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LEAPING CONVERGENTS OF HURWITZ CONTINUED FRACTIONS

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Abstract

Let $p_n/q_n = [a_0; a_1, \dots, a_n]$ be the n-th convergent of the continued fraction expansion of $[a_0; a_1, a_2, \dots]$. Leaping convergents are those of every r-th convergent p_{rn+i}/q_{rn+i} $(n = 0, 1, 2, \dots)$ for fixed integers r and i with $r \geq 2$ and $i = 0, 1, \dots, r-1$. The leaping convergents for the e-type Hurwitz continued fractions have been studied. In special, recurrence relations and explicit forms of such leaping convergents have been treated.

In this paper, we consider recurrence relations and explicit forms of the leaping convergents for some different types of Hurwitz continued fractions.

Keywords: Leaping convergents, Hurwitz continued fractions.

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1. Introduction

$\alpha = [a_0; a_1, a_2, \dots]$ denotes the regular (or simple) continued fraction expansion of a real α, where

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$$\alpha = a_0 + \theta_0, \quad a_0 = \lfloor \alpha \rfloor \,,$$
$$1/\theta_{n-1} = a_n + \theta_n, \quad a_n = \lfloor 1/\theta_{n-1} \rfloor \quad (n \ge 1)\,.$$

The n-th convergent of the continued fraction expansion of α is denoted by $p_n/q_n = [a_0; a_1, \dots, a_n]$. It is well-known that p_n and q_n satisfy the recurrence relation:

$$p_n = a_n p_{n-1} + p_{n-2} \quad (n \ge 0), \qquad p_{-1} = 1, \qquad p_{-2} = 0,$$
$$q_n = a_n q_{n-1} + q_{n-2} \quad (n \ge 0), \qquad q_{-1} = 0, \qquad q_{-2} = 1\,.$$

Leaping convergents are those of every r-th convergent p_{rn+i}/q_{rn+i} ($n = 0, 1, 2, \dots$) for fixed integers r and i with $r \ge 2$ and $i = 0, 1, \dots, r-1$. Leaping convergents of the continued fraction expansion of $e^{1/s}$ ($s \ge 1$) have been considered. This continued fraction is one of the typical Hurwitz continued fractions. Hurwitz continued fraction expansions have the form

$$\begin{aligned}&[a_0; a_1, \dots, a_n, \overline{Q_1(k), \dots, Q_p(k)}]_{k=1}^{\infty}\\ &\qquad = [a_0; a_1, \dots, a_n, Q_1(1), \dots, Q_p(1), Q_1(2), \dots, Q_p(2), Q_1(3), \dots]\,,\end{aligned}$$

where a_0 is an integer, $a_1, \dots, a_n$ are positive integers, $Q_1, \dots, Q_p$ are polynomials with rational coefficients which take positive integral values for $k = 1, 2, \dots$ and at least one of the polynomials is not constant. Various Hurwitz continued fractions are mentioned in Section 3.

Elsner [1] studied arithmetical properties of leaping convergents p_{3n+1}/q_{3n+1} for the continued fraction of $e = [2; \overline{1, 2k, 1}]_{k=1}^{\infty}$. Putting $P_n = p_{3n+1}$, $Q_n = q_{3n+1}$ ($n \ge 0$), $P_{-1} = P_{-2} = Q_{-1} = 1$, $Q_{-2} = -1$, $P_{-n} = P_{n-3}$ and $Q_{-n} = -Q_{n-3}$ ($n \ge 0$), then for any integer n

$$P_n = 2(2n+1)P_{n-1} + P_{n-2}, \quad Q_n = 2(2n+1)Q_{n-1} + Q_{n-2}\,.$$

The author [4] studied those p_{3n}/q_{3n} for $e^{1/s} = [1; \overline{s(2k-1)-1, 1, 1}]_{k=1}^{\infty}$ ($s \ge 2$). Putting $P_n = p_{3n}$, $Q_n = q_{3n}$ ($n \ge 0$), $P_{-n} = P_{n-1}$ and

$Q_{-n}=-Q_{n-1}$ $(n\ge 0)$, then for any integer n

$$P_n=2s(2n-1)P_{n-1}+P_{n-2},\quad Q_n=2s(2n-1)Q_{n-1}+Q_{n-2}\,.$$

In the latter case, $\hat{P}_n=p_{3n+1}$, $\hat{Q}_n=q_{3n+1}$, $\tilde{P}_n=p_{3n+2}$ and $\tilde{Q}_n=q_{3n+2}$ do not satisfy any recurrence relations of the type like $P_n=S_nP_{n-1}+P_{n-2}$. But they do some different type of relations. Moreover, all p's and q's are explicitly expressed in the aspect of leaping convergents in [5]. Namely, for $n\ge 1$ we have

