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ON L-IDEAL-BASED L-ZERO-DIVISOR GRAPHS

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Abstract

In a manner analogous to a commutative ring, the L-ideal-based L-zero-divisor graph of a commutative ring R can be defined as the undirected graph $\Gamma(\mu)$ for some L-ideal μ of R. The basic properties and possible structures of the graph $\Gamma(\mu)$ are studied.

Keywords: μ-Zero-divisor, L-zero-divisor graph, μ-diameter, μ-girth, μ-nilradical ideal, μ-domainlike ring.

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1. Introduction

Research on the theory of fuzzy sets has been witnessing an exponential growth; both within mathematics and in its applications. This ranges from traditional mathematical like logic, topology, algebra, analysis etc. to pattern recognition, information theory, artificial intelligence, neural networks and planning. Consequently, fuzzy set theory has emerged as a potential area of interdisciplinary research and fuzzy graph theory is of recent interest.

Zadeh in [20] introduced the notion of a fuzzy subset μ of a non-empty set X as a function from X to $[0,1]$. Goguen in [8] generalized the notion of fuzzy subset of X to that of an L-fuzzy subset, namely a function from X to a lattice L. In [14], Rosenfeld considered the fuzzification of algebraic structures. Liu [10], introduced and examined the notion of a fuzzy ideal of a ring. Since then several authors have obtained intersting results on

L-fuzzy ideals of a ring R and L-fuzzy modules (see [8, 9, 11, 18]). See [12] for a comprehensive survey of the literature on these developments.

Rosenfeld in [17] considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975. During the same time Yeh and Bang in [19] has also introduced various connectedness concepts in fuzzy graphs. After the pioneering work of Rosenfeld and Yeh and Bang in 1975, when some basic fuzzy graph theoretic concepts and applications have been indicated, several authors have been finding deeper results, and fuzzy analogues of many other graph theoretic concepts. This include fuzzy trees, fuzzy line graphs, operations on fuzzy graphs, automorphism of fuzzy graphs, fuzzy interval graphs, cycles and cocycles of fuzzy graphs, and meric aspects in fuzzy graphs.

Among the most interesting graphs are the zero-divisor graphs, because these involve both ring theory and graph theory. By studying these graphs we can gain a broader insight into the concepts and properties that involve both graphs and rings. It was Beck (see [6]) who first introduced the notion of a zero-divisor graph for commutative ring. This notion was later redefined by D.F. Anderson and P.S. Livingston in [1]. Since then, there has been a lot of interest in this subject and various papers were published establishing different properties of these graphs as well as relations between graphs of various extensions (see [2, 3, 4]). The notion of a zero-divisor graph were extended to non-commutative rings [15] and to commutative semirings in [7] and various properties were established in [15] and [7].

Let R be a commutative ring with identity and μ an L-ideal of R. In the present paper, we introduce and investigate the L-ideal-based L-zero-divisor graph of R, denoted by $\Gamma(\mu)$ (see Definition 3.2). We know (at least as far as we are aware) of no systematic study of L-fuzzy ideal-based L-zero-divisor graph in the ring context. By mean of the graph $\Gamma(\mu)$, we hope to begin such a study. There are two possible directions one can pursue. The first is to try to understand the possible shapes of the graph $\Gamma(\mu)$ as μ ranges over the class of L-ideals, and the second is to infer properties of the set of μ-zero-divisors of R (see Definition 3.1). In this paper we concentrate on the second direction. Here is a brief summary of our paper. We will make an intensive study of the notions of μ-Zero-divisors, μ-nilradical ideals, and L-zero-divisor graph of a commutative ring R. For example, we show that $\Gamma(\mu)$ is connected with $\mathrm{diam}(\Gamma(\mu)) \leq 3$. Furthermore, if $\Gamma(\mu)$ contains a cycle, then $\mathrm{gr}(\Gamma(\mu)) \leq 4$. Also, we study $\Gamma(\mu)$ for several classes of L-rings which generalize L-valuation domains to the context of rings with μ-zero-

divisors. These are L-rings with non-zero μ-zero-divisors that satisfy certain divisibility conditions between elements and equality conditions between the set of μ-zero-divisors in R and the set of μ-nilpotent elements in R. In this case, we completely characterize the μ-diameter and μ-girth of the L-zero-divisor graph of such L-rings (see Sections 3, 4, 5, and 6).

2. Preliminaries

Throughout this paper R is a commutative ring with identity and L stands for a complete lattice with least element 0 and greatest element 1. In order to make this paper easier to follow, we recall in this section various notions from graph theory and fuzzy commutative algebra theory which will be used in the sequel.

For a graph Γ by $E(\Gamma)$ and $V(\Gamma)$ we denote the set of all edges and vertices, respectively. We recall that a graph is connected if there exists a path connecting any two distinct vertices. At the other extreme, we say that Γ is totally disconnected if no two vertices of Γ are adjacent. The distance between two distinct vertices a and b, denoted by $\mathrm{d}(a,b)$, is the length of the shortest path connecting them (if such a path does not exist, then $d(a,a) = 0$ and $d(a,b) = \infty$). The diameter of graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is equal to $\sup\{d(a,b) : a,b \in V(\Gamma)\}$. A graph is complete if it is connected with diameter less than or equal to one. The girth of a graph Γ, denoted $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise; $\mathrm{gr}(\Gamma) = \infty$. Eccentricity of a vertex a is defined as a $\sup\{d(a,x) : x \in V(\Gamma)\}$. If the diameter of a graph is finite, it is interesting to see what is the smallest eccentricity of a vertex in Γ. Vertices of Γ with this smallest eccentricity form the center of this graph. Center of the graph is one of the so-called central sets of a graph. Therefore, notably, the graphs with finite diameter are very important.

If R is a commutative ring, let $Z(R)$ denote the set of zero-divisors of R and let $Z(R)^*$ denote the set of non-zero zero-divisors of R. We consider the undirected graph $\Gamma(R)$ with vertices in the set $V(\Gamma(R)) = Z(R)^*$, such that for distinct vertices a and b there is an edge connecting them if and only if $ab = 0$. Then $\Gamma(R)$ is connected with $\mathrm{diam}(\Gamma(R)) \leq 3$ ([1, Theorem 2.3]) and $\mathrm{gr}(\Gamma(R)) \leq 4$ ([13, (1.4)]). Thus $\mathrm{diam}(\Gamma(R)) = 0, 1, 2$, or 3 and $\mathrm{gr}(\Gamma(R)) = 3, 4$, or ∞.

Let R be a commutative ring and L stands for a complete lattice with least element 0 and greatest element 1. By an L-subset μ of a non-empty set X, we mean a function μ from X to L. If $L = [0,1]$, then μ is called

a fuzzy subset of X. L^X denotes the set of all L-subsets of X. We recall some definitions and lemmas from the book [12], which we need them for development of our paper.

Definition 2.1. A L-ring is a function $\mu : R \to L$, where $(R, +, .)$ is a ring, that satisfies:

(1) $\mu \neq 0$;

(2) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$ for every x, y in R;

(3) $\mu(xy) \geq \mu(x) \wedge \mu(y)$ for every x, y in R.

Definition 2.2. Let $\mu \in L^R$. Then μ is called an L-ideal of R if for every $x, y \in R$ the following conditions are satisfied:

(1) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$;

(2) $\mu(xy) \geq \mu(x) \vee \mu(y)$.

The set of all L-ideals of R is denoted by $LI(R)$.

Lemma 2.3. *Let R be a ring and $\mu \in LI(R)$. Then $\mu(x) \leq \mu(0)$ and $\mu(1) \leq \mu(x)$ for every x in R.*

3. Definitions and basic structures

We begin with the key definition of this paper.

Definition 3.1. Let R be a ring and $\mu \in LI(R)$. A μ-zero-divisor is an element $x \in R$ for which there exists $y \in R$ with $\mu(y) \neq \mu(0)$ such that $\mu(xy) = \mu(0)$.

The set of μ-zero-divisors in R will be denoted by $Z(\mu)$.

