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ON CONGRUENCE DISTRIBUTIVITY OF ORDERED ALGEBRAS WITH CONSTANTS

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Abstract

We define the order-congruence distributivity at 0 and order-congruence n-distributivity at 0 of ordered algebras with a nullary operation 0. These notions are generalizations of congruence distributivity and congruence n-distributivity. We prove that a class of ordered algebras with a nullary operation 0 closed under taking subalgebras and direct products is order-congruence distributive at 0 iff it is order-congruence n-distributive at 0. We also characterize such classes by a Mal'tsev condition.

Keywords: ordered algebra, n-distributivity, distributivity, Mal'tsev condition.

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1. Introduction

Following Huhn [13, 14, 15], a lattice is called *n-distributive* if it satisfies the so-called n-distributive identity

$$x \wedge \bigvee_{i=0}^{n} y_i \le \bigvee_{j=0}^{n} \left(x \wedge \bigvee_{\substack{i=0 \\ i\neq j}}^{n} y_i \right).$$

Notice that $n = 1$ renders the usual distributive law. The n-distributive identity has proved to be a very useful tool in several investigations, cf., for example, Huhn [13, 14, 15] and Herrmann-Huhn [12]. Also, n-distributivity and some of its generalizations have played an important role in studying certain classes of congruence varieties.

While there are n-distributive but non-distributive lattices for each $n \ge 2$, many known results state that n-distributivity (for congruences) implies distributivity for some particular cases. Our chief goal is to give a new result of this kind, which is a common generalization of two distinct earlier theorems.

Next, before formulating our main result, we survey the known "n-distributivity implies distributivity" type results in the following paragraphs. Notice that distributivity trivially implies n-distributivity, so instead of implication we can say that n-distributivity is equivalent to distributivity under specific circumstances. The operators of forming subalgebras, direct products, finite direct products (that is, direct products of finitely many algebras), finite subdirect products, finite subdirect powers and subdirect squares will be denoted by $\mathbf{S}$, $\mathbf{P}$, $\mathbf{P}_f$, $\mathbf{P}^s_f$, $\mathbf{Q}^s_f$, and $\mathbf{Q}^s_2$, respectively. The congruence lattice of an algebra $\mathbf{A}$ is denoted by $\operatorname{Con}\mathbf{A}$.

The story started with Nation [17], who proved that a variety of algebras is congruence distributive iff it is congruence n-distributive. In fact, he stated this result for certain more general identities not just for the n-distributive law.

Czédli [4] has noticed that, for each n, if $\mathcal{K}$ is an $\mathbf{SP}_f$-closed class of algebras and $\{\operatorname{Con}\mathbf{A} : \mathbf{A} \in \mathcal{K}\}$ is n-distributive, then it is distributive as well. Later, he improved this result by replacing $\mathbf{P}_f$ with $\mathbf{Q}_f$, see Proposition 1 of [5].

Much more can be stated in the presence of modularity. Using the Freese-Jónsson Amalgamation Property, FJAP in short, from Freese and

Jónsson [9], Czédli [4] has shown that if $\mathcal{K}$ is $\mathbf{Q}_2^s$-closed and $\{\mathrm{Con}\,\mathbf{A} : \mathbf{A} \in \mathcal{K}\}$ is n-distributive and modular, then it is distributive. (His result extends for several other identities, and also for the lattices of relative congruences and order-congruences, see Theorem 1 of [4], but the details are not relevant here.)

For classes of *ordered algebras* $\mathbf{A} = (A, F, \leq)$, see Bloom [2], the basic operations are assumed to be monotone with respect to the partial ordering $\leq$. Notice that ordered algebras are generalizations of algebras, since an algebra can be considered as an ordered algebra with the equality relation as a partial ordering on it.

The operators $\mathbf{H}$, $\mathbf{S}$, $\mathbf{P}$, etc. are defined in the natural way; in particular, homomorphisms are *monotone* mappings preserving the operations. *Order-congruences* of $\mathbf{A}$ are defined as kernels of homomorphisms from $\mathbf{A}$ to any other $\mathbf{B}$. These congruences form an algebraic lattice $\mathrm{Con}^{\leq}\,\mathbf{A}$. Notice, and see later, that $\mathrm{Con}^{\leq}\,\mathbf{A}$ is a complete meet-subsemilattice but not a sublattice of $\mathrm{Con}\,\mathbf{A}$. Generalizing Nation's above-mentioned result, Czédli and Lenkehegyi [8] proved that if $\mathcal{K}$ is an $\mathbf{SP}$-closed class of ordered algebras with $\{\mathrm{Con}^{\leq}\,\mathbf{A} : \mathbf{A} \in \mathcal{K}\}$ being n-distributive, then $\{\mathrm{Con}^{\leq}\,\mathbf{A} : \mathbf{A} \in \mathcal{K}\}$ is distributive.