$$p_{3n}=p^*_{3n-2}=\sum_{k=0}^n\frac{(2n-k)!}{k!(n-k)!}s^{n-k},$$

$$q_{3n}=q^*_{3n-2}=\sum_{k=0}^n(-1)^k\frac{(2n-k)!}{k!(n-k)!}s^{n-k},$$

$$p_{3n-1}=p^*_{3n-3}=n\sum_{k=0}^n\frac{(2n-k-1)!}{k!(n-k)!}s^{n-k},$$

$$q_{3n-1}=q^*_{3n-3}=\sum_{k=0}^{n-1}(-1)^k\frac{(2n-k-1)!}{k!(n-k-1)!}s^{n-k},$$

$$p_{3n-2}=p^*_{3n-4}=\sum_{k=0}^{n-1}\frac{(2n-k-1)!}{k!(n-k-1)!}s^{n-k},$$

$$q_{3n-2}=q^*_{3n-4}=n\sum_{k=0}^n(-1)^k\frac{(2n-k-1)!}{k!(n-k)!}s^{n-k}.$$

Note that all the six formulas for p^*_{3n-2}, p^*_{3n-3}, p^*_{3n-4}, q^*_{3n-2}, q^*_{3n-3} and q^*_{3n-4} correspond to $s=1$ in the continued fraction expansion of e.

In this paper, we consider recurrence relations and explicit forms of the leaping convergents for some different types of Hurwitz continued fractions.

2. Recurrence relations of leaping convergents

In [5, Theorem 2], some three term relations were shown for a more general continued fraction of $[1;\overline{T_1(k),T_2(k),T_3(k)}]_{k=1}^\infty$. In [6, Theorem 1] these results were further extended in the following form.

Lemma 1. *Let the continued fraction be given by*

$$[a_0;\overline{T_1(k),T_2(k),\dots,T_r(k)}\,]_{k=1}^\infty$$

with odd r, where each $T_\nu(k)$ ($\nu=1,2,\dots,r$) takes a positive integer for $k=1,2,\dots$. Let α, β, γ, δ, α', β', γ' and δ' be integers defined by

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=\begin{pmatrix}\alpha(n) & \beta(n)\\ \gamma(n) & \delta(n)\end{pmatrix}=\begin{pmatrix}T_2(n) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix}T_r(n) & 1\\ 1 & 0\end{pmatrix},$$

and

$$\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}=\begin{pmatrix}\alpha(n-1) & \beta(n-1)\\ \gamma(n-1) & \delta(n-1)\end{pmatrix}=\begin{pmatrix}T_2(n-1) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix}T_r(n-1) & 1\\ 1 & 0\end{pmatrix},$$

respectively. Then we have for $n\ge 2$

$$(\gamma' T_1(n)+\delta')x_n=U(n)x_{n-1}+(\gamma T_1(n+1)+\delta)x_{n-2}\,,$$

where $U(n)=(\gamma' T_1(n)+\delta')(\alpha T_1(n+1)+\beta)+\gamma'(\gamma T_1(n+1)+\delta)$, and $x_n=p_{rn+1}$ or $x_n=q_{rn+1}$.

By shifting the position from $T_1(n)$ to $T_\nu(n)$ ($\nu=2,\dots,r$), a more general result was shown in [6, Theorem 2].

Lemma 2. *Let the continued fraction be given by*

$$[a_0; \overline{T_1(k), T_2(k), \ldots, T_r(k)}\,]_{k=1}^{\infty},$$

where each $T_\nu(k)$ ($\nu = 1, 2, \ldots, r$) takes a positive integer for $k = 1, 2, \ldots$. Let $R_{i,j}(n)$ ($i = 1, 2, \ldots, r-1$; $j = 1, 2, 3, 4$) be integers defined by

$$\begin{pmatrix} R_{i,1}(n) & R_{i,2}(n) \\ R_{i,3}(n) & R_{i,4}(n) \end{pmatrix} = \begin{pmatrix} T_{i+1}(n-1) & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} T_r(n-1) & 1 \\ 1 & 0 \end{pmatrix}$$
$$\cdot \begin{pmatrix} T_1(n) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} T_2(n) & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} T_i(n) & 1 \\ 1 & 0 \end{pmatrix}$$

with

$$\begin{pmatrix} R_{0,1}(n) & R_{0,2}(n) \\ R_{0,3}(n) & R_{0,4}(n) \end{pmatrix} = \begin{pmatrix} T_1(n-1) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} T_2(n-1) & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} T_r(n-1) & 1 \\ 1 & 0 \end{pmatrix}.$$

Then we have for $n \ge 2$

$$\begin{aligned} &R_{i,3}(n)x_n \\ &= (R_{i,3}(n)R_{i,1}(n+1) + R_{i,4}(n)R_{i,3}(n+1))x_{n-1} + (-1)^{r-1}R_{i,3}(n+1)x_{n-2}, \end{aligned}$$

where $x_n = p_{rn+i}$ or $x_n = q_{rn+i}$.