Definition 3.2. Let R be a ring and $\mu \in LI(R)$. We define an undirected graph $\Gamma(\mu)$ with vertices $V(\Gamma(\mu)) = Z(\mu)^* = Z(\mu) - \mu_* = \{x \in Z(\mu) : \mu(x) \neq \mu(0)\}$, where distinct vertces x and y are adjacent if and only if $\mu(xy) = \mu(0)$, where $\mu_* = \{x \in R : \mu(x) = \mu(0)\}$.

Notation. For the graph $\Gamma(\mu)$ by $\mathrm{diam}(\Gamma(\mu))$, $\mathrm{gr}(\mu)$ and $d_\mu(a,b)$ we denote the diameter, the girth and the distance between two distinct vertices a and b, respectively.

Remark 3.3. Let R be a ring and $\mu \in LI(R)$. Clearly, if μ is a non-zero constant, then $\Gamma(\mu) = \emptyset$. So throughout this paper we shall assume unless otherwise stated, that μ is not a non-zero constant. Thus there is a non-zero element y of R such that $\mu(y) \neq \mu(0)$.

Definition 3.4. Let R be a ring and $\mu \in LI(R)$. We say μ is an L-integral domain if $Z(\mu) = \mu_*$.

Theorem 3.5. *Let R be a ring and $\mu \in LI(R)$. Then the following hold:*

(1) *If μ is one to one, then $\Gamma(R) = \Gamma(\mu)$.*

(2) *μ is an L-integral domain if and only if $\Gamma(\mu) = \emptyset$.*

Proof. This follows directly from the definitions. ■

Definition 3.6. Let R be a ring and $\mu \in LI(R)$. An element $a \in R$ is said to be μ-nilpotent precisely when there exists a positive integer n such that $\mu(a^n) = \mu(0)$.

The set of all μ-nilpotents of R is denoted by $\mathrm{nil}(\mu)$, and we set $\mathrm{nil}(\mu)^* = \mathrm{nil}(\mu) - \mu_*$.

Remark 3.7. Assume that R is a ring and $\mu \in LI(R)$. Let $\mu(x^n) = \mu(0)$ for some positive integer n. By Remark 3.3, there exists $0 \neq y \in R$ such that $\mu(y) \neq \mu(0)$. Then $\mu(x^n y) \geq \mu(x^n) \vee \mu(y) = \mu(0) \vee \mu(y) = \mu(0)$; so $\mu(x^n y) = \mu(0)$ by Lemma 2.3. Hence $x^n \in Z(\mu)$. In particular, if $\mu(x) = \mu(0)$, we conclude that $x \in Z(\mu)$. Moreover, $\mu(x^{n+k}) \geq \mu(x^n) \vee \mu(x^k) = \mu(0)$; hence $\mu(x^{n+k}) = \mu(0)$ for every positive integer k.

Example 3.8. Let $R = Z_8$ denote the ring of integers modulo 8. We define the mapping $\mu : R \to [0,1]$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x = \bar{0} \\ 1/2 & \text{if } x \neq \bar{0}. \end{cases}$$

Then $\mu \in LI(R)$ and $Z(\mu) = \mathrm{nil}(\mu) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$.

Lemma 3.9. *Let R be a ring and $\mu \in LI(R)$. Then The following hold:*

(1) $\mathrm{nil}(\mu)$ *is an ideal of R and* $\mu_* \subseteq \mathrm{nil}(\mu) \subseteq Z(\mu)$.

(2) *If $Z(\mu)$ is an ideal of R, then $Z(\mu)$ is a prime ideal of R.*

Proof. (1) Let $x, y \in \mathrm{nil}(\mu)$ and $r \in R$. Then $\mu(x^n) = \mu(0) = \mu(y^m)$ for some positive integers n, m. So there are integers $a_0, a_1, \ldots, a_{n+m}$ such that $\mu((x-y)^{m+n}) = \mu(a_0x^{n+m} + \cdots + a_mx^ny^m + \cdots + a_{n+m}y^{n+m}) \geq$

$$\mu(a_0x^{n+m}) \wedge \cdots \wedge \mu(a_{n+m}y^{n+m}) \geq \mu(0) \wedge \cdots \wedge \mu(0) = \mu(0),$$

and hence $\mu((x-y)^{m+n}) = \mu(0)$ by Lemma 2.3 and Remark 3.7. Thus $x - y \in \mathrm{nil}(\mu)$. Since $\mu((rx)^n) \geq \mu(r^n) \vee \mu(x^n) \geq \mu(r^n) \vee \mu(0) = \mu(0)$, we get $rx \in \mathrm{nil}(\mu)$. Thus $\mathrm{nil}(\mu)$ is an ideal of R. Clearly, $\mu_* \subseteq \mathrm{nil}(\mu)$. Finally, let $x \in \mathrm{nil}(\mu)$. By Remark 3.7, we may assume that $\mu(x) \neq \mu(0)$. Since $x \in \mathrm{nil}(\mu)^*$, let n ($n \geq 2$) be the least positive integer such that $\mu(x^n) = \mu(0)$. As $\mu(x^{n-1}) \neq \mu(0)$, $\mu(x) \neq \mu(0)$ and $\mu(xx^{n-1}) = \mu(0)$, we conclude that $x \in Z(R)$, as required.

(2) Let $x, y \in R$ be such that $xy \in Z(\mu)$. Then there exists $z \in R$ such that $\mu(z) \neq \mu(0)$ and $\mu(xyz) = \mu(0)$. Therefore, if $\mu(yz) = \mu(0)$, then $y \in Z(\mu)$. If $\mu(yz) \neq \mu(0)$, then $x \in Z(\mu)$. Thus $Z(\mu)$ is a prime ideal of R. ■

Theorem 3.10. *Let R be a ring and $\mu \in LI(R)$. Then the following hold:*

(1) *If $x \in \mathrm{nil}(\mu)^*$ and $y \in Z(\mu)^*$, then $d_\mu(x, y) \leq 2$ in $\Gamma(\mu)$.*

(2) *Let $x \in Z(\mu) - \mathrm{nil}(\mu)$, and let $y \in \mathrm{nil}(\mu)^*$ such that $x|zy^n$ for some positive integer n and $z \in R - Z(\mu)$. Then $d_\mu(x, y) \leq 2$ in $\Gamma(\mu)$.*

Proof. (1) We may assume that $x \neq y$ and $\mu(xy) \neq \mu(0)$. Since $y \in Z(\mu)^*$ and $\mu(xy) \neq \mu(0)$, there is a $z \in Z(\mu)^* - \{x\}$ such that $\mu(zy) = \mu(0)$. Let n be the least positive integer such that $\mu(x^nz) = \mu(0)$ since $x \in \mathrm{nil}(\mu)^*$. If $n = 1$, then $x - z - y$ is a path of length 2 from x to y. If $n \geq 2$, then $x - x^{n-1}z - y$ is a path between x and y. Thus $d_\mu(x, y) \leq 2$.

(2) We may assume that $x \neq y$ and $\mu(xy) \neq \mu(0)$. Since $x \in Z(\mu) - \mathrm{nil}(\mu)$ and $\mu(xy) \neq \mu(0)$, there is a $w \in Z(\mu)^* - \{x, y\}$ such that $\mu(xw) = \mu(0)$. Since $x|zy^n$ with $\mu(z) \neq \mu(0)$ (otherwise, $z \in \mu_* \subseteq Z(\mu)$, a contradiction) and $\mu(xw) = \mu(0)$, we get $\mu(zy^nw) = \mu(0)$. If $\mu(y^nw) \neq \mu(0)$, then $z \in Z(\mu)$, a contradiction. So we conclude that $\mu(y^nw) = \mu(0)$. Let m be

the least positive integer such that $\mu(wy^m) = \mu(0)$. If $m = 1$, then $x - w - y$ is a path of length 2 from x to y. If $m \geq 2$, then $x - y^{m-1}w - y$ is a path between x and y. Thus $d_\mu(x, y) \leq 2$ in $\Gamma(\mu)$. ■

Definition 3.11. Let R be a ring and $\mu \in LI(R)$. A μ-unit of R is an element $a \in R$ if $\mu(a) \neq \mu(0)$ and for which there exist $b \in R$ such that $\mu(ab) = \mu(1)$.

