\Gamma(R)$ is connected with $\mathrm{diam}(\Gamma(R)) \leq 3$ ([1, Theorem 2.3]) and $\mathrm{gr}(\Gamma(R)) \leq 4$ ([10, (1.4)]). Thus $\mathrm{diam}(\Gamma(R)) = 0, 1, 2$, or 3 and $\mathrm{gr}(\Gamma(R)) = 3, 4$, or ∞.

Let R be a commutative ring and L stands for a complete lattice with least element 0 and greatest element 1. By an L-subset μ of a non-empty set X, we mean a function μ from X to L. If $L = [0, 1]$, then μ is called a fuzzy subset of X. L^X denotes the set of all L-subsets of X. We recall some definitions and lemmas from the book [9], which we need for development of our paper.

Definition 2.1. An L-ring is a function $\mu : R \to L$, where $(R, +, .)$ is a ring, that satisfies:

(1) $\mu \neq 0$;

(2) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$ for every x, y in R;

(3) $\mu(xy) \geq \mu(x) \vee \mu(y)$ for every x, y in R.

Definition 2.2. Let $\mu \in L^R$. Then μ is called an L-ideal of R if for every $x, y \in R$ the following conditions are satisfied:

(1) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$;

(2) $\mu(xy) \geq \mu(x) \vee \mu(y)$.

The set of all L-ideals of R is denoted by $LI(R)$.

Lemma 2.3. *Let R be a ring and $\mu \in LI(R)$. Then $\mu(x) \leq \mu(0)$ for every x in R.*

Definition 2.4 [5, Definition 3.1]**.** Let R be a ring and $\mu \in LI(R)$. A μ-zero-divisor is an element $x \in R$ for which there exists $y \in R$ with $\mu(y) \neq \mu(0)$ such that $\mu(xy) = \mu(0)$.

The set of μ-zero-divisors in R will be denoted by $Z(\mu)$.

Definition 2.5 [5, Definition 3.2]**.** Let R be a ring and $\mu \in LI(R)$. We define an undirected graph $\Gamma(\mu)$ with vertices $V(\Gamma(\mu)) = Z(\mu)^* = Z(\mu) - \mu_* = \{x \in Z(\mu) : \mu(x) \neq \mu(0)\}$, where distinct vertces x and y are adjacent if and only if $\mu(xy) = \mu(0)$, where $\mu_* = \{x \in R : \mu(x) = \mu(0)\}$.

Notation. For the graph $\Gamma(\mu)$, we denote the diameter, the girth, and the distance between two distinct vertices a and b, by $\mathrm{diam}(\Gamma(\mu))$, $\mathrm{gr}(\mu)$ and $d_\mu(a,b)$, respectively.

Remark 2.6. Let R be a ring and $\mu \in LI(R)$. Clearly, if μ is a non-zero constant, then $\Gamma(\mu) = \emptyset$. So throughout this paper, we shall assume unless otherwise stated, that μ is not a non-zero constant. Thus there is a non-zero element y of R such that $\mu(y) \neq \mu(0)$.

Definition 2.7 [5, Definition 3.4]**.** Let R be a ring and $\mu \in LI(R)$. We say μ is an L-integral domain if $Z(\mu) = \mu_*$.

Definition 2.8 [5, Definition 3.6]**.** Let R be a ring and $\mu \in LI(R)$. An element $a \in R$ is said to be μ-nilpotent precisely when there exists a positive integer n such that $\mu(a^n) = \mu(0)$.

The set of all μ-nilpotents of R is denoted by $\mathrm{nil}(\mu)$, and we set $\mathrm{nil}(\mu)^* = \mathrm{nil}(\mu) - \mu_*$.

Theorem 2.9 [5, Theorem 3.16]**.** *Let R be a ring and $\mu \in LI(R)$. Then $\Gamma(\mu)$ is connected with* $\mathrm{diam}(\Gamma(\mu)) \leq 3$.

Theorem 2.10 [5, Theorem 3.17]**.** *Let R be a ring and $\mu \in LI(R)$. If $\Gamma(\mu)$ contains a cycle, then* $\mathrm{gr}(\Gamma(\mu)) \leq 4$.

3. Diameter and direct products

Before starting to describe the diameter of a finite direct product of L-rings, we will develop some tools that will be used in examining L-commutative rings, not necessarily with identity with L-zero-divisor graphs having diameter either 1 or 2. Compare the next lemma with [2, Lemma 2.2].

Lemma 3.1. *Let S be a commutative ring and $\mu \in LI(S)$ with* $\mathrm{diam}(\Gamma(\mu)) = 1$. *Then $S = Z(\mu)$ if and only if $\mu(S^2) = \mu(\{0\})$.*

Proof. Let $\mu(x^2) \neq \mu(0)$ for some $x \in S$. By assumption, there exists a non-zero element $y \in S$ such that $x \neq y$. Observe that $x + y \neq x$. Since $S = Z(\mu)$ and $\mathrm{diam}(\Gamma(\mu)) = 1$, we have $\mu(xy) = \mu(0)$. So, $\mu(x^2) = \mu(x^2+xy-xy) \geq \mu(x^2+xy) \wedge \mu(xy) = \mu(0) \wedge \mu(0) = \mu(0)$; hence $\mu(x^2) = \mu(0)$ by Lemma 2.3, a contradiction. The other implication is clear. ■

Example 3.2. Let $S = \mathbb{Z}$ denote the ring of integers. We define the mapping $\mu : S \to [0,1]$ by

$$\mu(x) = \begin{cases} 1/2 & \text{if } x \in 2\mathbb{Z} \\ 1/5 & \text{otherwise.} \end{cases}$$

Then $\mu \in LI(S)$ and $Z(\mu) = \mathbb{Z}$. Since $\mu(3^2) \neq \mu(0)$, we have $\mu(S^2) \neq \mu(\{0\})$. Therefore, the condition $\mathrm{diam}(\Gamma(\mu)) = 1$ is not superficial in Lemma 3.1

Compare the next lemma with [2, Lemma 2.3].