These relations are entailed from the regular continued fractions. On the contrary, it is not easy to find the continued fraction satisfying a given three term relation. In addition, such relations can not be directly applied to the simple continued fraction which period is not pure.

In [2], we consider the non-regular continued fractions in order to deal with some more general three term relations. In this paper we shall consider the leaping convergents of non-regular continued fractions and obtain their characteristics. As the case of non-periodic simple continued fractions, [2, Corollary 1] is stated as follows.

Lemma 3. *Given a regular continued fraction*

$$\alpha = [a_0; a_1, a_2, \dots, a_\rho, \overline{T_1(k), T_2(k), \dots, T_w(k)}\,]_{k=1}^{\infty}\,,$$

where $\rho \geq 0$ and $w \geq 1$ are fixed integers. Then for any integers r and i with $r \geq 2$ and $0 \leq \rho \leq i < \rho + r$

$$\begin{aligned}
&(-1)^{r-1} D_{r-1}(M-r)\cdot z_n\\
&\qquad + \big(D_{r-1}(M)D_r(M-r) + D_{r-1}(M-r)D_{r-2}(M+1)\big)\cdot z_{n-1}\\
&\qquad\qquad - D_{r-1}(M)\cdot z_{n-2} = 0 \quad \big(M = (n-1)r + i + 2\big)
\end{aligned}$$

holds for $z_n = p_{rn+i}$ and $z_n = q_{rn+i}$. Here, for positive integers a and l define $D_0(a) = 1$ and

$$D_l(a) = \begin{vmatrix}
-T(a) & -1 & 0 & & & \\
1 & -T(a+1) & -1 & & & \\
0 & 1 & -T(a+2) & & & \\
 & & & \ddots & -1 & \\
 & & & 1 & -T(a+l-2) & -1 \\
 & & & & 1 & -T(a+l-1)
\end{vmatrix},$$

where

$$T(a)=T_{w\{(a-\rho-1)/w\}+1}\left(\left\lceil\frac{a-\rho}{w}\right\rceil\right)$$

for a fixed positive integer w. $\{\cdot\}$ denotes the fractional part function and $\lceil\cdot\rceil$ the ceiling function.

3. Hurwitz continued fractions

Up to the present, the following numbers are well-known to yield Hurwitz continued fractions.

$$e^{1/s}=[1;\overline{(2k-1)s-1,1,1}\,]_{k=1}^{\infty}\quad(s\in\mathbb{Z},s>1)\,.$$

$$ae^{1/a}=[a+1;\overline{2a-1,2k,1}\,]_{k=1}^{\infty}\quad(a\in\mathbb{Z}_{+})\,.$$

$$\frac{1}{a}e^{1/a}=[0;a-1,2a,\overline{1,2k,2a-1}\,]_{k=1}^{\infty}\quad(a\in\mathbb{Z},a>1)\,.$$

$$e^{2}=[7;\overline{3k-1,1,1,3k,12k+6}\,]_{k=1}^{\infty}\,.$$

$$e^{2/s}=[1;\overline{3ks-\frac{5s+1}{2},12ks-6s,3ks-\frac{s+1}{2},1,1}\,]_{k=1}^{\infty}\quad(s:\mathrm{odd},s\geq 3)\,.$$

$$\sqrt{\frac{v}{u}}\tanh\frac{1}{\sqrt{uv}}=[0;\overline{(4k-3)u,(4k-1)v}\,]_{k=1}^{\infty}\quad(u,v\in\mathbb{Z}_{+})\,.$$

$$\frac{I_{(a/b)+1}\left(\dfrac{2}{b}\right)}{I_{a/b}\left(\dfrac{2}{b}\right)}=[0;\overline{a+kb}]_{k=1}^{\infty}\,,$$

where $I_{\lambda}(z)$ are the modified Bessel functions of the first kind, defined by

$$I_\lambda(z)=\sum_{n=0}^\infty\frac{(z/2)^{\lambda+2n}}{n!\Gamma(\lambda+n+1)}\,.$$

$$\sqrt{fracvu}\tan\frac{1}{\sqrt{uv}}=[0;u-1,\overline{1,(4k-1)v-2,1,(4k+1)u-2}\,]_{k=1}^\infty\,.$$

$$\frac{J_{(a/b)+1}\left(\dfrac{2}{b}\right)}{J_{a/b}\left(\dfrac{2}{b}\right)}=[0;a+b-1,\overline{1,a+(k+1)b-2}\,]_{k=1}^\infty\,,$$

where $J_\lambda(z)$ are the Bessel functions of the first kind, defined by

$$J_\lambda(z)=\left(\frac{z}{2}\right)^\lambda\sum_{n=0}^\infty\frac{(iz/2)^{2n}}{n!\Gamma(\lambda+n+1)}\,.$$

It seems that each one of the above belongs to one of the types, e-type (and/or e^2-type), *tanh*-type and tan-type. No concrete example where the degree of any polynomial exceeds 1 is known.