The set of all μ-units of R is denoted by $U(\mu)$.

Proposition 3.12. *Let R be a ring and $\mu \in LI(R)$. Then the following hold:*

(1) *If $x \in U(\mu)$, then $\mu(x) = \mu(1)$.*

(2) *If $x \in \mathrm{nil}(\mu)$, then $1 + x \in U(\mu)$.*

(3) *If $x^n \in U(\mu)$ for some positive integer n, then $x \in U(\mu)$.*

Proof. (1) Since x is a μ-unit, there is an element $y \in R$ such that $\mu(1) = \mu(xy) \geq \mu(x) \vee \mu(y) \geq \mu(x)$. Now the assertion follows from Lemma 2.3.

(2) By assumption, there exists a positive integer n such that $\mu(x^n) = \mu(0)$ and

$$\mu(1 + x^n) \geq \mu(1) \wedge \mu(x^n) = \mu(1) \wedge \mu(0) \geq \mu(1).$$

On the other hand, $\mu(1) = \mu(1 + x^n - x^n) \geq \mu(1 + x^n) \wedge \mu(x^n) = \mu(1 + x^n)$; hence $\mu(1) = \mu(1 + x^n) = \mu((1 + x)(1 + x + \dot{+}(-1)^{n-1}x^{n-1}))$. It follows that $1 + x$ is a μ-unit in R.

(3) Is clear. ■

Example 3.13. Let R denote the ring of integers modulo 8 and let μ be the fuzzy ideal in Example 3.8. Then $U(\mu) = \{\bar{1}, \bar{2}, \ldots, \bar{7}\}$ and $U(\mu) \cap Z(\mu) = \{\bar{2}, \bar{4}, \bar{6}\}$.

Theorem 3.14. *Let R be a ring and $\mu \in LI(R)$. Let $\mathrm{nil}(\mu)$ be a prime ideal of R with $\mathrm{nil}(\mu) \subsetneqq Z(\mu)$ and $Z(\mu) \cap U(\mu) = \emptyset$. Then the set $Z(\mu) - \mathrm{nil}(\mu)$ is not finite.*

Proof. Suppose not. If $x \in Z(\mu) - \mathrm{nil}(\mu)$, then there exist positive integers n, m $(n > m \geq 1)$ such that $x^m = x^n$, so $\mu(x^m(1 - x^{n-m})) = \mu(0)$; hence $x^m(1 - x^{n-m}) \in \mathrm{nil}(\mu)$. Then $\mathrm{nil}(\mu)$ prime gives $1 - x^{n-m} \in \mathrm{nil}(\mu)$, and so by Proposition 3.11, $x \in U(\mu) \cap Z(\mu)$, which is a contradiction, as needed. ■

Theorem 3.15. *Let R be a ring and $\mu \in LI(R)$. Then $V(\Gamma(\mu)) - \mathrm{nil}(\mu)$ is totally disconnected if and only if* $\mathrm{nil}(\mu)$ *is a prime ideal of R.*

Proof. Suppose that $V(\Gamma(\mu)) - \mathrm{nil}(\mu)$ is totally disconnected. Let $x, y \notin \mathrm{nil}(\mu)$ such that $xy \in \mathrm{nil}(\mu)$. So there exists a positive integer n such that $\mu(x^n y^n) = \mu(0)$. If $\mu(x^n) = \mu(0)$, then $x \in \mathrm{nil}(\mu)$, which is a contradiction. So we may assume that $\mu(x^n) \neq \mu(0)$ and $\mu(y^n) \neq \mu(0)$. If $x^n = y^n$, then $\mu(x^{2n}) = \mu(0)$; thus $x \in \mathrm{nil}(\mu)$, a contradiction. So we may assume that $x^n \neq y^n$. Thus $x^n, y^n \in V(\Gamma(\mu)) - \mathrm{nil}(\mu)$ and $x^n - y^n$ is a path from x^n to y^n in $\Gamma(\mu)$ and this is a contradiction. Thus $xy \notin \mathrm{nil}(\mu)$ and $\mathrm{nil}(\mu)$ is a prime ideal of R. Conversely, assume that $\mathrm{nil}(\mu)$ is a prime ideal of R, and let x and y be two distinct elements of $V(\Gamma(\mu)) - \mathrm{nil}(\mu)$. Suppose that $\mu(xy) = \mu(0)$. Then $xy \in \mathrm{nil}(\mu)$; hence either x or y belong to $\mathrm{nil}(\mu)$, which is a contradiction. ■

Example 3.16. Let $R = \mathbb{Z}$ denote the ring of integers. We define the mapping $\mu : R \to [0, 1]$ by

$$\mu(x) = \begin{cases} 1/2 & \text{if } x \in 2\mathbb{Z} \\ 1/5 & \text{otherwise} \end{cases}$$

Then $\mu \in LI(R)$, $Z(\mu) = \mathbb{Z}$ and $\mathrm{nil}(\mu) = 2\mathbb{Z}$. Since $\mathrm{nil}(\mu)$ is a prime ideal of R, we get $V(\Gamma(\mu)) - \mathrm{nil}(\mu)$ is totally disconnected by Theorem 3.15.

Theorem 3.17. *Let R be a ring and $\mu \in LI(R)$. Then $\Gamma(\mu)$ is connected with* $\mathrm{diam}(\Gamma(\mu)) \leq 3$.

Proof. Let x and y be distinct vertices of $\Gamma(\mu)$. We split the proof into five cases.

Case 1. $\mu(xy) = \mu(0)$. Then $x - y$ is a path in $\Gamma(\mu)$.

Case 2. $\mu(xy) \neq \mu(0)$, $\mu(x^2) = \mu(0)$, and $\mu(y^2) = \mu(0)$. Then

$$\mu(x(xy)) = \mu(x^2 y) \geq \mu(x^2) \vee \mu(y) = \mu(0) \vee \mu(y) = \mu(0).$$

Then Lemma 2.3 gives $\mu(x(xy)) = \mu(0)$. Similarly, $\mu(y(xy)) = \mu(0)$. Then $x - xy - y$ is a path in $\Gamma(\mu)$.

Case 3. $\mu(xy) \neq \mu(0)$, $\mu(x^2) = \mu(0)$, and $\mu(y^2) \neq \mu(0)$. Then there is an element $b \in Z(\mu)^* - \{x, y\}$ with $\mu(by) = \mu(0)$. If $\mu(bx) = \mu(0)$, then $x - b - y$ is a path between x and y. If $\mu(bx) \neq \mu(0)$, then $x - bx - y$ is a path. In either case there is a path between x and y.

Case 4. $\mu(xy) \neq \mu(0)$, $\mu(x^2) \neq \mu(0)$, and $\mu(y^2) = \mu(0)$. The proof of (4) is similar to that (3).

Case 5. $\mu(xy) \neq \mu(0)$, $\mu(x^2) \neq \mu(0)$, and $\mu(y^2) \neq \mu(0)$. Then there are $a, b \in Z(\mu)^* - \{x, y\}$ with $\mu(ax) = \mu(0) = \mu(by)$. If $a = b$, then $x - a - y$ is a path. If $a \neq b$ and $\mu(ab) \neq \mu(0)$, then $x - ab - y$ is a path. If $a \neq b$ and $\mu(ab) = \mu(0)$, then $x - a - b - y$ is a path between x and y.

Thus $\Gamma(\mu)$ is connected and $\mathrm{diam}(\Gamma(\mu)) \leq 3$. ■

Theorem 3.18. *Let R be a ring and $\mu \in LI(R)$. If $\Gamma(\mu)$ contains a cycle, then* $\mathrm{gr}(\Gamma(\mu)) \leq 4$.

Proof. Suppose not. Assume that $\Gamma(\mu)$ contains a cycle $x_0 - x_1 - \cdots - x_n - x_0$ such that $\mathrm{gr}(\Gamma(\mu)) > 4$ (so $n \geq 4$), $\mu(x_i x_j) \neq \mu(0)$ for all $i, j \in \{0, 1, \ldots, n\}$ with $|i - j| \geq 2$ and $\mu(x_i x_{i+1}) = \mu(0)$. We split the proof into three cases.