Lemma 3.3. *Let S be a ring and $\mu \in LI(S)$ such that* $\mathrm{diam}(\Gamma(\mu)) = 2$. *Suppose $Z(\mu)$ is a (not necessarily proper) subring of S. Then for all $x, y \in Z(\mu)$, there exists $z \in Z^*(\mu)$ such that $\mu(zx) = \mu(zy) = \mu(0)$.*

Proof. Let $x, y \in Z(\mu)$. We split the proof into three cases.

Case 1. $x = 0$ or $y = 0$. Assume that $x = 0$ and let $z \in Z^*(\mu)$. Then $\mu(zx) \geq \mu(x) \vee \mu(z) = \mu(0)$; so $\mu(zx) = \mu(0)$ by Lemma 2.3. Similarly, for $y = 0$.

Case 2. $x = y \neq 0$. If $\mu(xy) = \mu(0)$, we choose $z = x$, and if $\mu(xy) \neq \mu(0)$, then there exists $z \in Z(\mu)^*$ such that $\mu(zx) = \mu(zy) = \mu(0)$ since $\mathrm{diam}(\Gamma(\mu)) = 2$.

Case 3. $x \neq y$, $x \neq 0$ and $y \neq 0$. If $\mu(xy) \neq \mu(0)$, we are done. So we may assume that $\mu(xy) = \mu(0)$. If $x+y = 0$, then $\mu(xy) = \mu(-x^2) = \mu(x^2) = \mu(0)$. Therefore, $z = x$ yiels the desired element. So, suppose $\mu(x^2) \neq \mu(0)$, $\mu(y^2) \neq \mu(0)$ and $x + y \neq 0$. Let $X' = \{x' \in Z^*(\mu) : \mu(xx') = \mu(0)\}$ and $Y' = \{y' \in Z^*(\mu) : \mu(yy') = \mu(0)\}$. Observe that $x \in Y'$ and $y \in X'$; hence X' and Y' are nonempty. Now we show that $X' \cap Y' \neq \emptyset$. Suppose not. By assumption, we have $\mu(x(x+y)) \geq \mu(x^2) \wedge \mu(xy) = \mu(x^2) \wedge \mu(0) = \mu(x^2) \neq \mu(0)$ and $\mu(x^2) = \mu(x^2 + xy - xy) \geq \mu(x^2 + xy) \wedge \mu(0) = \mu(x(x + y))$; hence $\mu(x(x + y)) = \mu(x^2) \neq \mu(0)$. It follows that $x + y \notin X'$. Similarly, $x + y \notin Y'$. Since $\mathrm{diam}(\Gamma(\mu)) = 2$ and $Z(\mu)$ is a subring, there exists $w \in Z(\mu)^*$ such that $\mu(xw) = \mu(w(x + y)) = \mu(0)$. Also, we have $\mu(yw) = \mu(yw + xw - xw) \geq \mu(yw + xw) \wedge \mu(xw) = \mu(0)$; so $\mu(yw) = \mu(0)$ by Lemma 2.3. Then $w \in X' \cap Y'$, which is a contradiction. Now since $X' \cap Y' \neq \emptyset$, choose $z \in X' \cap Y'$. ∎

Remark 3.4. Assume that $(L_1, \leq_1), (L_2, \leq_2), \ldots, (L_n, \leq_n)$ are complete lattices with least element 0 and greatest element 1 and let $L = L_1 \times L_2 \times \cdots \times L_n$. We set up a partial order on L as follows: for $X = (x_1, x_2, \ldots, x_n), Y = (y_1, y_2, \ldots, y_n \in L$, we write $X \leq Y$ if and only if $x_i \leq_i y_i$ for every $i = 1, 2, \ldots, n$. It is straightforward to check that $\leq$ is a partial order on L. Furthermore, if we define

$$X \vee Y = (x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n)$$

$$X \wedge Y = (x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_n \wedge y_n),$$

then an inspection will show that L is a complete lattice with least element 0 and greatest element 1.

Lemma 3.5. *Assume that $L_1, L_2, \ldots, L_n$ $(n \geq 2)$ are as in Remark* 3.4 *and let $R_1, R_2, \ldots, R_n$ be commutative rings, $\mu_i \in L_iI(R_i)$ for every $i = 1, \ldots, n$, $X = (x_1, x_2, \ldots, x_n) \in R = R_1 \times R_2 \times \cdots \times R_n$, $L = L_1 \times L_2 \times \cdots \times L_n$ and $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$. Then the mapping $\mu : R \to L$ defined by $\mu(X) = (\mu_1(x_1), \mu_2(x_2), \mu_n(x_n))$ is an L-ideal of R.*

Proof. Let $X = (x_1, x_2, \ldots, x_n), Y = (y_1, y_2, \ldots, y_n \in R$. Then $\mu(X - Y) = (\mu_1(x_1 - y_1), \ldots, \mu_n(x_n - y_n))$. By Remark 3.4, we have $\mu(X) \wedge \mu(Y) = (\mu_1(x_1) \wedge \mu_1(y_1), \ldots, \mu_n(x_n) \wedge \mu(y_n)$; so $\mu(X - Y) \geq \mu(X) \wedge \mu(Y)$ since for each i, $\mu_i(x_i - y_i) \geq_i \mu_i(x_i) \wedge \mu_i(y_i)$ (also see the definition of $\leq$). Similarly, $\mu(XY) \geq \mu(X) \vee \mu(Y)$. ∎

Remark 3.6. Throughout this section, we shall assume, unless otherwise stated, that R, L, and μ are as described in Remark 3.4 and Lemma 3.5.

Compare the next theorem with [2, Theorem 3.3].

Theorem 3.7. *Let R, L, and μ be as in Remark* 3.6, *and let $\mu_i \in L_iI(R_i)$ such that $R_n = Z(\mu_n)$ and $\mu_1, \ldots, \mu_{n-1}$ are* L*-integral domains. Then the following hold:*

(i) *If* $\mathrm{diam}(\Gamma(\mu_n)) \leq 2$, *then* $\mathrm{diam}(\Gamma(\mu)) = 2$.