Following Chajda [3], assume that $\mathcal{K}$ is a class of algebras of type τ and 0 is a fixed nullary operation symbol in τ. For k-ary lattice terms s and t we say that the lattice identity $s \leq t$ *holds for congruences of* $\mathcal{K}$ *at 0*, if for every $\mathbf{A} \in \mathcal{K}$ and $\Theta_1, \ldots, \Theta_k \in \mathrm{Con}\,\mathbf{A}$, the $s(\Theta_1, \ldots, \Theta_k)$-class of 0 is included in the $t(\Theta_1, \ldots, \Theta_k)$-class of 0. If $s \leq t$ is the n-distributive (distributive) identity, then we say that $\mathcal{K}$ is *congruence n-distributive at* 0 (*congruence distributive at* 0). Czédli [5] has proved that if $\mathcal{K}$ is an $\mathbf{SQ}_f$-closed class with a constant 0 and $\mathcal{K}$ is congruence n-distributive at 0 (or, more generally, some other identity of Nation [17] holds for congruences of $\mathcal{K}$ at 0), then $\mathcal{K}$ is congruence distributive at 0.

If $\mathcal{S}$ denotes the variety of meet-semilattices with 0, then, according to Chajda [3], $\mathcal{S}$ is congruence-distributive at 0. However, Czédli [5] made the surprising discovery that the dual of the distributive law does not hold for congruences of $\mathcal{S}$ at 0. This indicates the difficulty when considering lattice identities for congruences at 0.

For a class of *ordered* algebras of type τ with a constant 0 in τ, the above-defined notions are self-explanatory. Our main result generalizes both [8] and a part of [5] as follows.

Theorem 1 (Main theorem). *Let $n \geq 2$, and let $\mathcal{K}$ be an* $\mathbf{SP}$*-closed class of ordered algebras with a constant* 0*. Then the n-distributive law holds for order-congruences of $\mathcal{K}$ at* 0 *iff so does the distribuitve law.*

Many lattice identities can be characterized by Mal'tsev conditions, see Jónsson [16] for a first survey. All these identities mentioned in [16], and later in Chapter 13 of Freese and McKenzie [10] imply modularity. Later, Czédli and Horváth [6] (see also Czédli, Horváth and Lipparini [7]) proved that if a lattice identity implies modularity, then it can be characterized by a Mal'tsev condition. Somehow, modularity seems to be relevant when dealing with Mal'tsev conditions.

As a by-product of the proof of Theorem 1, we will give a Mal'tsev condition to characterize the class $\mathcal{K}$ from the theorem. Then the Mal'tsev condition from Chajda [3] and that from Czédli and Lenkehegyi [8] will become straightforward corollaries. Since order-congruence distributivity at 0 trivially implies order-congruence modularity (that is, $\alpha \wedge (\beta \vee (\alpha \wedge \gamma)) \leq (\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$ at 0, the relevance of modularity remains untouched even at 0.

2. Order-congruences

In the introduction we defined ordered algebras and order-congruences on them. According to the following lemma, order-congruences of an ordered algebra are congruences of the corresponding algebra. Sometimes all congruences are order-congruences as well, e.g. in case of lattices equipped with the usual ordering. However the additive group of integers with the usual ordering has many congruences, but it has only the two trivial order-congruences.

The following two lemmas describe order-congruences and their join. The proofs can be found in Czédli-Lenkehegyi [8]. From now on, the join of order-congruences will be understood in $\mathrm{Con}^{\leq} \mathbf{A}$ without any notational warning.