Case 1. $x_1 x_{n-1} \neq x_0$ and $x_1 x_{n-1} \neq x_n$. Then $\mu(x_0 x_n) = \mu(0)$ and $\mu(x_1 x_{n-1}) \neq \mu(0)$ since $|n - 2| \geq 2$, and we have

$$\mu(x_0 x_1 x_{n-1}) \geq \mu(x_0 x_1) \vee \mu(x_{n-1}) = \mu(0) \vee \mu(x_{n-1}) = \mu(0).$$

Thus $\mu(x_0 x_1 x_{n-1}) = \mu(0)$ by Lemma 2.3. Similarly, $\mu(x_1 x_{n-1} x_n) = \mu(0)$. So $x_0 - x_1 x_{n-1} - x_n - x_0$ is a cycle of length 3.

Case 2. $x_1 x_{n-1} = x_0$. Since $\mu(x_0^2) = \mu(x_0 x_1 x_{n-1}) \geq$

$$\mu(x_0 x_1) \vee \mu(x_{n-1}) = \mu(0) \vee \mu(x_{n-1}) = \mu(0),$$

we get $\mu(x_0^2) = \mu(x_0 x_1 x_{n-1}) = \mu(0)$. We claim that there is an element y of R such that $\mu(x_0 y) \neq \mu(0)$ and $x_0 y \neq x_0$. Suppose not. Then for every $y \in R$, either $\mu(x_0 y) = \mu(0)$ or $x_0 y = x_0$. Take $y = x_3$. Then by

assumption, $\mu(x_0x_3) \neq \mu(0)$ and $x_0x_3 \neq x_0$ (if $x_0x_3 = x_0$, then $\mu(x_0x_2) = \mu(x_0x_2x_3) \geq \mu(x_0) \vee \mu(x_2x_3) = \mu(0)$, so $\mu(x_0x_2) = \mu(0)$ by Lemma 2.3, a contradiction), which is a contradiction. So there is an element y of R such that $\mu(x_0y) \neq \mu(0)$ and $x_0y \neq x_0$.

If $x_0y \neq x_1$, then $\mu(x_0x_1y) \geq \mu(x_0x_1) \vee \mu(y) = \mu(0) \vee \mu(y) = \mu(0)$, so $\mu(x_0x_1y) = \mu(0)$. Thus, we have either $x_0 - x_1 - x_0y - x_0$ is a cycle. Similarly, if $x_0y = x_1$, then $x_0y \neq x_n$ and $\mu(x_nx_0y) = \mu(0) = \mu(x_0yx_0)$. Thus $x_0 - x_n - x_0y - x_0$ is a 3-cycle in $\Gamma(\mu)$.

Case 3. $x_1x_{n-1} = x_n$. This necessarily forces $\mu(x_n^2) = \mu(0)$ and there exists an element $y \in R$ such that $\mu(x_ny) \neq \mu(0)$ and $x_ny \neq x_n$. If $x_ny \neq x_n$, then $x_n - x_ny - x_{n-1} - x_n$ is a cycle of length 3, and if $x_ny = x_n$, then $x_n - x_0 - x_ny - x_n$ is a 3-cycle in $\Gamma(\mu)$. Thus, every case leads to a contradiction. ■

4. $\Gamma(\mu)$ when $\mu \in \Re_R$

Let R be a commutative ring with identity. A prime ideal P of R is called a divided prime ideal of R if $P \subseteq Rx$ for all $x \in R-P$. Let $\Re_R = \{\mu \in LI(R) : \mathrm{nil}(\mu)$ is a non-zero divided prime ideal of $R\}$. We are interested in the case where the L-ideal μ satisfies $\mu_* \neq \mathrm{nil}(\mu) \subseteq Rz$ for all $z \in Z(\mu) - \mathrm{nil}(\mu)$. In particular, this condition holds when $\mu \in \Re_R$. In this case, we show that $\mathrm{nil}(\mu)$ is a divided prime ideal of R when $\mathrm{nil}(\mu) \subsetneqq Z(\mu)$.

Theorem 4.1. *Let R be a ring and $\mu \in LI(R)$ with $\mu_* \neq \mathrm{nil}(\mu) \subseteq Rz$ for all $z \in Z(\mu) - \mathrm{nil}(\mu)$. Then the following hold:*

(1) $\mathrm{nil}(\mu)$ *is a prime ideal of R.*

(2) $\mathrm{nil}(\mu) \subseteq \bigcap_{n\geq 1} Rz^n$ *for all $z \in Z(\mu) - \mathrm{nil}(\mu)$.*

(3) *If* $\mathrm{nil}(\mu) \subsetneqq Z(\mu)$, *then* $\mathrm{nil}(\mu)$ *is a divided prime ideal of R.*

Proof. (1) If $Z(\mu) = \mathrm{nil}(\mu)$, then $\mathrm{nil}(\mu)$ is a prime ideal of R by Lemma 3.9. So we may assume that $\mathrm{nil}(\mu) \subsetneqq Z(\mu)$ and nil)($\mu \subseteq Rz$ for all $z \in Z(\mu) - \mathrm{nil}(\mu)$. Suppose that $\mathrm{nil}(\mu)$ is not prime. Then there exist $x, y \in Z(\mu) - \mathrm{nil}(\mu)$ such that $xy \in \mathrm{nil}(\mu)$. So there exists a positive integer n such that $\mu(x^ny^n) = \mu(0)$. It is easy to see that there exists a positive integer $m < n$ such that $\mu(x(x^my^n) = \mu(0)$ with $\mu(x^my^n) \neq \mu(0)$; hence $x \in Z(\mu)$. Similarly, $y \in Z(\mu)$. Since $x, y \notin \mathrm{nil}(\mu)$, we have $\mu(x^k) \neq \mu(0)$

and $\mu(y^k) \neq \mu(0)$ for all positive integer k. As $x \in Z(\mu)$, there is an element $x' \in R$ with $\mu(x') \neq \mu(0)$ such that $\mu(xx') = \mu(0)$. Since $\mu(x^2) \neq \mu(0)$ and $\mu(x^2x') = \mu(0)$ (because $\mu(x^2x') \geq \mu(xx') \vee \mu(x) = \mu(0) \vee \mu(x) = \mu(0)$, and so we have equality by Lemma 2.3), we get $x^2 \in Z(\mu) - \mathrm{nil}(\mu)$; hence $\mathrm{nil}(\mu) \subseteq Rx^2$. Thus $xy = x^2d$ for some $d \in R$. Moreover, since $\mu(0) = \mu(x^{2n}d^n)$ and $\mu((xd)^{2n})) \geq \mu(x^{2n}d^n) \vee \mu(d^n) = \mu(0)$, we have $\mu((xd)^{2n})) = \mu(0)$ by Lemma 2.3; so $xd \in \mathrm{nil}(\mu)$. Thus $y - xd \notin \mathrm{nil}(\mu)$ since $y \notin \mathrm{nil}(\mu)$. As $\mu(x(y - xd)) = \mu(xy - x^2y) = \mu(0)$ and $\mu(x) \neq \mu(0)$, we must have $y - xd \in Z(\mu) - \mathrm{nil}(\mu)$. Thus $\mathrm{nil}(\mu) \subseteq R(y - xd)$, and hence $x\mathrm{nil}(\mu) \subseteq R(x(y - xd)) = \{0\}$. Let $z \in \mathrm{nil}(\mu) - \mu_* \subseteq Rx^2$. Then $z = x^2r$ for some $r \in R$. Then $z \in \mathrm{nil}(\mu)$ gives $\mu(x^{2k}r^k) = \mu(0)$ for some positive integer k; hence $\mu((xr)^{2k}) \geq \mu(x^{2k}r^k) \vee \mu(r^k) = \mu(0)$. It then follows from Lemma 2.3 that $xr \in \mathrm{nil}(\mu)$. Thus $z = x(xr) \in x\mathrm{nil}(\mu) = \{0\}$, a contradiction. Hence $\mathrm{nil}(\mu)$ is a prime ideal of R.