(ii) *If* $\mathrm{diam}(\Gamma(\mu_n)) = 3$, *then* $\mathrm{diam}(\Gamma(\mu)) = 3$.

Proof. (i) Let $x = (x_1, \dots, x_n) \in R$ and $y_n \in R_n^*$. Then $\mu(x(0, 0, \dots, y_n)) = \mu(0)$ since $R_n = Z(\mu_n)$; hence $Z(\mu) = R$. If $z_n \in Z^*(\mu_n)$, then

$$\mu((1, 1, 0, \dots, 0)(1, 1, \dots, 1, z_n)) \neq \mu(0);$$

so $d_\mu((1, 1, 0, \dots, 0), (1, 1, \dots, 1, z_n)) \geq 2$. Now if $\operatorname{diam}(\Gamma(\mu_n)) \leq 2$, then for $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in R$ we have either $\mu(ab) = \mu(0)$ or for some $c_n \in R_n^*$ we get $\mu(a(0, 0, \dots, c_n)) = \mu(0) = \mu(b(0, \dots, c_n))$ using Lemma 3.3 in the diameter two case. So we have $\operatorname{diam}(\Gamma(\mu)) = 2$. If $\operatorname{diam}(\Gamma(\mu_n)) = 3$, then there exist $x_n, y_n \in R_n^*$ such that $d_{\mu_n}(x_n, y_n) = 3$. Then for $b_i \in R_i^*$ $(1 \leq i \leq n-1)$ we have $d_\mu((b_1, \dots, b_{n-1}, x_n), (b_1, \dots, b_{n-1}, y_n)) = 3$, as required. ■

For the remainder of the section, we assume that $R_1, R_2, \dots, R_{n-1}$ and R_n are rings, not necessarily with identity, such that $Z^*(\mu_1), \dots, Z^*(\mu_{n-1})$ and $Z^*(\mu_n)$ are nonempty. Compare the next theorem with [2, Theorem 3.4].

Theorem 3.8. *Let R, L, and μ be as in Remark* 3.6, *and let $\mu_i \in L_i I(R_i)$ such that* $\operatorname{diam}(\Gamma(\mu_i)) = 1$ *for all $i = 1, \dots, n$. Then the following hold:*

(i) $\operatorname{diam}(\Gamma(\mu)) = 1$ *if and only if $\mu_i(R_i^2) = \mu_i(\{0\})$ for every $i \in \{1, 2, \dots, n\}$.*

(ii) $\operatorname{diam}(\Gamma(\mu)) = 2$ *if and only if $\mu_i(R_i^2) = \mu_i(\{0\})$ and $\mu_j(R_j^2) \neq \mu_j(\{0\})$ for some $i, j \in \{1, 2, \dots, n\}$.*

(iii) $\operatorname{diam}(\Gamma(\mu)) = 3$ *if and only if $\mu_i(R_i^2) \neq \mu_i(\{0\})$ for every $i \in \{1, 2, \dots, n\}$.*

Proof. (i) Assume that $\mu_i(R_i^2) = \mu_i(\{0\})$ for all i, and let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in Z^*(\mu)$. Then

$$\mu(xy) = (\mu_1(x_1 y_1), \dots, \mu_n(x_n y_n)) = (\mu_1(0), \dots, \mu_n(0)) = \mu(0);$$

hence $\operatorname{diam}(\Gamma(\mu)) = 1$. Conversely, assume that $\mu_j(R_j^2) \neq \mu_j(\{0\})$ for some $j \in \{1, 2, \dots, n\}$. Then $\mu_j(x_j y_j) \neq \mu_j(0)$ for some $x_j, y_j \in R_j$. Let $z_i \in Z^*(\mu_i)$ for $i \neq j$. Set $X = (0, \dots, x_j, \dots, 0)$, $Y = (0, \dots, y_j, \dots, 0)$ and $Z = (0, \dots, z_i, \dots, 0)$. Then $\mu(XZ) = \mu(YZ) = \mu(0)$; hence $X - Z - Y$ is a path of length 2 from X to Y in $Z^*(\mu)$, which is a contradiction.

(ii) Let $\mu_i(R_i^2) = \mu_i(\{0\})$ and $\mu_j(R_j^2) \neq \mu_j(\{0\})$ for some $i, j \in \{1, 2, \ldots, n\}$. Then $\mathrm{diam}(\Gamma(\mu)) \neq 1$ by (i). Let $c_i \in Z^*(\mu_i)$, and set $c = (0, \ldots, c_i, \ldots, 0)$. For any $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in Z^*(\mu)$, at worst we have $x - c - y$ is a path from x to y in $Z^*(\mu)$. So, $\mathrm{diam}(\Gamma(\mu)) \leq 2$. The result then follows from (i). Conversely, assume that $\mathrm{diam}(\Gamma(\mu)) = 2$. If $\mu_i(R_i^2) = \mu_i(\{0\})$, then $R_i = Z(\mu_i)$ for all $i = 1, \ldots, n$ (see Lemma 3.1); so $\mathrm{diam}(\Gamma(\mu)) = 1$ by (i), a contradiction. If for each i, $Z(\mu_i) \neq R_i$, then there must exist $x_i \in R_i$ with $x_i \notin Z(\mu_i)$ for all $i = 1, \ldots, n$. For each i, let $z_i \in Z^*(\mu_i)$. So there is an element $w_i \in Z^*(\mu_i)$ such that $\mu_i(z_i w_i) = \mu_i(0)$ for all i. If $a = (z_1, x_2, \ldots, x_n)$ and $b = (x_1, z_2, x_3, \ldots, x_n)$, then $\mu(a(w_1, 0, \ldots, 0)) = \mu(0)$ and $\mu(b(0, w_2, 0, \ldots, 0)) = \mu(0)$; hence $a, b \in Z^*(\mu)$. Since $\mu(ab) \neq \mu(0)$, we get $\mathrm{d}_\mu(a, b) > 1$. As $\mathrm{diam}(\Gamma(\mu)) = 2$, there exists $c = (c_1, \ldots, c_n) \in Z^*(\mu)$ such that $\mu(ac) = \mu(bc) = \mu(0)$. It follows that there exists i $(1 \leq i \leq n)$ such that $x_i \in Z^*(\mu_i)$, a contradiction. Thus the proof is complete.