Lemma 2. *Let Θ be a binary relation on an ordered algebra $\mathbf{A} = (A, F, \leq)$. Then Θ is an order-congruence iff it is a congruence of the algebra (A, F) and, for all $a, b \in A$, the pair (a, b) belongs to Θ iff there exist elements $a_0, \ldots, a_u, b_0, \ldots, b_v \in A$ such that*

$$a = a_0 \Theta a_1 \leq a_2 \Theta a_3 \leq \ldots a_u = b;$$

$$b = b_0 \Theta b_1 \leq b_2 \Theta b_3 \leq \ldots b_v = a.$$

Let $\Theta_1, \ldots, \Theta_n$ be order-congruences of an ordered algebra $\mathbf{A}$. Then, for $a, b \in A$, let the notation $a \xrightarrow{\Theta_1,\ldots,\Theta_n} b$ stand for

$$(a,b) \in \bigcup_{k=1}^{\infty} (\Theta_1 \circ \cdots \circ \Theta_n \circ \leq)^k .$$

Notice that Lemma 2 implies $\Theta = \{(a,b) \in A^2 : a \xrightarrow{\Theta} b \text{ and } b \xrightarrow{\Theta} a\}$, provided Θ is an order-congruence of A.

Lemma 3. *Let $\Theta_1, \ldots, \Theta_n$ be order-congruences of an ordered algebra $\mathbf{A}$. Then*

$$\Theta_1 \vee \ldots \vee \Theta_n = \Big\{(a,b) \in A^2 : a \xrightarrow{\Theta_1,\ldots,\Theta_n} b \text{ and } b \xrightarrow{\Theta_1,\ldots,\Theta_n} a\Big\}.$$

The following lemma is evident.

Lemma 4. *Let $\Theta_1, \ldots, \Theta_n$ be order-congruences of an ordered-algebra $\mathbf{A}$, and let $t(x)$ be a unary algebraic function of the same type. Then $a \xrightarrow{\Theta_1,\ldots,\Theta_n} b$ implies $t(a) \xrightarrow{\Theta_1,\ldots,\Theta_n} t(b)$ for all $a, b \in A$.*

3. Further results and proofs

From now on, let us fix an **SP**-closed class $\mathcal{K}$ of ordered algebras of type τ with a constant 0. For a positive integer n, let X be the set of variables $X = \{x_1, \ldots, x_{n+1}\}$, and let $x_0 = 0$. Let $\mathbf{F}_{\mathcal{K}}(X)$ denote the *$\mathcal{K}$-free ordered algebra over X*. Note that the definition of $\mathcal{K}$-free ordered algebras is the same as in case of algebras, and the same proof shows that $\mathcal{K}$-free ordered algebras are in $\mathbf{SP}\mathcal{K}$, cf. Grätzer [11] or Birkhoff [1]. For arbitrary indices $i, j \leq n+1$, let Θ_{ij} denote the smallest order-congruence of $\mathbf{F}_{\mathcal{K}}(X)$ containing (x_i, x_j), and let $\overline{\Theta_i}$ denote the following order-congruence:

$$\overline{\Theta_i} = \bigvee_{\substack{h=0\\ h\neq i}}^{n} \Theta_{n-h,n-h+1}.$$

For an algebra $\mathbf{A} \in \mathcal{K}$ and an order-congruence $\alpha \in \mathrm{Con}^{\leq} \mathbf{A}$, let $[0]\alpha$ denote the α-class of 0. From now on, for a lattice term t, we will use the following notation:

$$t\Big(\vec{x}, \underbrace{y}_{h}, \vec{z}\Big) := t\Big(\underbrace{x}_{1}, \dots, \underbrace{x}_{h-1}, \underbrace{y}_{h}, \underbrace{z}_{h+1}, \dots, \underbrace{z}_{n+1}\Big).$$

Proposition 5. *Let $\mathcal{K}$ be an* **SP***-closed class of ordered algebras with a constant* 0*, and let* n *be a positive integer. Then the following four conditions are equivalent:*

(i) $\mathrm{Con}^{\leq}\,\mathbf{A}$ *is* n*-distributive at* 0 *for all* $\mathbf{A} \in \mathcal{K}$*;*

(ii) $x_{n+1} \in [0] \bigvee_{k=0}^{n} \left(\Theta_{0,n+1} \wedge \overline{\Theta_k}\right)$ *in* $\mathbf{F}_{\mathcal{K}}(X)$*;*