(2) Let $z \in Z(\mu) - \mathrm{nil}(\mu)$. Then there exists $z' \in R$ with $\mu(z') \neq \mu(0)$ and $\mu(zz') = \mu(0)$. Let n $(n \geq 2)$ be an integer. Then $\mu(z^nz') \geq \mu(z^{n-1}) \vee \mu(zz') = \mu(0)$, so $\mu(z^nz') = \mu(0)$. Thus $z^n \in Z(\mu)$ for all positive integer n. Since $\mathrm{nil}(\mu)$ is a prime ideal of R by part (1) above, and thus $\mathrm{nil}(\mu) \subseteq Rz^n$ for all integers $n \geq 1$. Hence $\mathrm{nil}(\mu) \subseteq \bigcap_{n\geq 1} Rz^n$.

(3) Let $z \in R - \mathrm{nil}(\mu)$ and $w \in Z(\mu) - \mathrm{nil}(\mu)$. There is an element $w' \in R$ with $\mu(w') \neq \mu(0)$ and $\mu(ww') = \mu(0)$. Since $\mu(ww'z) \geq \mu(ww') \vee \mu(z) = \mu(0)$, we coclude that $\mu(ww'z) = \mu(0)$;- thus $wz \in Z(\mu)$. So $wz \in Z(\mu) - \mathrm{nil}(\mu)$ since $\mathrm{nil}(\mu)$ is prime, and thus $\mathrm{nil}(\mu) \subseteq Rwz \subseteq Rz$. ■

Corollary 4.2. *Let R be a ring and $\mu \in LI(R)$. Then the following statements are equivalent:*

(1) $\mu_* \neq \mathrm{nil}(\mu) \subseteq Rz$ *for all* $z \in Z(\mu) - \mathrm{nil}(\mu)$ *and* $\mathrm{nil}(\mu) \subsetneqq Z(\mu)$.
[(2)] $\mu \in \Re_R$ *and* $\mathrm{nil}(\mu) \subsetneqq Z(\mu)$.

Proof. Apply Theorem 4.1. ■

Definition 4.3. Let R be a ring, $\mu \in LI(R)$, and J an ideal of R. Then the subset $\mathrm{ann}_\mu(J)$, the μ-annihilator of J with respect to μ, is defined by

$$\mathrm{ann}_\mu(J) = \{y \in R : \mu(yJ) = \mu(0)\} = \{y \in R : \mu(xy) = \mu(0) \text{ for all } x \in J\}.$$

Lemma 4.4. *Let R be a ring, $\mu \in LI(R)$, and J an ideal of R. Then $\mathrm{ann}_\mu(J)$ is an ideal of R.*

Proof. The proof is straightforward. ■

Theorem 4.5. *Let* $\mu \in \Re_R$, $\mathrm{nil}(\mu) \subsetneqq Z(\mu)$, *and* $N(\mu) = \{x \in R : \mu(x^2) = \mu(0)\}$.

(1) *If* $\mu(xy) = \mu(0)$ *for* $x \in Z(\mu) - \mathrm{nil}(\mu)$ *and* $y \in R$, *then* $y \in N(\mu) \subseteq \mathrm{nil}(\mu)$ *and* $\mathrm{ann}_\mu(x) \subseteq \mathrm{ann}_\mu(\mathrm{nil}(\mu))$. *Then the following hold:*

(2) $\mathrm{nil}(\mu)$ *is infinite.*

(3) $V(\Gamma(\mu)) - \mathrm{nil}(\mu)$ *is totally disconnected.*

Proof. (1) By hypothesis, $xy \in \mathrm{nil}(\mu)$, so $y \in \mathrm{nil}(\mu)$ since $\mathrm{nil}(\mu)$ is a divided prime ideal of R by Theorem 4.1; hence $\mathrm{nil}(\mu) \subseteq Rx$. Thus $y^2 \in y\mathrm{nil}(\mu) \subseteq R(xy)$; thus $y^2 = xyr$ for some $r \in R$. Furthermore, $\mu(y^2) \geq \mu(xy) \vee \mu(r) = \mu(0)$. This implies that $\mu(y^2) = \mu(0)$ by Lemma 2.3. Hence $y \in N(\mu)$. Let $z \in \mathrm{ann}_\mu(x)$. Then $\mu(xz) = \mu(0)$, so $z\mathrm{nil}(\mu) \subseteq R(xz)$. Suppose that $u \in \mathrm{nil}(\mu)$. Then $uz \in z\mathrm{nil}(\mu) \subseteq R(xz)$; thus $uz = xzs$ for some $s \in R$. Therefore, since $\mu(uz) \geq \mu(xz) \vee \mu(s) = \mu(0)$, we get $\mu(uz) = \mu(0)$; hence $z \in \mathrm{ann}_\mu(\mathrm{nil}(\mu))$, as needed.

(2) Suppose not. Let $x \in Z(\mu) - \mathrm{nil}(\mu)$. Then there exists $z \in R$ with $\mu(z) \neq \mu(0)$ and $\mu(xz) = \mu(0)$. By part (1) above, $z \in \mathrm{nil}(\mu)^*$. It then follows from Theorem 4.1 that $\mathrm{nil}(\mu) \subseteq \bigcap_{n\geq 1} R(x^2)$. So for every positive integer n, we must have $z \in R(x^n)$. Then for each positive integer n, we have $z = z_n x^n$ for some $z_n \in R$. Note that $z_n \in \mathrm{nil}(\mu)^*$ since $\mathrm{nil}(\mu)$ is a prime ideal of R and $x^n \notin \mathrm{nil}(\mu)$. Since $\mathrm{nil}(\mu)$ is finite, there exist positive integers $n > m$ such that $z_m = z_n$, so $z = z_n x^n = z_m x^n = x^{n-m}(z_m x^m) = x^{n-m} z$. Moreover, $\mu(z) = \mu(x^{n-m} z) \geq \mu(xz) \vee \mu(x^{n-m-1}) = \mu = (0) \vee \mu(x^{n-m-1}) = \mu(0)$; hence $\mu(z) = \mu(0)$, which is a contradiction.

(3) Since $\mathrm{nil}(\mu)$ is a prime ideal of R, the graph $V(\Gamma(\mu)) - \mathrm{nil}(\mu)$ is totally disconnected by Theorem 3.15. ■

5. L-chained rings

In this section, we continue the investigation of $\Gamma(\mu)$ when R is a chained ring and $\mu \in LI(R)$. We say that a ring R is a chained ring if the (principal) ideals of R are linearly ordered (by inclusion), equivalently, if either $x|y$ or $y|x$ for all $x, y \in R$.

Lemma 5.1. *Let R be a ring and $\mu \in LI(R)$. If $N(\mu) = \{x \in R : \mu(x^2) = \mu(0)\}$ and $x \in \mathrm{nil}(\mu) - N(\mu)$, then $\mu(xy) = \mu(0)$ for some $y \in N(\mu)^* - \{x\}$, where $N(\mu)^* = N(\mu) - \mu_*$.*

Proof. Let n ($n \geq 3$) be the least positive integer such that $\mu(x^n) = \mu(0)$ and let $y = x^{n-1}$. Then $\mu(xy) = \mu(0)$, $\mu(y) \neq \mu(0)$, and $\mu(y^2) = \mu(x^{2n-2})$. It follows from Lemma 2.3 that $\mu(x^{2n-2}) = \mu(x^n x^{n-2}) \geq \mu(x^n) \vee \mu(x^{n-2}) = \mu(0) \vee \mu(x^{n-2}) = \mu(0)$; hence $\mu(y^2) = \mu(0)$ by Lemma 2.3. Clearly, $x \neq y$ since $\mu(x^2) \neq \mu(0)$, and the proof is complete. ■

Proposition 5.2. *Let R be a chained ring, $\mu \in LI(R)$, $N(\mu) = \{x \in R : \mu(x^2) = \mu(0)\}$ and $x, y \in R$.*