Recently, the author obtained more general Hurwitz continued fractions of three types. In [5] and [7], the author constituted more general forms of Hurwitz continued fractions of e-type, namely, some extended forms of the continued fractions of $e^{1/s}$, $ae^{1/a}$ and $\frac{1}{a}e^{1/a}$.

$$[0;\overline{u(a+kb)-1,1,v-1}]_{k=1}^\infty$$

$$=\frac{\displaystyle\sum_{n=0}^\infty u^{-2n-1}v^{-2n}b^{-n}(n!)^{-1}\prod_{i=1}^{n+1}(a+bi)^{-1}}{\displaystyle\sum_{n=0}^\infty b^{-n}(n!)^{-1}\left((uv)^{-2n}\prod_{i=1}^{n}(a+bi)^{-1}-(uv)^{-2n-1}\prod_{i=1}^{n+1}(a+bi)^{-1}\right)}$$

$$=\frac{{}_0F_1\left(;\dfrac{a}{b}+2;\dfrac{1}{u^2v^2b^2}\right)}{uv(a+b){}_0F_1\left(;\dfrac{a}{b}+1;\dfrac{1}{u^2v^2b^2}\right)-{}_0F_1\left(;\dfrac{a}{b}+2;\dfrac{1}{u^2v^2b^2}\right)}$$

and

$$[0;\overline{v-1,1,u(a+kb)-1}]_{k=1}^{\infty}$$

$$=\frac{\displaystyle\sum_{n=0}^{\infty}b^{-n}(n!)^{-1}\left(u^{-2n}v^{-2n-1}\prod_{i=1}^{n}(a+bi)^{-1}+u^{-2n-1}v^{-2n-2}\prod_{i=1}^{n+1}(a+bi)^{-1}\right)}{\displaystyle\sum_{n=0}^{\infty}(uv)^{-2n}b^{-n}(n!)^{-1}\prod_{i=1}^{n}(a+bi)^{-1}}$$

$$=\frac{{}_0F_1\left(;\dfrac{a}{b}+2;\dfrac{1}{u^2v^2b^2}\right)}{uv(a+b){}_0F_1\left(;\dfrac{a}{b}+1;\dfrac{1}{u^2v^2b^2}\right)-{}_0F_1\left(;\dfrac{a}{b}+2;\dfrac{1}{u^2v^2b^2}\right)}.$$

It was e-type Hurwitz continued fractions that several recurrence relations and explicit forms have been studied about the leaping convergents in [1], [3, 4] and [5]. In the next two sections, we introduce more general Hurwitz continued fractions of tanh-type, tan-type, and obtain the explicit forms of the corresponding leaping convergents. In Section 6 we show the explicit forms of the leaping convergents of $e^{2/s}$, which are slightly different from those of e-type. In Section 7 we prove the theorem presented in the next section.

4. EXPLICIT FORMS FOR THE CONVERGENTS OF tanh-TYPE HURWITZ CONTINUED FRACTIONS

In [3], [7] the author obtained a generalized *tanh*-type Hurwitz continued fraction as

$$[0;\overline{u(a+(2k-1)b),v(a+2kb)}\,]_{k=1}^{\infty}$$

$$=\frac{\displaystyle\sum_{n=0}^{\infty}(n!)^{-1}u^{-n-1}(vb)^{-n}\prod_{i=1}^{n+1}(a+bi)^{-1}}{\displaystyle\sum_{n=0}^{\infty}(n!)^{-1}(uvb)^{-n}\prod_{i=1}^{n}(a+bi)^{-1}}$$

$$= \frac{{}_0F_1\left(;\frac{a}{b}+2;\frac{1}{uvb^2}\right)}{u(a+b){}_0F_1\left(;\frac{a}{b}+1;\frac{1}{uvb^2}\right)},$$

where

$${}_0F_1(;c;z) = \sum_{n=0}^{\infty} \frac{1}{(c)_n}\frac{z^n}{n!}$$

is the confluent hypergeometric limit function with $(c)_n = c(c+1)\dots(c+n-1)$ $(n \geq 1)$ and $(c)_0 = 1$. This continued fraction includes the cases of

$$\sqrt{\frac{v}{u}}\tanh\frac{1}{\sqrt{uv}} \qquad \text{and} \qquad \frac{I_{(a/b)+1}\left(\frac{2}{b}\right)}{I_{a/b}\left(\frac{2}{b}\right)}.$$

By applying Lemmata 1 and 2, the three term relation about the leaping convergents of $[0;\overline{u(a+(2k-1)b), v(a+2kb)}\,]_{k=1}^{\infty}$ is given by

$$\begin{aligned}\big(a+(n-2)b\big)p_n = \Big(&\big(a+(n-2)b\big)\big(a+(n-1)b\big)(a+nb)uv \\ &+ 2\big(a+(n-1)b\big)\Big)p_{n-2} - (a+nb)p_{n-4} \quad (n \geq 4)\,.\end{aligned}$$

The same relation also holds for q's instead of p's. Now, such p's and q's can be expressed explicitly as follows.