(iii) *there are finite* u *and* v *and* $(n+1)$*-ary terms* s_{ik}, t_{jk} *(*$i \leq u$*,* $j \leq v$ *and* $k \leq n+1$*) such that* $\mathcal{K}$ *satisfies the following identities*

$$s_{00}(x_1, \dots, x_{n+1}) = 0 \qquad \text{and } s_{u,n+1}(x_1, \dots, x_{n+1}) = x_{n+1};$$

$$s_{ik}(x_1, \dots, x_n, 0) = 0;$$

$$t_{00}(x_1, \dots, x_{n+1}) = x_{n+1} \text{ and } t_{v,n+1}(x_1, \dots, x_{n+1}) = 0;$$

$$t_{jk}(x_1, \dots, x_n, 0) = 0;$$

$$s_{i,n+1}(x_1, \dots, x_{n+1}) \leq s_{i+1,0}(x_1, \dots, x_{n+1}) \text{ if } i < u;$$

$$t_{j,n+1}(x_1, \dots, x_{n+1}) \leq t_{j+1,0}(x_1, \dots, x_{n+1}) \text{ if } j < v;$$

$$s_{ik}(\vec{0}, \underbrace{x}_{n-k+1}, \vec{x}) = s_{i,k+1}(\vec{0}, \underbrace{x}_{n-k+1}, \vec{x}) \text{ if } k < n+1;$$

$$t_{jk}(\vec{0}, \underbrace{x}_{n-k+1}, \vec{x}) = t_{j,k+1}(\vec{0}, \underbrace{x}_{n-k+1}, \vec{x}) \text{ if } k < n+1.$$

(iv) $\mathrm{Con}^{\leq}\,\mathbf{A}$ *is distributive at* 0 *for all* $\mathbf{A} \in \mathcal{K}$.

Proof. (i) $\Rightarrow$ (ii). The class $\mathcal{K}$ is closed under $\mathbf{S}$ and $\mathbf{P}$, therefore $\mathcal{K}$ contains $\mathbf{F}_{\mathcal{K}}(X)$. Using the n-distributivity, condition (i) implies condition (ii) trivially:

$$x_{n+1} \in [0]\Theta_{0,n+1} \wedge \bigvee_{i=0}^{n} \Theta_{i,i+1} \subseteq [0]\bigvee_{i=0}^{n} \left(\Theta_{0,n+1} \wedge \overline{\Theta_i}\right).$$

(ii) $\Rightarrow$ (iii). Condition (ii) means that $(0, x_{n+1}) \in \bigvee_{i=0}^{n} \left(\Theta_{0,n+1} \wedge \overline{\Theta_i}\right)$. Using Lemma 3 we obtain $(n+1)$-ary terms s_{ik}, t_{jk} ($i \leq u$, $j \leq v$ and $k \leq n+1$) such that the following relations hold in $\mathbf{F}_{\mathcal{K}}(X)$:

(1) $s_{00}(x_1, \ldots, x_{n+1}) = 0$ and $s_{u,n+1}(x_1, \ldots, x_{n+1}) = x_{n+1}$

(2) $s_{ik}(x_1, \ldots, x_{n+1}) \, \Theta_{0,n+1} \wedge \overline{\Theta_k} \, s_{i,k+1}(x_1, \ldots, x_{n+1})$ if $k < n+1$

(3) $s_{i,n+1}(x_1, \ldots, x_{n+1}) \leq s_{i+1,0}(x_1, \ldots, x_{n+1})$ if $i < u$

(4) $t_{00}(x_1, \ldots, x_{n+1}) = x_{n+1}$ and $t_{v,n+1}(x_1, \ldots, x_{n+1}) = 0$

(5) $t_{jk}(x_1, \ldots, x_{n+1}) \, \Theta_{0,n+1} \wedge \overline{\Theta_k} \, t_{j,k+1}(x_1, \ldots, x_{n+1})$ if $k < n+1$

(6) $t_{j,n+1}(x_1, \ldots, x_{n+1}) \leq t_{j+1,0}(x_1, \ldots, x_{n+1})$ if $j < v$

Using Lemma 2 and the relations above, we obtain

(7) $$0 \, \Theta_{0,n+1} \, s_{ik}(x_1, \ldots, x_{n+1});$$

(8) $$x_{n+1} \, \Theta_{0,n+1} \, t_{jk}(x_1, \ldots, x_{n+1}).$$

Let us consider the monotone homomorphism $\phi : \mathbf{F}_{\mathcal{K}}(X) \to \mathbf{F}_{\mathcal{K}}(X)$ defined by the equations $x_h\phi = x_h$ ($h \leq n$) and $x_{n+1}\phi = 0$. Both $\Theta_{0,n+1}$ and $\ker\phi$ are order-congruences, therefore $(0, x_{n+1}) \in \ker\phi$ implies $\Theta_{0,n+1} \subseteq \ker\phi$.