(iii) Follows from (i) and (ii). ■

Compare the next theorem with [2, Theorem 3.5].

Theorem 3.9. *Let R, L, and μ be as in Remark* 3.6, *and let* $\mu_i \in L_i I(R_i)$ *such that* $\mathrm{diam}(\Gamma(\mu_i)) = 2$ *for all* $i = 1, \ldots, n$. *Then the following hold:*

(i) $\mathrm{diam}(\Gamma(\mu)) \neq 1$.

(ii) $\mathrm{diam}(\Gamma(\mu)) = 2$ *if and only if* $R_i = Z(\mu_i)$ *for some* $i \in \{1, 2, \ldots, n\}$.

(iii) $\mathrm{diam}(\Gamma(\mu)) = 3$ *if and only if* $R_i \neq Z(\mu_i)$ *for every* $i \in \{1, 2, \ldots, n\}$.

Proof. (i) Since $\mathrm{diam}(\Gamma(\mu_n)) = 2$, there exist distinct $y_n, w_n \in Z^*(\mu_n)$ with $\mu_n(y_n w_n) \neq \mu_n(0)$. Set $a = (0, 0, \ldots, y_n)$ and $b = (0, 0, \ldots, w_n)$. Then $\mu(ab) = (\mu_1(0), \ldots, \mu_{n-1}(0), \mu_n(y_n w_n)) \neq \mu(0)$. Therefore $\mathrm{diam}(\Gamma(\mu)) > 1$.

(ii) Assume that $R_i = Z(\mu_i)$ for some $i \in \{1, 2, \ldots, n\}$. Since $R_i = Z(\mu_i)$, for $x_i, y_i \in Z(\mu_i)$ there exists $z_i \in Z^*(\mu_i)$ such that $\mu_i(x_i z_i) = \mu_i(y_i z_i) = \mu_i(0)$ by Lemma 3.3. So, for any $x = (x_1, \ldots, x_n), y = y_1, \ldots, y_n)$ $\in Z^*(\mu)$, there exists $z = (0, 0, \ldots, 0, z_i, 0, \ldots, 0) \in Z^*(\mu)$ such that $\mu(xz) = \mu(yz) = \mu(0)$. If without loss of generality $y = z$, we have $\mu(xy) = \mu(0)$. Therefore, $\mathrm{diam}(\Gamma(\mu)) \leq 2$. By (i), it must be that $\mathrm{diam}(\Gamma(\mu)) = 2$. Conversely, suppose that $\mathrm{diam}(\Gamma(\mu)) = 2$ and $R_i \neq Z(\mu_i)$ for all $i \in \{1, 2, \ldots, n\}$. Let $e_i \in Z(\mu_i)$ and $m_i \in R_i - Z(\mu_i)$ for all i. Set $a = (e_1, m_2, \ldots, m_n)$ and $b = (m_1, e_2, m_3, \ldots, m_n)$. Then $\mu(ab) \neq \mu(0)$. Since $\mathrm{diam}(\Gamma(\mu)) = 2$, there exists $z = (z_1, \dots, z_n) \in Z^*(\mu)$ such that $\mu(az) = \mu(bz) = \mu(0)$. Then $\mu_1(e_1z_1) = \mu_1(0)$, $\mu_i(m_iz_i) = \mu_i(0)$ $(2 \le i \le n)$, $\mu_1(m_1z_1) = \mu_1(0)$, $\mu_2(e_2z_2) = \mu_2(0)$ and $\mu_i(m_iz_i) = \mu_i(0)$ $(3 \le i \le n)$, which is a contradiction.

(iii) Follows from (i) and (ii). ∎

Compare the next theorem with [2, Theorem 3.9].

Theorem 3.10. *Let R, L, and μ be as in Remark* 3.6, *and let $\mu_i \in L_iI(R_i)$ such that* $\mathrm{diam}(\Gamma(\mu_i)) = 3$ *for all $i = 1, \dots, n$. Then* $\mathrm{diam}(\Gamma(\mu)) = 3$.

Proof. Since for each i, $\mathrm{diam}(\Gamma(\mu_i)) = 3$, there exist distinct $x_i, y_i \in Z^*(\mu_i)$ with $\mu_i(x_iy_i) \neq \mu_i(0)$ and there is no $z_i \in Z^*(\mu_i)$ such that $\mu_i(x_iz_i) = \mu_i(y_iz_i) = \mu_i(0)$. Consider $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. Now for each $i \in \{1, \dots, n\}$, there are elements $x'_i, y'_i \in Z^*(\mu_i)$ such that $\mu_i(x_ix'_i) = \mu_i(0)$ and $\mu_i(y_iy'_i) = \mu_i(0)$; hence $x, y \in Z^*(\mu)$. Since $\mu(xy) \neq \mu(0)$, we have $\mathrm{diam}(\Gamma(\mu)) > 1$. If $\mathrm{diam}(\Gamma(\mu)) = 2$, there exists $a = (a_1, \dots, a_n) \in Z^*(\mu)$ such that $\mu(ax) = \mu(ay) = \mu(0)$. Then for each $i \in \{1, \dots, n\}$, we have $\mu_i(x_ia_i) = \mu_i(y_ia_i) = \mu_i(0)$, a contradiction. Thus $\mathrm{diam}(\Gamma(\mu)) = 3$. ∎

Compare the next theorem with [2, Theorem 3.8].