Theorem 1. *Let p_n/q_n be the n-th convergent of the continued fraction*

$$[0;\overline{u(a+(2k-1)b), v(a+2kb)}\,]_{k=1}^{\infty} = \frac{{}_0F_1\left(;\frac{a}{b}+2;\frac{1}{uvb^2}\right)}{u(a+b){}_0F_1\left(;\frac{a}{b}+1;\frac{1}{uvb^2}\right)}.$$

Then, for $n = 1, 2, \ldots$ we have

$$p_{2n-1} = \sum_{k=0}^{n-1} \binom{n+k-1}{2k} \left(\prod_{i=n-k+1}^{n+k} (a+bi) \right) (uv)^k,$$

$$p_{2n} = \sum_{k=0}^{n-1} \binom{n+k}{2k+1} \left(\prod_{i=n-k+1}^{n+k+1} (a+bi) \right) u^k v^{k+1},$$

$$q_{2n-1} = \sum_{k=0}^{n-1} \binom{n+k}{2k+1} \left(\prod_{i=n-k}^{n+k} (a+bi) \right) u^{k+1} v^k,$$

$$q_{2n} = \sum_{k=0}^{n} \binom{n+k}{2k} \left(\prod_{i=n-k+1}^{n+k} (a+bi) \right) (uv)^k.$$

If $a = -1$ and $b = 2$, then by $\prod_{i=\mu}^{\tau}(2i-1) = (2\tau-1)!!/(2\mu-3)!!$ we have the following.

Corollary 1. *Let p_n/q_n be the n-th convergent of the continued fraction par $\sqrt{v/u}\tanh 1/\sqrt{uv} = [0; \overline{(4k-3)u, (4k-1)v}\ ]_{k=1}^{\infty}$. Then, for $n = 1, 2, \ldots$ we have*

$$p_{2n-1} = \sum_{k=0}^{n-1} \binom{2n+2k-1}{4k} \binom{4k}{2k} \frac{(2k)!}{2^{2k}} (uv)^k,$$

$$p_{2n} = \sum_{k=0}^{n-1} \binom{2n+2k+1}{4k+2} \binom{4k+2}{2k+1} \frac{(2k+1)!}{2^{2k+1}} u^k v^{k+1},$$

$$q_{2n-1}=\sum_{k=0}^{n-1}\binom{2n+2k}{4k+2}\binom{4k+2}{2k+1}\frac{(2k+1)!}{2^{2k+1}}u^{k+1}v^k,$$

$$q_{2n}=\sum_{k=0}^{n}\binom{2n+2k}{4k}\binom{4k}{2k}\frac{(2k)!}{2^{2k}}(uv)^k.$$

5. Explicit forms for the convergents of tan-type Hurwitz continued fractions

In [3], [7] the author also obtained a generalized *tan*-type Hurwitz continued fraction as

$$[0;u(a+b)-1,\overline{1,v(a+2kb)-2,1,u(a+(2k+1)b)-2}\,]$$

$$=\frac{\displaystyle\sum_{n=0}^{\infty}(-1)^n(n!)^{-1}u^{-n-1}(vb)^{-n}\prod_{i=1}^{n+1}(a+bi)^{-1}}{\displaystyle\sum_{n=0}^{\infty}(-1)^n(n!)^{-1}(uvb)^{-n}\prod_{i=1}^{n}(a+bi)^{-1}}$$

$$=\frac{{}_0F_1\left(;\dfrac{a}{b}+2;\dfrac{-1}{uvb^2}\right)}{u(a+b){}_0F_1\left(;\dfrac{a}{b}+1;\dfrac{-1}{uvb^2}\right)},$$

including the cases of

$$\sqrt{\frac{v}{u}}\tan\frac{1}{\sqrt{uv}}\qquad\text{and}\qquad\frac{J_{(a/b)+1}\left(\dfrac{2}{b}\right)}{J_{a/b}\left(\dfrac{2}{b}\right)}.$$