(1) *If $\mu(xy) = \mu(0)$, then either $x \in N(\mu)$ or $y \in N(\mu)$.*

(2) *If $x, y \in N(\mu)$, then $\mu(xy) = \mu(0)$.*

(3) *If $x, y \in Z(\mu) - N(\mu)$, then $\mu(xy) \neq \mu(0)$.*

(4) *If $x \in Z(\mu)^*$, then $\mu(xy) = \mu(0)$ for some $y \in N(\mu)^*$.*

(5) *If $x_1, x_2, \ldots, x_n \in Z(\mu)^*$, then there is a $y \in N(\mu)^*$ such that $\mu(x_i y) = \mu(0)$ for every integer i, $1 \leq i \leq n$.*

(6) *$N(\mu)$ is an ideal of R.*

(7) *$N(\mu)$ is a prime ideal of R if and only if $N(\mu) = \mathrm{nil}(\mu)$.*

Proof. (1) We may assume that $x|y$. Then $y = ax$ for some $a \in R$; hence $\mu(y^2) = \mu(axy) \geq \mu(a) \vee \mu(xy) = \mu(a) \vee \mu(0) = \mu(0)$. Thus $\mu(y^2) = \mu(0)$ by Lemma 2.3; so $y \in N(\mu)$.

(2) We may assume that $x|y$. Then $y = ax$ for some $a \in R$; so $\mu(xy) = \mu(ax^2) \geq \mu(a) \vee \mu(x^2) = \mu(0)$. Thus $\mu(xy) = \mu(0)$.

(3) follows from the case (1) above.

(4) If $x \in N(\mu)$, then let $y = x$. If $x \in Z(\mu) - N(\mu)$, then there exists $y \in R$ with $\mu(y) \neq \mu(0)$ such that $\mu(xy) = \mu(0)$. By the case (3) above, we must have $y \in N(\mu)^*$.

(5) Since R is a chained ring, there is an integer j, $1 \leq j \leq n$, such that $x_j | x_i$ for all i, $1 \leq i \leq n$. By the case (4) above, there exists $y \in N(\mu)^*$ such that $\mu(x_j y) = \mu(0)$; hence $\mu(x_i y) = \mu(0)$ for all i, $1 \leq i \leq n$.

(6) Let $x, y \in N(\mu)$ and $r \in R$. Then $\mu(r^2 x^2) \geq \mu(r^2) \vee \mu(x^2) = \mu(r^2) \vee \mu(0) = \mu(0)$; so $\mu(r^2 x^2) = \mu(0)$ by Lemma 2.3. Thus $rx \in N(\mu)$. Now we need only show that $x+y \in N(\mu)$. By assumption, $\mu(x^2) = \mu(0) = \mu(y^2)$ and

$\mu(xy) = \mu(0)$ by part (2). Thus $\mu((x+y)^2) \geq \mu(x^2) \vee \mu(y^2) \vee \mu(2xy) = \mu(0)$; so $\mu((x+y)^2) = \mu(0)$ by Lemma 2.3. Thus $N(\mu)$ is an ideal of R.

(7) Let $N(\mu)$ is a prime ideal of R. Since the inclusion $N(\mu) \subseteq \mathrm{nil}(\mu)$ is clear, we will prove the reverse inclusion. Let $x \in \mathrm{nil}(\mu)$. Then $\mu(x^n) = \mu(0)$ for some positive integer n. Let m ($m \geq 3$) be the least positive integer such that $\mu(x^m) = \mu(0)$, and let $y = x^m$. Then $\mu(y^2) = \mu(x^{2m}) = \mu(0)$; hence $N(\mu)$ prime gives $x \in N(\mu)$, and so we have equality. Conversely, assume that $xy \in N(\mu)$ for some $x, y \in R$. Then by part (1) above, either $x^2 \in N(\mu) = \mathrm{nil}(\mu)$ or $y^2 \in N(\mu) = \mathrm{nil}(\mu)$; thus either $x \in N(\mu)$ or $y \in N(\mu)$, as needed. ■

Theorem 5.3. *Let R be a chained ring and $\mu \in LI(R)$. If $N(\mu) = \{x \in R : \mu(x^2) = \mu(0)\}$, then $V(\Gamma(\mu)) - N(\mu)$ is totally disconnected.*

Proof. Apply Proposition 5.2. ■

Theorem 5.4. *Let R be a chained ring and $\mu \in LI(R)$. Then* $\mathrm{diam}(\Gamma(\mu)) \leq 2$.

Proof. If $|Z(\mu)^*| = 1$, then $\mathrm{diam}(\Gamma(\mu)) = 0$. So we may assume that $|Z(\mu)^*| \geq 2$. Let $N(\mu) = \{x \in R : \mu(x^2) = \mu(0)\}$, and let $x, y \in Z(\mu)^*$ with $x \neq y$. If $x, y \in N(\mu)$, then $\mu(xy) = \mu(0)$ by Proposition 5.2 (2), and thus $d_\mu(x, y) = 1$. If $x \in N(\mu)$ and $y \notin N(\mu)$, then $\mu(yz) = \mu(0)$ for some $z \in N(\mu)^*$ by Proposition 5.2 (4), and $\mu(xz) = \mu(0)$ by Proposition 5.2 (2); hence $x - z - y$ is a path from x to y. Thus $d_\mu(x, y) \leq 2$. Finally, let $x, y \notin N(\mu)$. Then $\mu(xz) = \mu(yz) = \mu(0)$ by Proposition 5.2 (5). Thus $d_\mu(x, y) \leq 2$, and hence $\mathrm{diam}(\Gamma(\mu)) \leq 2$. ■

Theorem 5.5. *Let R be a chained ring and $\mu \in LI(R)$ with $Z(\mu) \neq \{0\}$, and let $N(\mu) = \{x \in R : \mu(x^2) = \mu(0)\}$. Then exactly one of the following three cases must occur:*

(1) $|Z(\mu)^*| = 1$. *In this case,* $\mathrm{diam}(\Gamma(\mu)) = 0$*;*

(2) $|Z(\mu)^*| \geq 2$ *and* $N(\mu) = Z(\mu)$. *In this case,* $\mathrm{diam}(\Gamma(\mu)) = 1$*;*

(3) $|Z(\mu)^*| \geq 2$ *and* $N(\mu) \subsetneqq Z(\mu)$. *In this case,* $\mathrm{diam}(\Gamma(\mu)) = 2$.

Proof. This follows directly from Proposition 5.2 and Theorem 5.4. ■

Theorem 5.6. *Let R be a chained ring and $\mu \in LI(R)$ with $N(\mu) = \{x \in R : \mu(x^2) = \mu(0)\}$. Then exactly one of the following three cases must occur:*

(1) $|N(\mu)^*| = 1$. *In this case,* $\mathrm{gr}(\Gamma(\mu)) = \infty$*;*

(2) $|N(\mu)^*| = 2$ *and* $N(\mu) = Z(\mu)$. *In this case,* $\mathrm{gr}(\Gamma(\mu)) = \infty$*;*

(3) $|N(\mu)^*| = 2$ *and* $N(\mu) \subsetneqq Z(\mu)$. *In this case,* $\mathrm{gr}(\Gamma(\mu)) = 3$*;*

(4) $|N(\mu)^*| \geq 3$. *In this case,* $\mathrm{gr}(\Gamma(\mu)) = 3$.

Proof. (1) Let $N(\mu)^* = \{x\}$. If $N(\mu)^* = Z(\mu)^*$, then $\mathrm{gr}(\Gamma(\mu)) = \infty$. If $N(\mu)^* \subsetneqq Z(\mu)^*$, then $\Gamma(\mu)$ is a star graph with center x by Proposition 5.2. Thus $\mathrm{gr}(\Gamma(\mu)) = \infty$.

(2) By hypothesis, $|Z(\mu)^*| = 2$; hence $\mathrm{gr}(\Gamma(\mu)) = \infty$.

