# PRE-STRONGLY SOLID VARIETIES OF COMMUTATIVE SEMIGROUPS 

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#### Abstract

Generalized hypersubstitutions are mappings from the set of all fundamental operations into the set of all terms of the same language do not necessarily preserve the arities. Strong hyperidentities are identities which are closed under the generalized hypersubstitutions and a strongly solid variety is a variety which every its identity is a strong hyperidentity. In this paper we give an example of pre-strongly solid varieties of commutative semigroups and determine the least and the greatest pre-strongly solid variety of commutative semigroups.


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## 1. Introduction

Hyperidentities were invented by Aczel, Belousov and Taylor. The notion of hyperidentities and solid varieties of a given type as well as derived algebras of given type were invented by E. Graczyńska and D. Schweigert in [3]. An identity $t \approx t^{\prime}$ of terms of any type $\tau$ is called a hyperidentity for an algebra $\underline{A}=\left(A ;\left(f_{i}^{A}\right)_{i \in I}\right)$ if $t \approx t^{\prime}$ holds identically for every choice of $n$-ary term operation to represent $n$-ary operation symbols occurring in $t$ and $t^{\prime}$. A variety which every its identity is a hyperidentity is called solid variety. Hyperidentities can be characterized more precisely using the concept of a hypersubstitution which was introduced by K. Denecke, D. Lau, R. Pöschel and D. Schweigert. A hypersubstitution of type $\tau$ is a mapping $\sigma:\left\{f_{i} \mid i \in\right.$ $I\} \longrightarrow W_{\tau}(X)$ which assigns to every $n_{i}$-ary operation symbol $f_{i}$ an $n_{i}$-ary term. The set of all hypersubstitutions of type $\tau$ is denoted by $\operatorname{Hyp}(\tau)$. For every $\sigma \in \operatorname{Hyp}(\tau)$ induces a mapping $\hat{\sigma}: W_{\tau}(X) \longrightarrow W_{\tau}(X)$ by the following steps:
(i) $\hat{\sigma}[x]:=x$, for any variable $x \in X$, and
(ii) $\hat{\sigma}\left[f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right]:=\sigma\left(f_{i}\right)\left(\hat{\sigma}\left[t_{1}\right], \ldots, \hat{\sigma}\left[t_{n_{i}}\right]\right)$, where $\hat{\sigma}\left[t_{j}\right], 1 \leq j \leq n_{i}$ are already defined.

A binary operation $\circ_{h}$ on $\operatorname{Hyp}(\tau)$ is defined by $\sigma_{1} \circ_{h} \sigma_{2}:=\hat{\sigma_{1}} \circ \sigma_{2}$ for every $\sigma_{1}, \sigma_{2} \in \operatorname{Hyp}(\tau)$ where $\circ$ is the natural composition of mappings. Let $\sigma_{i d}$ be the hypersubstitution where $\sigma_{i d}\left(f_{i}\right)=f_{i}\left(x_{1}, \ldots, x_{n_{i}}\right)$. It turns out that $\left(H y p(\tau) ; \circ_{h}, \sigma_{i d}\right)$ is a monoid with $\sigma_{i d}$ is an identity element.
S. Leeratanavalee and K. Denecke generalized the concepts of hypersubstitutions, hyperidentities and solid varieties to generalized hypersubstitutions, strong hyperidentities and strongly solid varieties [4]. A generalized hypersubstitution of type $\tau$ is a mapping $\sigma:\left\{f_{i} \mid i \in I\right\} \longrightarrow W_{\tau}(X)$ from the set of all $n_{i}$-ary operation symbols into the set of all terms built up by elements of the alphabet $X:=\left\{x_{1}, x_{2}, \ldots\right\}$ and operation symbols from $\left\{f_{i} \mid i \in I\right\}$ which does not necessarily preserve the arity.