Apply Lemma 3 to the leaping convergents of

$$[0; u(a+b)-1, \overline{1, v(a+2kb)-2, 1, u(a+(2k+1)b)-2}]_{k=1}^{\infty}\,.$$

Note that $T(M) = T_3(n) = 1$, $T(M+1) = T_4(n) = u\bigl(a+(2n+1)b\bigr)-2$, $T(M+2) = T_1(n+1) = 1$ and $T(M+3) = T_2(n+1) = v\bigl(a+(2n+2)b\bigr)-2$. For simplicity, put $s_n = a+nb$. Then the three term recurrence relation for $i = 2$ is given by

$$\begin{aligned} s_{2n-1}p_{4n+2} = \bigl(s_{2n-1}(us_{2n+1}-1) + s_{2n+1}(uvs_{2n-1}s_{2n} - us_{2n-1} - 1)\bigr)p_{4n-2} \\ - s_{2n+1}p_{4n-6} \quad (n \ge 2)\,. \end{aligned}$$

The same relation also holds for q's instead of p's. The relations for $i = 1$, 3 and 4 are similarly obtained as follows.

$$\begin{aligned} &(uvs_{2n-2}s_{2n-1} - vs_{2n-2} - us_{2n-1})p_{4n+1} \\ &\quad = \bigl((uvs_{2n-2}s_{2n-1} - vs_{2n-2} - us_{2n-1})(us_{2n+1}-1) \\ &\qquad + (uvs_{2n}s_{2n+1} - vs_{2n} - us_{2n+1})(uvs_{2n-2}s_{2n-1} - us_{2n-1} - 1)\bigr)p_{4n-3} \\ &\qquad\quad - (uvs_{2n}s_{2n+1} - vs_{2n} - us_{2n+1})p_{4n-7} \quad (n \ge 2)\,, \end{aligned}$$

$$\begin{aligned} &(uvs_{2n-1}s_{2n} - us_{2n-1} - vs_{2n})p_{4n+3} \\ &\quad = \bigl((uvs_{2n-1}s_{2n} - us_{2n-1} - vs_{2n})(vs_{2n+2}-1) \\ &\qquad + (uvs_{2n+1}s_{2n+2} - us_{2n+1} - vs_{2n+2})(uvs_{2n-1}s_{2n} - vs_{2n} - 1)\bigr)p_{4n-1} \\ &\qquad\quad - (uvs_{2n+1}s_{2n+2} - us_{2n+1} - vs_{2n+2})p_{4n-5} \quad (n \ge 2)\,, \end{aligned}$$

$$\begin{aligned} s_{2n}p_{4n+4} = \bigl(s_{2n}(vs_{2n+2}-1) + s_{2n+2}(uvs_{2n}s_{2n+1} - vs_{2n} - 1)\bigr)p_{4n} \\ - s_{2n+2}p_{4n-4} \quad (n \ge 2)\,. \end{aligned}$$

Now, such p's and q's can be expressed explicitly as follows.

Theorem 2. *Let p_n/q_n be the n-th convergent of the continued fraction*

$$[0; u(a+b)-1, \overline{1, v(a+2kb)-2, 1, u(a+(2k+1)b)-2}\,]$$
$$= \frac{{}_0F_1\left(;\frac{a}{b}+2; \frac{-1}{uvb^2}\right)}{u(a+b){}_0F_1\left(;\frac{a}{b}+1; \frac{-1}{uvb^2}\right)}\,.$$

Then, for $n = 1, 2, \ldots$ we have

$$p_{4n-3} = P_2(n) - P_1(n-1), \qquad p_{4n-2} = P_2(n),$$
$$q_{4n-3} = Q_2(n) - Q_1(n-1), \qquad q_{4n-2} = Q_2(n),$$
$$p_{4n-1} = P_1(n) - P_2(n), \qquad p_{4n} = P_1(n),$$
$$q_{4n-1} = Q_1(n) - Q_2(n), \qquad q_{4n} = Q_1(n)$$

where

$$P_1(n) = \sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{n+k}{2k+1} \left(\prod_{i=n-k+1}^{n+k+1} (a+bi) \right) u^k v^{k+1}\,,$$

$$P_2(n) = \sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{n+k-1}{2k} \left(\prod_{i=n-k+1}^{n+k} (a+bi) \right) (uv)^k\,,$$

$$Q_1(n) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{2k} \left(\prod_{i=n-k+1}^{n+k} (a+bi) \right) (uv)^k\,,$$

$$Q_2(n) = \sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{n+k}{2k+1} \left(\prod_{i=n-k}^{n+k} (a+bi) \right) u^{k+1} v^k\,.$$

If $a=-1$ and $b=2$, then we have the following.