(3) Let $N(\mu)^* = \{x, y\}$. If $y \neq -x$, then $\mu((x+y)^2) \geq \mu(x^2) \wedge \mu(y^2) \wedge \mu(xy) \wedge \mu(xy) = \mu(0)$ (note that by Proposition 5.2 (2), $\mu(xy) = 0$); so $\mu((x+y)^2) = \mu(0)$ by Lemma 2.3. It follows that $x + y \in N(\mu)^*$. Thus, either $x + y = x$ or $x + y = y$, a contradiction. So we may assume that $y = -x$. If $z \in Z(\mu)^* - N(\mu)^*$, then $x - y - z - x$ is a triangle since by Proposition 5.2 (4), $\mu(xz) = \mu(0) = \mu(yz)$, so $\mathrm{gr}(\Gamma(\mu)) = 3$.

(4) If $|N(\mu)^*| \geq 3$, then $\mathrm{gr}(\Gamma(\mu)) = 3$ by 5.2 (2). ■

6. L-domainlike rings

We say that a ring R is domainlike ring if $Z(R) = \mathrm{nil}(R)$ [5]. In this section, we investigate the properties of $\Gamma(\mu)$, where R is a μ-domainlike ring and $\mu \in LI(R)$. We say that a ring R is μ-domainlike ring if $Z(\mu) = \mathrm{nil}(\mu)$.

Proposition 6.1. *Let R be a ring and $\mu \in LI(R)$. Let $x, y \in \mathrm{nil}(\mu)^*$ be distinct with $\mu(xy) \neq \mu(0)$. Then there is a path of length 2 from x to y in* $\mathrm{nil}(\mu)^* \subseteq Z(\mu)^*$.

Proof. Since $\mu(xy) \neq \mu(0)$ and $x \in \mathrm{nil}(\mu)^*$, let n ($n \geq 2$) be the least positive integer such that $\mu(x^n y) = \mu(0)$. Also, since $\mu(x^{n-1}y) \neq \mu(0)$ and $y \in \mathrm{nil}(\mu)^*$, let m ($m \geq 2$) be the l–=east positive integer such that $\mu(x^{n-1}y^m) = \mu(0)$. Then $\mu(0) \neq \mu(x^{n-1}y^{m-1}) \in \mathrm{nil}(\mu)^*$. Thus $x - x^{n-1}y^{m-1} - y$ is a path of length 2 from x to y in $\mathrm{nil}(\mu)^*$. ■

Theorem 6.2. *Let R be a μ-domainlike ring and $\mu \in LI(R)$. Then* $\mathrm{diam}(\Gamma(\mu)) \leq 2$.

Proof. Apply Proposition 6.1. ■

Lemma 6.3. *Let R be a ring and $\mu \in LI(R)$. If $|Z(\mu)^*| \geq 3$ and there exist a, b in $Z(\mu)^*$ such that $\mu(ab) = \mu(0) = \mu(a^2) = \mu(b^2)$, then* $\mathrm{gr}(\Gamma(\mu)) = 3$.

Proof. By assumption, if $\mathrm{diam}(\Gamma(\mu)) = 1$, then there exist x_1, x_2 and x_3 in $Z(\mu)^*$ such that $\mu(x_1x_2) = \mu(x_2x_3) = \mu(x_3x_1) = \mu(0)$; hence $x_1-x_2-x_3-x_1$ is a cycle of length 3. So we may assume that $\mathrm{diam}(\Gamma(\mu)) > 1$. Then there exists some $c \in Z(\mu)^* - \{a, b\}$ such that (without loss of generality) $\mu(ac) = \mu(0) \neq \mu(bc)$. Since $\mu(a(a+b)) \geq \mu(a^2) \wedge \mu(ab) = \mu(0) \wedge \mu(0) = \mu(0)$ and $\mu(b(a+b)) \geq \mu(ab) \wedge \mu(ab) = \mu(0)$, we must have $\mu(a(a+b)) = \mu(0) = \mu(b(a+b))$ by Lemma 2.3. Now we will show that $\mu(a+b) \neq \mu(0)$. Suppose not. Then $\mu(c(a+b)) \geq \mu(c) \vee \mu(a+b) = \mu(0)$; so $\mu(c(a+b)) = \mu(0)$. Therefore, $\mu(bc) = \mu(bc+ac-ac) = \mu(c(a+b)-ac) \geq \mu(c(a+b)) \wedge \mu(ac) = \mu(0)$; thus $\mu(bc) = \mu(0)$, and this is a contradiction. So $a+b \in Z(\mu)^*$. Thus $a-b-a+b-a$ is a cycle of length 3, as required. ■

Lemma 6.4. *Let R be a ring and $\mu \in LI(R)$, and let $a, b \in Z(\mu)^*$ be such that $\mu(ab) = \mu(0) = \mu(a^3) = \mu(b^3)$, $\mu(a^2) \neq \mu(0)$ and $\mu(b^2) \neq \mu(0)$. Then* $\mathrm{gr}(\Gamma(\mu)) = 3$.

Proof. By hypothesis, $\mu(ab^2) \geq \mu(ab) \vee \mu(b) = \mu(0) \vee \mu(b) = \mu(0)$. Then $\mu(ab^2) = \mu(0)$ by Lemma 2.3, and $b^2 \neq a$ (otherwise, $b^4 = a^2$ and $\mu(a^2) = \mu(b^4) \geq \mu(b^3) \vee \mu(b) = \mu(0)$; so $\mu(a^2) = \mu(0)$, a contradiction). Similarly, $b^2 \neq b$ and $b^2 \neq 0$. Thus $b-a-b^2-b$ is a 3-cycle in $\Gamma(\mu)$, and hence $\mathrm{gr}(\Gamma(\mu)) = 3$. ■

Lemma 6.5. *Let R be a ring and $\mu \in LI(R)$, and let $a, \in Z(\mu)^*$ be such that $\mu(a^n) = \mu(0)$ and $\mu(a^{n-1}) \neq \mu(0)$ for some $n \geq 4$. Then* $\mathrm{gr}(\Gamma(\mu)) = 3$.

Proof. Let a be an element $Z(\mu)^*$ such that $\mu(a^n) = \mu(0)$ and $\mu(a^{n-1}) \neq \mu(0)$ for some $n \geq 5$. If $k > n$, then $\mu(a^k) \geq \mu(a^n) \vee \mu(a^{k-n}) = \mu(0)$; so $\mu(a^k) = \mu(0)$ by Lemma 2.3. Then $a^{n-3} - a^{n-2} - a^{n-1} - a^{n-3}$, and hence $\mathrm{gr}(\Gamma(\mu)) = 3$. If there exists some $a \in Z(\mu)^*$ with $\mu(a^4) = \mu(0)$ and $\mu(a^3) \neq \mu(0)$, then consider the element $a^2 + a^3$. If $a^2 + a^3 = a^3$, then $\mu(a^2) = \mu(0)$ and $\mu(a^3) \geq \mu(a) \vee \mu(a^2) = \mu(0)$; hence $\mu(a^3) = \mu(0)$, a contradiction. Thus $a^2+a^3 \neq a^3$. Similarly, $a^2+a^3 \neq a^2$. If $a^2+a^3 = 0$, then $\mu(a^3) = \mu(a(-a^3) = \mu(-a^4) = \mu(0)$; which is a contradiction. Therefore, $a^2 + a^3 \neq 0$. Clearly, $a^2 \neq 0$ (otherwise, $\mu(a^3) = \mu(0)$, a contradiction). If $\mu(a^2+a^3) = \mu(0)$, then $\mu(a^3+a^4) \geq \mu(a) \vee \mu(a^2+a^3) = \mu(a) \vee \mu(0) = \mu(0)$; so $\mu(a^3 + a^4) = \mu(0)$. It then follows that

$$\mu(a^3) = \mu(a^3 + a^4 - a^4) \geq \mu(a^3 + a^4) \vee \mu(-a^4) = \mu(0).$$

Therefore, $\mu(a^3) = \mu(0)$, which is a contradiction. So, $\mu(a^2 + a^3) \neq \mu(0)$. Thus, we get the cycle $a^2 - a^3 - (a^2 + a^3) - a^2$ with length 3. Thus, $\mathrm{gr}(\Gamma(\mu)) = 3$. ■

Theorem 6.6. *Let R be a μ-domainlike ring and $\mu \in LI(R)$. If $\Gamma(\mu)$ contains a cycle, then* $\mathrm{gr}(\Gamma(\mu)) = 3$.