Theorem 3.11. *Let R, L, and μ be as in Remark* 3.6, *and let $\mu_i \in L_iI(R_i)$ such that* $\mathrm{diam}(\Gamma(\mu_i)) = 1$, $\mathrm{diam}(\Gamma(\mu_j)) = 2$ *for some $i, j \in \{1, \dots, n\}$, and there is no $k \in \{1, \dots, n\}$ with* $\mathrm{diam}(\Gamma(\mu_k)) = 3$. *Then the following hold:*

(i) $\mathrm{diam}(\Gamma(\mu)) \neq 1$.

(ii) $\mathrm{diam}(\Gamma(\mu)) = 2$ *if and only if $R_i = Z(\mu_i)$ for some $i \in \{1, 2, \dots, n\}$.*

(iii) $\mathrm{diam}(\Gamma(\mu)) = 3$ *if and only if $R_i \neq Z(\mu_i)$ for every $i \in \{1, 2, \dots, n\}$.*

Proof. (i) Same as Theorem 3.9 (i).

(ii) Let $R_i = Z(\mu_i)$ and $\mathrm{diam}(\Gamma(\mu_i)) = 1$. Thus we have $\mu_i(R_i^2) = \mu_i(\{0\})$ by Lemma 3.1. Let $x_i \in R_i^*$. Since $\mu((0, \dots, 0, x_i, 0, \dots, 0)(y_1, y_2, \dots, y_n)) = \mu(0)$ for all $(y_1, y_2, \dots, y_n) \in Z^*(\mu)$, we have $\mathrm{diam}(\Gamma(\mu)) \le 2$. It follows from (i) that $\mathrm{diam}(\Gamma(\mu)) = 2$. Conversely, assume that $\mathrm{diam}(\Gamma(\mu)) = 2$. Suppose $R_i \neq Z(\mu_i)$ for every $i \in \{1, 2, \dots, n\}$. Then for each i, $R_i \neq Z(\mu_i)$. Without loss of generality, let $z_1 \in Z^*(\mu_1)$. Then there exists $w_1 \in Z^*(\mu_1)$ such that $\mu_1(z_1w_1) = \mu_1(0)$. For each i, let $r_i \in R_i - Z(\mu_i)$, and set $a = (r_1, 0, \dots, 0), b = (0, r_2, 0, \dots, 0), c = (z_1, 0, \dots, 0)$ and $d = (w_1, r_2, r_3, \dots, r_n)$. Then $a - b - c - d$ is a path of length 3. Since a

is annihilated only by an element of the form $(0,0,\ldots,0,t_n)$ and d is annihilated by an element of the form $(s_1,0,\ldots,0)$ with $\mu_1(s_1w_1)=\mu_1(0)$, there is no path of length 2 from a to d. Hence $\operatorname{diam}(\Gamma(\mu))=3$, a contradiction.

(iii) By (i) and (ii). ∎

Compare the next theorem with [2, Theorem 3.6].

Theorem 3.12. *Let R, L, μ be as in Remark* 3.6, *and let $\mu_i \in L_iI(R_i)$ such that* $\operatorname{diam}(\Gamma(\mu_i))=1$, $\operatorname{diam}(\Gamma(\mu_j))=3$ *for some $i,j\in\{1,\ldots,n\}$, and there is no $k\in\{1,\ldots,n\}$ with* $\operatorname{diam}(\Gamma(\mu_k))=2$. *Then the following hold:*

(i) $\operatorname{diam}(\Gamma(\mu))\neq 1$.

(ii) $\operatorname{diam}(\Gamma(\mu))=2$ *if and only if $R_i=Z(\mu_i)$ and* $\operatorname{diam}(\Gamma(\mu_i))=1$ *for some $i\in\{1,2,\ldots,n\}$.*

(iii) $\operatorname{diam}(\Gamma(\mu))=3$ *if and only if there is no $k\in\{1,\ldots,n\}$ with $R_k\neq Z(\mu_k)$ and* $\operatorname{diam}(\Gamma(\mu_k))=1$.

Proof. (i) Same as Theorem 3.9 (i).

(ii) ($\Leftarrow$) Same as Theorem 3.11 (ii)). Conversely, assume that $\operatorname{diam}(\Gamma(\mu))=2$; we show that $\operatorname{diam}(\Gamma(\mu_i))=1$ and $R_i=Z(\mu_i)$ for some $i\in\{1,2,\ldots,n\}$. Suppose either $\operatorname{diam}(\Gamma(\mu_i))\neq 1$ or $R_i\neq Z(\mu_i)$ for every $i\in\{1,2,\ldots,n\}$. Let $i_1,\ldots,i_k$ be such that $\operatorname{diam}(\Gamma(\mu_{i_r}))=1$ $(1\le r\le k)$, and let $j_1,\ldots,j_t$ be such that $\operatorname{diam}(\Gamma(\mu_{j_s}))=3$ $(1\le s\le t)$. Since for each s $(1\le s\le t)$, $\operatorname{diam}(\Gamma(\mu_{j_s}))=3$, there exist distinct $x_{j_s},y_{j_s}\in Z^*(\mu_{j_s})$ with $\mu_{j_s}(x_{j_s}y_{j_s})\neq\mu_{j_s}(0)$ such that there is no $z_{j_s}\in Z^*(\mu_{j_s})$ with $\mu_{j_s}(x_{j_s}z_{j_s})=\mu_{j_s}(y_{j_s}z_{j_s})=\mu_{j_s}(0)$. Moreover, for each s $(1\le s\le t)$, there must exist $x'_{j_s},y'_{j_s}\in Z^*(\mu_{j_s})$ with $\mu_{j_s}(x_{j_s}x'_{j_s})=\mu_{j_s}(0)$ and $\mu_{j_s}(y_{j_s}y'_{j_s})=\mu_{j_s}(0)$. Now for each r $(1\le r\le k)$, let $m_{i_r}\in R_{i_r}-Z(\mu_{i_r})$. Set $c=(m_{i_1},\ldots,x_{j_1},\ldots,x_{j_t},\ldots,0)$ and $d=(m_{i_1},\ldots,y_{j_1},\ldots,y_{j_t},\ldots,0)$. Then

$$\mu(c(0,\ldots,x'_{j_1},0,\ldots,0))=\mu(0);$$

so $c\in Z^*(\mu)$. Similarly, $d\in Z^*(\mu)$. As $\mu(cd)\neq\mu(0)$ and $\operatorname{diam}(\Gamma(\mu))=2$, there must be some $e=(e_1,\ldots,e_n)\in Z^*(\mu)$ such that $\mu(ce)=\mu(de)=\mu(0)$. But this is a contradiction, as needed.