Corollary 2. *Let p_n/q_n be the n-th convergent of the continued fraction $\sqrt{v/u}\tan 1/\sqrt{uv} = [0; u-1, \overline{1, (4k-1)v-2, 1, (4k+1)u-2}\,]_{k=1}^{\infty}$. Then, for $n=1,2,\dots$ we have*

$$\begin{aligned} p_{4n-3} &= P_2^*(n) - P_1^*(n-1), & p_{4n-2} &= P_2^*(n),\\ q_{4n-3} &= Q_2^*(n) - Q_1^*(n-1), & q_{4n-2} &= Q_2^*(n),\\ p_{4n-1} &= P_1^*(n) - P_2^*(n), & p_{4n} &= P_1^*(n),\\ q_{4n-1} &= Q_1^*(n) - Q_2^*(n), & q_{4n} &= Q_1^*(n), \end{aligned}$$

where

$$P_1^*(n) = \sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{2n+2k+1}{4k+2} \binom{4k+2}{2k+1} \frac{(2k+1)!}{2^{2k+1}} u^k v^{k+1},$$

$$P_2^*(n) = \sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{2n+2k-1}{4k} \binom{4k}{2k} \frac{(2k)!}{2^{2k}} (uv)^k,$$

$$Q_1^*(n) = \sum_{k=0}^{n} (-1)^{n-k} \binom{2n+2k}{4k} \binom{4k}{2k} \frac{(2k)!}{2^{2k}} (uv)^k,$$

$$Q_2^*(n) = \sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{2n+2k}{4k+2} \binom{4k+2}{2k+1} \frac{(2k+1)!}{2^{2k+1}} u^{k+1} v^k.$$

6. EXPLICIT FORMS FOR THE CONVERGENTS OF $e^{2/s}$

A three term relation of $e^{2/s}$ was given in [6]. Namely, for $n \ge 2$

$$A(n)x_n = U(n)x_{n-1} + B(n)x_{n-2}\,,$$

where $x_n = p_{5n+1}$ or $x_n = q_{5n+1}$, and

$$\begin{aligned} A(n) &= \gamma(n-1)T_1(n) + \delta(n-1) \\ &= 18\big(2s+1\big)^2 n^2 - 36\big(2s+1\big)^2 n + 2\big(35s^2+35s+9\big)\,, \end{aligned}$$

$$\begin{aligned} U(n) &= (\gamma(n-1)T_1(n) + \delta(n-1))(\alpha(n)T_1(n+1) + \beta(n)) \\ &\quad + \gamma(n-1)(\gamma(n)T_1(n+1) + \delta(n)) \\ &= 6(2s+1)(2n-1)\big(324\big(2s+1\big)^4 n^4 - 648\big(2s+1\big)^4 n^3 \\ &\quad + 18\big(2s+1\big)^2\big(68s^2+68s+19\big)n^2 + 18\big(2s+1\big)^2\big(4s^2+4s-1\big)n \\ &\quad - \big(140s^4+280s^3+190s^2+50s+3\big)\big) \end{aligned}$$

and

$$\begin{aligned} B(n) &= \gamma(n)T_1(n+1) + \delta(n) \\ &= 18\big(2s+1\big)^2 n^2 - 2s(s+1)\,. \end{aligned}$$

In this section we shall show the explicit forms for those of $e^{2/s}$. If $s > 1$ is odd, the continued fraction expansion of $e^{2/s}$ is given by

$$e^{2/s} = [1; \overline{\frac{(6k-5)s-1}{2}, (12k-6)s, \frac{(6k-1)s-1}{2}, 1, 1}\,]_{k=0}^{\infty}\,.$$

Let p_n/q_n be the n-th convergent of this continued fraction expansion.

Theorem 3. *For $n = 1, 2, \ldots$ we have*

$$p_{5n-4} = \frac{1}{(3n-2)!} \sum_{k=0}^{3n-2} (3n+k-2)! \binom{3n-2}{k} \frac{s^k}{2^{k+1}}\,,$$

$$p_{5n-3} = \frac{1}{(3n-1)!} \sum_{k=0}^{3n-1} (3n+k-1)! \binom{3n-1}{k} \left(\frac{s}{2}\right)^k\,,$$

$$p_{5n-2} = \frac{1}{(3n-1)!} \sum_{k=0}^{3n-1} (3n+k)! \binom{3n-1}{k} \left(\frac{s}{2}\right)^{k+1}\,,$$

$$p_{5n-1} = \frac{1}{(3n-1)!} \sum_{k=0}^{3n} (3n+k-1)! \binom{3n}{k} \left(\frac{s}{2}\right)^k\,,$$

$$p_{5n} = \frac{1}{(3n)!} \sum_{k=0}^{3n} (3n+k)! \binom{3n}{k} \left(\frac{s}{2}\right)^k\,,$$

and

$$q_{5n-4} = \frac{1}{(3n-2)!} \sum_{k=0}^{3n-2} (-1)^{3n-k-2} (3n+k-2)! \binom{3n-2}{k} \frac{s^k}{2^{k+1}}\,,$$

$$q_{5n-3} = \frac{1}{(3n-1)!} \sum_{k=0}^{3n-1} (-1)^{3n-k-1} (3n+k-1)! \binom{3n-1}{k} \left(\frac{s}{2}\right)^k\,,$$

$$q_{5n-2} = \frac{1}{(3n-1)!} \sum_{k=0}^{3n} (-1)^{3n-k} (3n+k-1)! \binom{3n}{k} \left(\frac{s}{2}\right)^k\,,$$

$$q_{5n-1} = \frac{1}{(3n-1)!} \sum_{k=0}^{3n-1} (-1)^{3n-k-1} (3n+k)! \binom{3n-1}{k} \left(\frac{s}{2}\right)^{k+1}\,,$$

$$q_{5n} = \frac{1}{(3n)!} \sum_{k=0}^{3n} (-1)^{3n-k} (3n+k)! \binom{3n}{k} \left(\frac{s}{2}\right)^k\,.$$