Using this and the relations in (7) and (8), we obtain that the following identities hold in $\mathbf{F}_{\mathcal{K}}(X)$, hence they hold in $\mathcal{K}$, too:

$$\begin{aligned} s_{ik}(x_1,\dots,x_n,0) &= s_{ik}(x_1\phi,\dots,x_{n+1}\phi)\\ &= s_{ik}(x_1,\dots,x_{n+1})\phi = 0\phi = 0, \end{aligned} \tag{9}$$

$$\begin{aligned} t_{jk}(x_1,\dots,x_n,0) &= t_{jk}(x_1\phi,\dots,x_{n+1}\phi)\\ &= t_{jk}(x_1,\dots,x_{n+1})\phi = x_{n+1}\phi = 0. \end{aligned} \tag{10}$$

Let us fix an index $k < n+1$ and let us consider the monotone homomorphism $\psi : \mathbf{F}_{\mathcal{K}}(X) \to \mathbf{F}_{\mathcal{K}}(x)$ defined by the equations: $x_{n-h}\psi = 0$ if $k \leq h$ and $x_{n-h+1}\psi = x$ if $h \leq k$. Both $\overline{\Theta_k}$ and $\ker\psi$ are order-congruences, moreover $(x_{n-h}, x_{n-h+1}) \in \ker\psi$ if $h \neq k$, therefore $\overline{\Theta_k} \subseteq \ker\psi$. Using this and the relations in (2) and (5), we obtain that the following identities hold in $\mathbf{F}_{\mathcal{K}}(X)$, hence they hold in $\mathcal{K}$, too:

$$\begin{aligned} s_{ik}\Big(\vec{0}, \overbrace{x}^{n-k+1}, \vec{x}\Big) &= s_{ik}(x_1\psi,\dots,x_{n+1}\psi) = s_{ik}(x_1,\dots,x_{n+1})\psi\\ &= s_{i,k+1}(x_1,\dots,x_{n+1})\psi = s_{i,k+1}(x_1\psi,\dots,x_{n+1}\psi)\\ &= s_{i,k+1}\Big(\vec{0}, \overbrace{x}^{n-k+1}, \vec{x}\Big);\ \text{and similarly,} \end{aligned} \tag{11}$$

$$t_{jk}\Big(\vec{0}, \overbrace{x}^{n-k+1}, \vec{x}\Big) = t_{j,k+1}(\vec{0}, \overbrace{x}^{n-k+1}, \vec{x}). \tag{12}$$

The identities in (1), (3), (4), (6), (9), (10), (11) and (12) show that condition (ii) implies condition (iii).

(iii) ⇒ (iv). Let us fix an ordered algebra $\mathbf{A} \in \mathcal{K}$ and order-congruences $\alpha, \beta, \gamma \in \mathrm{Con}^{\leq}\mathbf{A}$. We have to prove that $[0]\alpha \wedge (\beta \vee \gamma) \subseteq [0](\alpha\wedge\beta)\vee(\alpha\wedge\gamma)$. Suppose that $b \in [0]\alpha \wedge (\beta \vee \gamma)$. By Lemma 3, $(0, b) \in \alpha \wedge (\beta \vee \gamma)$ means that $(0, b) \in \alpha$ and

$$0 \xrightarrow{\beta,\gamma} b \text{ and } b \xrightarrow{\beta,\gamma} 0. \tag{13}$$

By condition (iii), we have $(n+1)$-ary terms s_{ik} and t_{jk}. For $i \leq u$, $j \leq v$ and $k \leq n+1$, let us consider the following unary algebraic functions:

$$s_{ik}\Big(\vec{0}, \overbrace{x}^{n+1-k}, \vec{b}\Big),\ s_{i0}\Big(\vec{x}, \overbrace{x}^{n}, \vec{0}\Big) \text{ and } t_{jk}\Big(\vec{0}, \overbrace{x}^{n+1-k}, \vec{b}\Big),\ t_{j0}\Big(\vec{x}, \overbrace{x}^{n}, \vec{0}\Big). \tag{14}$$