(iii) Since $\Gamma(\mu)$ is connected and $\operatorname{diam}(\Gamma(\mu))\le 3$, the diameter of $\Gamma(R)$ is either 2 or 3 by (i). If $\operatorname{diam}(\Gamma(R))=2$, then by (ii), $\operatorname{diam}(\Gamma(R_i))=1$ and $R_i=Z(R_i)$ for some $i\in\{1,\ldots,n\}$, which is a contradiction. Thus $\operatorname{diam}(\Gamma(R))=3$. The proof of the other implication is clear. ∎

Compare the next theorem with [2, Theorem 3.7].

Theorem 3.13. *Let R, L, and μ be as in Remark* 3.6, *and let $\mu_i \in L_i I(R_i)$ such that* $\operatorname{diam}(\Gamma(\mu_i)) = 2$, $\operatorname{diam}(\Gamma(\mu_j)) = 3$ *for some $i, j \in \{1, \dots, n\}$, and there is no $k \in \{1, \dots, n\}$ with* $\operatorname{diam}(\Gamma(\mu_k)) = 1$. *Then the following hold:*

(i) $\operatorname{diam}(\Gamma(\mu)) \neq 1$.

(ii) $\operatorname{diam}(\Gamma(\mu)) = 2$ *if and only if $R_i = Z(\mu_i)$ and* $\operatorname{diam}(\Gamma(\mu_i)) = 2$ *for some $i \in \{1, 2, \dots, n\}$.*

(iii) $\operatorname{diam}(\Gamma(\mu)) = 3$ *if and only if there is no $k \in \{1, \dots, n\}$ with $R_k \neq Z(\mu_k)$ and* $\operatorname{diam}(\Gamma(\mu_k)) = 2$.

Proof. (i) Same as Theorem 3.9 (i).

(ii) ($\Leftarrow$) Same as in proof of Theorem 3.9 (ii)). Conversely, assume that $\operatorname{diam}(\Gamma(\mu)) = 2$; we show that $\operatorname{diam}(\Gamma(\mu_i)) = 2$ and $R_i = Z(\mu_i)$ for some i. Suppose not. Let $i_1, \dots, i_k$ be such that $\operatorname{diam}(\Gamma(\mu_{i_r})) = 2$ $(1 \leq r \leq k)$, and let $j_1, \dots, j_t$ be such that $\operatorname{diam}(\Gamma(\mu_{j_s})) = 3$ $(1 \leq s \leq t)$. Since for each s $(1 \leq s \leq t)$, $\operatorname{diam}(\Gamma(\mu_{j_s})) = 3$, there exist distinct $x_{j_s}, y_{j_s} \in Z^*(\mu_{j_s})$ with $\mu_{j_s}(x_{j_s} y_{j_s}) \neq \mu_{j_s}(0)$. Moreover, for each s $(1 \leq s \leq t)$, there must exist $x'_{j_s}, y'_{j_s} \in Z^*(\mu_{j_s})$ with $\mu_{j_s}(x_{j_s} x'_{j_s}) = \mu_{j_s}(0)$ and $\mu_{j_s}(y_{j_s} y'_{j_s}) = \mu_{j_s}(0)$. Now for each r $(1 \leq r \leq k)$, let $m_{i_r} \in R_{i_r} - Z(\mu_{i_r})$. Set $c = (m_{i_1}, \dots, x_{j_1}, \dots, x_{j_t}, \dots, 0)$ and $d = (m_{i_1}, \dots, y_{j_1}, \dots, y_{j_t}, \dots, 0)$. Then $\mu(c(0, \dots, x'_{j_1}, 0, \dots, 0)) = \mu(0)$, so $c \in Z^*(\mu)$. Similarly, $d \in Z^*(\mu)$. As $\mu(cd) \neq \mu(0)$ and $\operatorname{diam}(\Gamma(\mu)) = 2$, there must be some $e = (e_1, \dots, e_n) \in Z^*(\mu)$ such that $\mu(ce) = \mu(0) = \mu(de)$. But this is a contradiction, as required.

(iii) By (i) and (ii). ■

Theorem 3.14. *Let R, L, and μ be as in Remark* 3.6, *and let $\mu_i \in L_i I(R_i)$ such that* $\operatorname{diam}(\Gamma(\mu_i)) = 1$, $\operatorname{diam}(\Gamma(\mu_j)) = 2$, *and* $\operatorname{diam}(\Gamma(\mu_k)) = 3$. *Then the following hold:*

(i) $\operatorname{diam}(\Gamma(\mu)) \neq 1$.

(ii) $\operatorname{diam}(\Gamma(\mu)) = 2$ *if and only if* $\operatorname{diam}(\Gamma(\mu_i)) \leq 2$ *and $R_i = Z(\mu_i)$ for some $i \in \{1, 2, \dots, n\}$.*

(iii) $\operatorname{diam}(\Gamma(\mu)) = 3$ *if and only if there is no $k \in \{1, 2, \dots, n\}$ with* $\operatorname{diam}(\Gamma(\mu_k)) \leq 2$ *and $R_k = Z(\mu_k)$.*

Proof. (i) Is clear.

(ii) Let $\mathrm{diam}(\Gamma(\mu_i)) \leq 2$ and $R_i = Z(\mu_i)$ for some $i \in \{1, 2, \ldots, n\}$. If $\mathrm{diam}(\Gamma(\mu_i)) = 1$ and $R_i = Z(\mu_i)$ for some i, then by a similar argument as in Theorem 3.11 (ii), we get $\mathrm{diam}(\Gamma(\mu)) = 2$. If $\mathrm{diam}(\Gamma(\mu_i)) = 2$ and $R_i = Z(\mu_i)$ for some i, then by a similar argument as in Theorem 3.12 (ii), we obtain $\mathrm{diam}(\Gamma(\mu)) = 2$. Conversely, assume that $\mathrm{diam}(\Gamma(\mu)) = 2$. It is easy to see from Theorem 3.13 (ii) that $\mathrm{diam}(\Gamma(\mu_i)) \leq 2$ and $R_i = Z(\mu_i)$ for some $i \in \{1, 2, \ldots, n\}$.