Note that the forms of q_{5n-2} and q_{5n-1} are interchanged to that of p_{5n-2} and p_{5n-1} in addition to the minus signs for every second term. Let p_n^*/q_n^* be the n-th convergent of the continued fraction of $e^2 = [7; \overline{3k-1, 1, 1, 3k, 12k+6}\ ]_{k=1}^{\infty}$. Then, for $n \ge 0$ we have

$$\frac{p_n^*}{q_n^*} = \frac{p_{n+2}}{q_{n+2}}\,,$$

where p_n/q_n is the n-th convergent of the continued fraction of $e^{2/s}$ mentioned above.

7. Proof of Theorem 1

We shall prove Theorem 1 by induction. The basic recurrence relation $p_n = a_n p_{n-1} + p_{n-2}$ $(n \ge 0)$ is used repeatedly. The proofs of other theorems are also done by induction in similar manners and omitted. The first initial values match the result because $p_0 = 0$, $p_1 = 1$ and $p_2 = (a+2b)v$. Suppose that the identities hold for p_{2n-1} and p_{2n}. Since

$$\bigl(a+(2n+1)b\bigr)\binom{n+k-1}{2k-1} + \binom{n+k-1}{2k}\bigl(a+b(n-k+1)\bigr) = \binom{n+k}{2k}\bigl(a+b(n+k+1)\bigr)\,,$$

we have

$$u\bigl(a+(2n+1)b\bigr)\binom{n+(k-1)}{2(k-1)+1}\left(\prod_{i=n-(k-1)+1}^{n+(k-1)+1}(a+bi)\right)u^{k-1}v^k$$

$$+\binom{n+k-1}{2k}\left(\prod_{i=n-k+1}^{n+k}(a+bi)\right)(uv)^k$$

$$=\binom{n+k}{2k}\left(\prod_{i=n-k+2}^{n+k+1}(a+bi)\right)(uv)^k\,.$$

Together with

$$\binom{n-1}{0} = 1 = \binom{n}{0}$$

and

$$\bigl(a+(2n+1)b\bigr)\binom{2n-1}{2n-1}\prod_{i=2}^{2n}(a+bi) = \binom{2n}{2n}\prod_{i=2}^{2n+1}(a+bi)\,,$$

we obtain

$$\begin{aligned} p_{2n+1} &= u\bigl(a+(2n+1)b\bigr)p_{2n} + p_{2n-1} \\ &= \sum_{k=0}^{n}\binom{n+k}{2k}\left(\prod_{i=n-k+2}^{n+k+1}(a+bi)\right)(uv)^k\,. \end{aligned}$$

Next, suppose that the identities hold for p_{2n} and p_{2n+1}. Since

$$\begin{aligned} &\bigl(a+(2n+2)b\bigr)\binom{n+k}{2k} + \binom{n+k}{2k+1}\bigl(a+b(n-k+1)\bigr) \\ &\qquad\qquad = \binom{n+k+1}{2k+1}\bigl(a+b(n+k+2)\bigr)\,, \end{aligned}$$

we have

$$\begin{aligned} &v\bigl(a+(2n+2)b\bigr)\binom{n+k}{2k}\left(\prod_{i=n-k+2}^{n+k+1}(a+bi)\right)(uv)^k \\ &\qquad + \binom{n+k}{2k+1}\left(\prod_{i=n-k+1}^{n+k+1}(a+bi)\right)u^k v^{k+1} \end{aligned}$$

$$= \binom{n+k+1}{2k+1} \left(\prod_{i=n-k+2}^{n+k+2} (a+bi) \right) u^k v^{k+1} .$$

Together with

$$\big(a+(2n+2)b\big)\binom{2n}{2n} \prod_{i=2}^{2n+1} (a+bi) = \binom{2n+1}{2n+1} \prod_{i=2}^{2n+2} (a+bi) ,$$

we obtain

$$p_{2n+2} = v\big(a+(2n+2)b\big)p_{2n+1} + p_{2n}$$

$$= \sum_{k=0}^{n} \binom{n+k+1}{2k+1} \left(\prod_{i=n-k+2}^{n+k+2} (a+bi) \right) u^k v^{k+1} .$$

The proof for the identities of q's are similarly done and omitted.

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