Using the identities in condition (iii) and using Lemma 4 for (13) and (14), we obtain:

$$\begin{aligned} s_{00}(0,\dots,0,b) = s_{00}\Big(\vec{0}, \overbrace{0}^{n}, \vec{b}\Big) &= s_{01}\Big(\vec{0}, \overbrace{0}^{n}, \vec{b}\Big) \xrightarrow{\beta,\gamma} s_{01}\Big(\vec{0}, \overbrace{b}^{n}, \vec{b}\Big) = \\ s_{01}\Big(\vec{0}, \overbrace{0}^{n-1}, \vec{b}\Big) &= s_{02}\Big(\vec{0}, \overbrace{0}^{n-1}, \vec{b}\Big) \xrightarrow{\beta,\gamma} s_{02}\Big(\vec{0}, \overbrace{b}^{n-1}, \vec{b}\Big) = \\ &\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ s_{0n}\Big(\vec{0}, \overbrace{b}^{2}, \vec{b}\Big) &= s_{0,n+1}\Big(\vec{0}, \overbrace{b}^{2}, \vec{b}\Big) \xrightarrow{\beta,\gamma} s_{0,n+1}\Big(\vec{b}, \overbrace{b}^{2}, \vec{b}\Big) = \\ s_{0,n+1}(b,\dots,b) &\leq s_{10}(b,\dots,b) \xrightarrow{\beta,\gamma} s_{10}(0,\dots,0,b) \end{aligned}$$

Now, $(0,b) \in \alpha$ and condition (iii) yield that the values of the unary algebraic functions in (14) are always in $[0]\alpha$. Hence

$$s_{00}(0,\dots,0,b) \xrightarrow{\alpha\wedge\beta,\alpha\wedge\gamma} s_{0,n+1}(b,\dots,b) \xrightarrow{\alpha\wedge\beta,\alpha\wedge\gamma} s_{10}(0,\dots,0,b).$$

Repeating the calculations above for $i = 1,\dots,u$, we obtain

$$s_{i-1,n+1}(b,\dots,b) \xrightarrow{\alpha\wedge\beta,\alpha\wedge\gamma} s_{i0}(0,\dots,0,b) \xrightarrow{\alpha\wedge\beta,\alpha\wedge\gamma} s_{i,n+1}(b,\dots,b).$$

Hence

$$0 = s_{00}(0,\dots,0,b) \xrightarrow{\alpha\wedge\beta,\alpha\wedge\gamma} s_{u,n+1}(b,\dots,b) = b. \tag{15}$$

Similarly, condition (iii), (13) and (14) imply

$$b = t_{00}(0, \dots, 0, b) \xrightarrow{\alpha\wedge\beta,\alpha\wedge\gamma} t_{v,n+1}(b, \dots, b) = 0. \tag{16}$$

Using Lemma 3 for (15) and (16), we obtain $(0, b) \in (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$, hence $b \in [0](\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$. This shows that (iii) implies (iv).

(iv) $\Rightarrow$ (i). We have to show that order-congruence distributivity at 0 implies order-congruence n-distibutivity at 0. It follows from the following observation. For any order-congruences $\alpha, \beta_0, \dots, \beta_n$ of an ordered algebra $\mathbf{A}$ that is order-congruence distributive at 0, we have

$$[0]\alpha \wedge \bigvee_{i=0}^{n} \beta_i = [0]\alpha \wedge \bigvee_{j=0}^{n} \bigvee_{\substack{i=0\\ i\neq j}}^{n} \beta_i \subseteq [0] \bigvee_{j=0}^{n} \left(\alpha \wedge \bigvee_{\substack{i=0\\ i\neq j}}^{n} \beta_i \right).$$

■

***Proof of Theorem* 1.** Notice that the equivalence of (i) and (iv) in Proposition 5 is the statement of Theorem 1. ■

Now, we show that Theorem 1 implies Czédli-Lenkehegyi's result [8] that congruence n-distributivity and congruence distributivity are equivalent on **SP**-closed classes of ordered algebras. First, we need some definitions.