Proof. Since $\Gamma(\mu)$ contains a cycle, $|Z(\mu)^*| \geq 3$ and $\mathrm{diam}(\Gamma(\mu)) \neq 0$. So by Theorem 6.2, either $\mathrm{diam}(\Gamma(\mu)) = 1$ or $\mathrm{diam}(\Gamma(\mu)) = 2$. If $\mathrm{diam}(\Gamma(\mu)) = 1$, then there exist x_1, x_2 and x_3 in $Z(\mu)^*$ such that $\mu(x_1x_2) = \mu(x_2x_3) = \mu(x_3x_1) = \mu(0)$; hence $x_1 - x_2 - x_3 - x_1$ is a cycle with length 3, and so $\mathrm{gr}(\Gamma(\mu)) = 3$.

For the remainder of the proof we will assume that $\mathrm{diam}(\Gamma(\mu)) = 2$. As $\Gamma(\mu)$ contains a cycle and $\mathrm{diam}(\Gamma(\mu)) = 2$, we may assume that $|Z(\mu)^*| \geq 4$. Let $a \in Z(\mu)^*$. Since $Z(\mu) = \mathrm{nil}(\mu)$, there exists a positive integer n such that $\mu(a^n) = \mu(0)$, but $\mu(a^{n-1}) \neq \mu(0)$. If $n \geq 4$, then $\mathrm{gr}(\Gamma(\mu)) = 3$ by Lemma 6.5.

Now suppose that for all $a \in Z(\mu)^*$ we have $\mu(a^3) = \mu(0)$. Since $\mathrm{diam}(\Gamma(\mu)) = 2$, there exist a, b and c in $Z(\mu)^*$ such that $d_\mu(a, b) = 2$ and $\mu(ac) = \mu(bc) = \mu(0)$. We split the proof into three cases.

Case 1. $\mu(a^2) = \mu(0) \neq \mu(b^2)$. If $\mu(a^2) = \mu(0) = \mu(c^2)$, then by Lemma 6.3, $\mathrm{gr}(\Gamma(\mu)) = 3$. So we may assume that $\mu(c^2) \neq \mu(0)$. Since $\mu(b^2) \neq \mu(0)$, $\mu(b^3) = \mu(0)$ and $\mu = (c^3) = \mu(0)$, Lemma 6.4 gives $\mathrm{gr}(\Gamma(\mu)) = 3$.

Case 2. $\mu(a^2) = \mu(0) = \mu(b^2)$. If $\mu(c^2) = \mu(0)$, then again Lemma 6.3 gives $\mathrm{gr}(\Gamma(\mu)) = 3$. So we may assume that $\mu(c^2) \neq \mu(0)$. Since $\mu(c^2) \neq \mu(0)$, we get $\mu(c) \neq \mu(0)$; hence $c^2 \in Z(\mu)^*$ (note that $\mu(c^3) = \mu(0)$). Clearly, either $c^2 \in Z(\mu)^* - \{a\}$ or $c^2 \in Z(\mu)^* - \{b\}$ (otherwise, $c^2 = a = b$, a contradiction). Since $\mu(c^2a) \geq \mu(ac) \vee \mu(c) = \mu(0) \vee \mu(c) = \mu(0)$, we get $\mu(ac^2) = \mu(0)$ by Lemma 2.3. Similarly, $\mu(c^2b) = \mu(0)$. If $c^2 \in Z(\mu)^* - \{a\}$, then $c - c^2 - a - c$ is a cycle of length 3. If $c^2 \in Z(\mu)^* - \{b\}$, then $c - c^2 - b - c$ is a cycle of length 3; hence in this case, $\mathrm{gr}(\Gamma(\mu)) = 3$.

Case 3. $\mu(a^2) \neq \mu(0)$ and $\mu(b^2) \neq \mu(0)$. If $\mu(c^2) \neq \mu(0)$, then $\mathrm{gr}(\Gamma(\mu)) = 3$ by Lemma 6.4. So we may assume that $\mu(c^2) = \mu(0)$. If there exists an element $x \in Z(\mu)^*$ such that $c \neq x$, $\mu(x^2) = \mu(0)$, and $x - a - c$ or $x - b - c$, then by an identical argument as the Case 2; we have $\mathrm{gr}(\Gamma(\mu)) = 3$. Since $Z(\mu) = \mathrm{nil}(\mu)$ is an ideal of R by Lemma 3.9, we have that $c + c \in Z(\mu)$.

Since $\mu(c^2) = \mu(0)$, we have $\mu((c+c)^2) \geq \mu(c^2) \wedge \mu(c^2) \wedge \mu(2c^2) = \mu(0)$; so $\mu((c+c)^2) = \mu(0)$. Clearly, $c+c \neq c$. If $c+c \neq 0$, let $x = c+c$, and we get $\mathrm{gr}(\Gamma(\mu)) = 3$. Now suppose $c+c = 0$. If either a^2 or b^2 is not equal to c, let $x = a^2$ or $x = b^2$, and again we get $\mathrm{gr}(\Gamma(\mu)) = 3$. So we may assume that $a^2 = b^2 = c$. By hypothesis, $|Z(\mu)^*| \geq 4$, $\mathrm{diam}(\Gamma(\mu)) = 2$, and $\mu(x^3) = \mu(0)$ for all $x \in Z(\mu)^*$. So there exists $d \in Z(\mu)^*$ such that either $\mu(da) = \mu(0)$, $\mu(db) = \mu(0)$ or $\mu(dc) = \mu(0)$ (otherwise, $\Gamma(\mu)$ is not connected, and this a contradiction). If $\mu(ad) = \mu(0)$, if $\mu(db) = \mu(0)$, if $\mu(dc) = \mu(0)$ and $\mu(d^2) = \mu(0)$ or if $\mu(dc) = \mu(0)$ and $d^2 \neq c$, we can appeal to previous cases to obtain $\mathrm{gr}(\Gamma(\mu)) = 3$. Now suppose $\mu(dc) = \mu(0)$ and $d^2 = c$. If $ab = a$, then $\mu(a^2) = \mu(a^2b^2) = \mu(c^2) = \mu(0)$, which is a contradiction. Similarly, $ab \neq b$. Clearly, $\mu(ab) \neq \mu(0)$. Thus, $ab = c$, for the otherwise we would let $x = ab$ above and have $\mathrm{gr}(\Gamma(\mu)) = 3$. Similarly, $ad = bd = c$. Therefore, $a(b-d) = 0$. If $b-d \neq c$, we again have $\mathrm{gr}(\Gamma(\mu)) = 3$. So suppose that $b = d+c$. Similarly, $b(d-a) = 0$. Again, if $d-a \neq c$, we will have $\mathrm{gr}(\Gamma(\mu)) = 3$. Now, if $d = a+c$, we have $b = d+c = a+c+c = a$, which is a contradiction. Thus, every case leads to $\mathrm{gr}(\Gamma(\mu)) = 3$. ■

Example 6.7. (1) Let R and μ be as in Example 3.8. Then R is a μ-domainlike ring, $\mathrm{gr}(\Gamma((\mu)) = \infty$ and $\mathrm{diam}(\Gamma(\mu)) = 2$ since $\bar{2}-\bar{4}-\bar{6}$ is a path of length 2 from $\bar{2}$ to $\bar{6}$ in $Z(\mu)^*$ (see Theorem 6.2).

(2) Let R and μ be as in Example 3.16. By Lemma 6.3, since $\mu(2.4) = \mu(0) = \mu(2^2) = \mu(4^2)$, we must have $\mathrm{gr}(\Gamma(\mu)) = 3$. Moreover, as $2-4-6-8$ is a path between 2 and 8, we have $\mathrm{diam}(\Gamma(\mu)) = 3$ by Theorem 3.17.

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