(iii) Follows from (i) and (ii). ∎

Example 3.15. Let $R_1 = Z_8$ denote the ring of integers modulo 8, $R_2 = Z_{25}$ the ring of integers modulo 25, $R_3 = Z_6$ the ring of integers modulo 6, $R_4 = Z$ the ring of integers and $R_5 = Z_{24}$ the ring of integers modulo 24. We define the mappings $\mu_1 : R_1 \to [0, 1]$ by

$$\mu_1(x) = \begin{cases} 1 & \text{if } x = \bar{0} \\ 1/2 & \text{otherwise} \end{cases}$$

$\mu_2 : R_2 \to [0, 1]$ by

$$\mu_2(x) = \begin{cases} 1 & \text{if } x = \bar{0} \\ 1/3 & \text{otherwise} \end{cases}$$

$\mu_3 : R_3 \to [0, 1]$ by

$$\mu_3(x) = \begin{cases} 1 & \text{if } x = \bar{0} \\ 1/4 & \text{otherwise} \end{cases}$$

$\mu_4 : R_4 \to [0, 1]$ by

$$\mu_4(x) = \begin{cases} 1/2 & \text{if } x \in 2\mathbb{Z} \\ 1/5 & \text{otherwise.} \end{cases}$$

and $\mu_5 : R_5 \to [0, 1]$ by

$$\mu_5(x) = \begin{cases} 1 & \text{if } x = \bar{0} \\ 1/2 & \text{otherwise.} \end{cases}$$

Then for each i $(1 \leq i \leq 5)$, $\mu_i \in LI(R_i)$, and $Z(\mu_1) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$, $Z(\mu_2) = \{\bar{0}, \bar{5}, \bar{10}, \bar{15}, \bar{20}\}$, $Z(\mu_3) = \{\bar{0}, \bar{2}, \bar{3}, \bar{4}\}$, $Z(\mu_4) = Z$, and $Z(\mu_5) = \{\bar{0}, \bar{2}, \bar{3}, \bar{4}, \bar{6}, \bar{8}, \bar{9}, \bar{10}, \bar{12}, \bar{14}, \bar{15}, \bar{16}, \bar{18}, \bar{20}, \bar{21}, \bar{22}\}$.

(1) Since $\mathrm{diam}(\Gamma(\mu_1)) = 2$, $R_1 \neq Z(\mu_1)$, and $\mathrm{diam}(\Gamma(\mu_4)) = 3$, we get $\mathrm{diam}(\Gamma(\mu_1 \times \mu_4)) = 3$ by Theorem 3.13 (iii).

(2) Since $\operatorname{diam}(\Gamma(\mu_2)) = 1$, $R_2 \neq Z(\mu_2)$, and $\operatorname{diam}(\Gamma(\mu_4)) = 3$, we have $\operatorname{diam}(\Gamma(\mu_2 \times \mu_4)) = 3$ by 3.12 (iii).

(3) As $\operatorname{diam}(\Gamma(\mu_4)) = 3 = \operatorname{diam}(\Gamma(\mu_5))$, we obtain $\operatorname{diam}(\Gamma(\mu_4 \times \mu_5)) = 3$ by Theorem 3.10.

(4) As $\operatorname{diam}(\Gamma(\mu_1)) = 2 = \operatorname{diam}(\Gamma(\mu_3))$, $R_1 \neq Z(\mu_1)$, and $R_3 \neq Z(\mu_3)$, we obtain $\operatorname{diam}(\Gamma(\mu_1 \times \mu_3)) = 3$ by Theorem 3.9 (iii).

(5) Since $\operatorname{diam}(\Gamma(\mu_1)) = 2$, $\operatorname{diam}(\Gamma(\mu_2)) = 1$, $R_1 \neq Z(\mu_1)$, and $R_2 \neq Z(\mu_2)$, we have $\operatorname{diam}(\Gamma(\mu_1 \times \mu_2)) = 3$ by Theorem 3.11 (iii).

(6) An inspection will show that $\operatorname{diam}(\Gamma(\mu_2 \times \mu_3 \times \mu_4)) = 3$ by Theorem 3.14 (iii).

4. Girth and direct products

We continue to use the notation already established; so R, L and μ are as the Remark 3.6. We are now ready to turn our attention toward describing the girth of L-zero-divisor graph of a direct product of L-commutative rings not necessarily with identity. Compare the next theorem with [2, Theorem 4.1].

Theorem 4.1. *Let R, L, and μ be as in Remark* 3.6, *and let $\mu_i \in L_iI(R_i)$ for $i = 1, \dots, n$. Then $\operatorname{gr}(\Gamma(\mu)) = 3$ if and only if one* (*or both*) *of the following hold:*

(i) $|Z^*(\mu_i)| \geq 2$ *for some* $i \in \{1, 2, \dots, n\}$

(ii) $|\operatorname{nil}(\mu_i)^*| \geq 1$ *and* $|\operatorname{nil}(\mu_j)^*| \geq 1$ *for some* $i, j \in \{1, 2, \dots, n\}$ *with* $i \neq j$.