For lattice identities λ and μ, let $\lambda \vDash_{\mathrm{oc}} \mu\,(\mathbf{SP})$ mean that for every **SP**-closed class $\mathcal{K}$ of ordered algebras, if λ holds for order-congruences of $\mathcal{K}$ then μ holds for order-congruences of $\mathcal{K}$. Similarly, let $\lambda \vDash_{\mathrm{oc0}} \mu\,(\mathbf{SP})$ mean that for every **SP**-closed class $\mathcal{K}$ of ordered algebras, if λ holds for order-congruences of $\mathcal{K}$ at 0 then μ holds for order-congruences of $\mathcal{K}$ at 0. The following Lemma implies Czédli–Lenkehegyi's result [8].

Lemma 6. *If* $\lambda \vDash_{\mathrm{oc0}} \mu\,(\mathbf{SP})$ *then* $\lambda \vDash_{\mathrm{oc}} \mu\,(\mathbf{SP})$.

Proof. Let $\mathcal{K}$ be an **SP**-closed class of ordered algebras of type τ such that λ holds for order-congruences of $\mathcal{K}$. We want to show that μ holds for order-congruences of $\mathcal{K}$. Add a new operation symbol 0 to the type of $\mathcal{K}$. This way we obtain $\tau_0 = \tau \cup \{0\}$. Let $\mathcal{K}_0$ denote the class of ordered algebras of type τ_0 whose τ-reducts are in $\mathcal{K}$. Notice that $\mathcal{K}_0$ is **SP**-closed.

Now, λ holds for order-congruences of $\mathcal{K}_0$, therefore λ holds for order-congruences of $\mathcal{K}_0$ at 0. By the assumption, μ holds for order-congruences of $\mathcal{K}_0$ at 0. Suppose μ is in the form $s \leq t$ where s and t are n-ary lattice terms. Let $\Theta_1, \ldots, \Theta_n \in \mathrm{Con}^{\leq} \mathbf{A}$ for some $\mathbf{A} \in \mathcal{K}$ and let $(u, v) \in s(\Theta_1, \ldots, \Theta_n)$. Setting $0 = u$ we turn $\mathbf{A}$ into a τ_0-algebra $\mathbf{A}_0 \in \mathcal{K}_0$. Then $v \in [0]s(\Theta_1, \ldots, \Theta_n) \subseteq [0]t(\Theta_1, \ldots, \Theta_n)$ yields $(u, v) \in t(\Theta_1, \ldots, \Theta_n)$. This shows that μ holds for order-congruences of $\mathcal{K}$. ■

Using Theorem 1 for arbitrary n and condition (i) and (iii) of Proposition 5 for $n = 1$, we obtain a "nice" Mal'tsev condition for order-congruence n-distributivity (and order-congruence distributivity) at 0. Notice that by the reflexivity of order-congruences, $u = v$ can be supposed in condition (iii) of Proposition 5.

Corollary 7. *Let $\mathcal{K}$ be an* **SP***-closed class of ordered algebras with a constant* 0*, and let n be a positive integer. Then the following three conditions are equivalent:*

(i) $\mathrm{Con}^{\leq} \mathbf{A}$ *is n-distributive at* 0 *for all* $\mathbf{A} \in \mathcal{K}$*;*

(ii) $\mathrm{Con}^{\leq} \mathbf{A}$ *is distributive at* 0 *for all* $\mathbf{A} \in \mathcal{K}$*;*

(iii) *there is a finite k and there are* 2*-ary terms s_{ij}, t_{ij} ($i \leq k$ and $j \leq 2$) such that $\mathcal{K}$ satisfies*

$$s_{00}(x, y) = 0,\ s_{k2}(x, y) = y \text{ and } s_{ij}(x, 0) = 0;$$

$$t_{00}(x, y) = y,\ t_{k2}(x, y) = 0 \text{ and } t_{ij}(x, 0) = 0;$$

$$s_{i2}(x, y) \leq s_{i+1,0}(x, y) \text{ if } i < k;$$

$$t_{i2}(x, y) \leq t_{i+1,0}(x, y) \text{ if } i < k;$$

$$s_{i0}(0, x) = s_{i1}(0, x), \quad s_{i1}(x, x) = s_{i2}(x, x);$$

$$t_{i0}(0, x) = t_{i1}(0, x), \quad t_{i1}(x, x) = t_{i2}(x, x).$$

As ordered algebras are generalizations of algebras, Corollary 7 gives a Mal'tsev condition for congruence distributivity at 0. The Mal'tsev condition found by Chajda [3] can be deduced from Corollary 7.

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