Proof. If (i) holds, there exists $i \in \{1, 2, \dots, n\}$ such that $|Z^*(\mu_i)| \geq 2$. Since $\Gamma(\mu_i)$ is connected by Theorem 2.9, there must exist $a_i, b_i \in Z^*(\mu_i)$ such that $\mu_i(a_i b_i) = \mu_i(0)$. Then

$$(0, \dots, 0, a_i, \dots, 0) - (0, \dots, b_i, \dots, 0) - (0, \dots, c_j, \dots, 0) - (0, \dots, a_i, \dots, 0)$$

is a cycle of length 3, where $c_j \in Z^*(\mu_j)$ and $i \neq j$. If (ii) holds, let $a_i \in R_i^*$ and $b_j \in R_j^*$ with $\mu_i(a_i^2) = \mu_i(0)$ and $\mu_j(b_j^2) = \mu_j(0)$. We may assume that $j > i$. Then

$$\begin{aligned}&(0, \dots, a_i, \dots, 0) - (0, \dots, a_i, \dots, b_j, \dots, 0)\\ &-(0, \dots, b_j, \dots, 0) - (0, \dots, a_i, \dots, 0)\end{aligned}$$

is a cycle of length 3. Conversely, suppose, without loss of generality, R_i has no μ_i-nilpotent elements for $i \in \{2, 3, \dots, n\}$. If $|Z^*(\mu_i)| < 2$, then

$|Z^*(\mu_i)| = 0$ $(2 \leq i \leq n)$. Let $(a_1, \ldots, a_n) - (b_1, \ldots, b_n) - (c_1, \ldots, c_n) - (d_1, \ldots, d_n) - (a_1, \ldots, a_n)$ be a cycle in $\Gamma(\mu)$. Since $|Z^*(\mu_i)| = 0$ for each i $(2 \leq i \leq n)$, there must exist $b_1, c_1 \in R_1$ such that $\mu_1(b_1) \neq \mu(0)$, $\mu_1(c_1) \neq \mu(0)$, and $\mu_1(b_1c_1) = \mu_1(0)$; hence $b_1, c_1 \in Z^*(\mu_1)$. Thus, $|Z^*(\mu_1)| \geq 2$. ■

Compare the next theorem with [2, Theorem 4.2].

Theorem 4.2. *Let R, L, and μ be as in Remark* 3.6, *and let $\mu_i \in L_iI(R_i)$ for $i = 1, 2$. Then* $\mathrm{gr}(\Gamma(\mu)) = 4$ *if and only if both of the following hold:*

(i) $|R_1| \geq 3$ *and* $|R_2| \geq 3$.

(ii) *Without loss of generality, μ_1 is an* L*-integral domain and* $|Z^*(\mu_2)| \leq 1$.

Proof. $(\Leftarrow)$ Clearly, $\mathrm{gr}(\Gamma(\mu)) \neq 3$ by Theorem 4.1 (i) and (ii). Now let $x_1, x_2 \in R_1^*$ be distinct; let $y_1, y_2 \in R_2^*$ be distinct. Then $(x_1, 0) - (0, y_1) - (x_2, 0) - (0, y_2) - (x_1, 0)$ is a cycle. Thus, $\mathrm{gr}(\Gamma(\mu)) = 4$. Conversely, assume that $\mathrm{gr}(\Gamma(\mu)) = 4$. Then Theorem 4.1 gives $|Z^*(\mu_1)| \leq 1$ and $|Z^*(\mu_2)| \leq 1$. Without loss of generality, assume μ_2 is not an L-integral domain; so there exists $x \in Z(\mu_2)$ such that $\mu_2(x) \neq \mu_2(0)$. It follows that $|Z^*(\mu_2)| = |\mathrm{nil}(\mu_2)^*| = 1$. If μ_1 is not an L-integral domain, then $|Z^*(\mu_1)| = |\mathrm{nil}(\mu_1)^*| = 1$. Thus $\mathrm{gr}(\Gamma(\mu)) = 3$, a contradiction. Therefore μ_1 is an L-integral domain; so $Z^*(\mu_1) = \emptyset$. Now a cycle must have the form $(x_1, y_1) - (0, y_2) - (x_2, y_3) - (0, y_4) - (x_1, y_1)$. In this cycle, y_2 and y_4 must be nonzero and distinct. Thus, $|R_2| \geq 3$. If either x_1 or x_2 is zero, then $|Z^*(\mu_2) \geq 2$; whence $\mathrm{gr}(\Gamma(\mu)) = 3$ by Theorem 4.1, a contradiction. If $x_1 = x_2$, then y_1 and y_3 are distinct. If $y_3 = 0$, then $y_1, y_2, y_4 \in Z^*(\mu_2)$, implying $y_1 = y_2 = y_4$, a contradiction. If $y_3 \neq 0$, then $y_2, y_3, y_4 \in Z^*(\mu_2)$, implying $y_2 = y_3 = y_4$, another contradiction. Therefore we must have $x_1 \neq x_2$ and $|R_1| \geq 3$. ■

Example 4.3. (1) Let R_2 and μ_2 be as in Example 3.15. Then $\mathrm{nil}(\mu_2)^* = Z^*(\mu_2) = \{\bar{5}, \bar{10}, \bar{15}, \bar{20}\}$. Since $|Z^*(\mu_i)| \geq 2$, $\mathrm{gr}(\Gamma(\mu_2) \times \mu) = 3$ for every $\mu \in LI(S)$ by Theorem 4.1.

(2) Let $R_1 = Z_7$ denote the ring of integers modulo 7 and $R_2 = Z_4$ the ring of integers modulo 4. We define the mappings $\mu_1 : R_1 \to [0, 1]$ by

$$\mu_1(x) = \begin{cases} 1 & \text{if } x = \bar{0} \\ 1/3 & \text{otherwise.} \end{cases}$$

and $\mu_2 : R_2 \to [0,1]$ by

$$\mu_2(x) = \begin{cases} 1 & \text{if } x = \bar{0} \\ 1/4 & \text{otherwise.} \end{cases}$$

Then for each i $(1 \leq i \leq 2)$, $\mu_i \in LI(R_i)$, $Z^*(\mu_1) = \emptyset$ (so μ_1 is an L-integral domain) and $Z^*(\mu_2) = \{\bar{2}\}$. Since $|Z^*(\mu_2)| = 1$, we get $\mathrm{gr}(\Gamma(\mu_1 \times \mu_2)) = 4$ by Theorem 4.2.

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