We denoted the set of all generalized hypersubstitutions of type $\tau$ by $\operatorname{Hyp}_{G}(\tau)$. To define a binary operation on $H y p_{G}(\tau)$, we defined firstly the concept of generalized superposition of terms $S^{m}: W_{\tau}(X)^{m+1} \longrightarrow W_{\tau}(X)$ by the following steps:

$$
\text { for any term } t \in W_{\tau}(X),
$$

(i) if $t=x_{j}, 1 \leq j \leq m$, then
$S^{m}\left(x_{j}, t_{1}, \ldots, t_{m}\right):=t_{j}$,
(ii) if $t=x_{j}, m<j \in \mathbb{N}$, then
$S^{m}\left(x_{j}, t_{1}, \ldots, t_{m}\right):=x_{j}$,
(iii) if $t=f_{i}\left(s_{1}, \ldots, s_{n_{i}}\right)$, then
$S^{m}\left(t, t_{1}, \ldots, t_{m}\right):=f_{i}\left(S^{m}\left(s_{1}, t_{1}, \ldots, t_{m}\right), \ldots, S^{m}\left(s_{n_{i}}, t_{1}, \ldots, t_{m}\right)\right)$.
Then the generalized hypersubstitution $\sigma$ can be extended to a mapping $\hat{\sigma}: W_{\tau}(X) \longrightarrow W_{\tau}(X)$ by the following steps:
(i) $\hat{\sigma}[x]:=x \in X$,
(ii) $\hat{\sigma}\left[f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right]:=\quad S^{n_{i}}\left(\sigma\left(f_{i}\right), \hat{\sigma}\left[t_{1}\right], \ldots, \hat{\sigma}\left[t_{n_{i}}\right]\right)$, for any $n_{i}$-ary operation symbol $f_{i}$ where $\hat{\sigma}\left[t_{j}\right], 1 \leq j \leq n_{i}$ are already defined.

We defined a binary operation $\circ_{G}$ on $H y p_{G}(\tau)$ by $\sigma_{1} \circ_{G} \sigma_{2}:=\hat{\sigma}_{1} \circ \sigma_{2}$ where $\circ$ denotes the usual composition of mappings and $\sigma_{1}, \sigma_{2} \in H y p_{G}(\tau)$. Let $\sigma_{i d}$ be the hypersubstitution mapping which maps each $n_{i}$-ary operation symbol $f_{i}$ to the term $f_{i}\left(x_{1}, \ldots, x_{n_{i}}\right)$. It turns out that $\left(H y p_{G}(\tau) ; \circ_{G}, \sigma_{i d}\right)$ is a monoid and the monoid $\left(\operatorname{Hyp}(\tau) ;{ }_{G}, \sigma_{i d}\right)$ of all arity preserving hypersubstitutions of type $\tau$ forms a submonoid of $\left(H y p_{G}(\tau) ;{ }_{G}, \sigma_{i d}\right)$.

If $\underline{M}$ is a submonoid of $\operatorname{Hyp}_{G}(\tau)$ then an identity $t \approx t^{\prime}$ is called an $M$-strong hyperidentity if $\hat{\sigma}[t] \approx \hat{\sigma}\left[t^{\prime}\right]$ are identities for every $\sigma \in M$. A variety $V$ is called $M$-strongly solid if every identity in it is an $M$-strong hyperidentity. In case of $M=\operatorname{Hyp}_{G}(\tau)$ we will call a strong hyperidentity and strongly solid respectively.

## 2. $\quad V$-PROPER GENERALIZED HYPERSUBSTITUTIONS AND NORMAL FORMS

In 2007, S. Leeratanavalee and S. Phatchat generalized the concept of $V$ proper hypersubstitutions and normal forms of hypersubstitutions introduced by J. Płonka [5] to $V$-proper generalized hypersubstitutions and normal forms of generalized hypersubstitutions.

Definition 2.1 ([5]). Let $V$ be a variety of type $\tau$. A generalized hypersubstitution $\sigma$ of type $\tau$ is called a $V$-proper generalized hypersubstitution if for every identity $s \approx t$ of $V$, the identity $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ also holds in $V$. We use $P_{G}(V)$ for the set of all $V$-proper generalized hypersubstitutions of type $\tau$.

Proposition 2.2 ([5]). For any variety $V$ of type $\tau$, $\left(P_{G}(V) ;{ }_{G}, \sigma_{i d}\right)$ is a submonoid of $\left(\operatorname{Hyp}_{G}(\tau) ; \circ_{G}, \sigma_{i d}\right)$.

Definition 2.3 ([5]). Let $V$ be a variety of type $\tau$. Two generalized hypersubstitutions $\sigma_{1}$ and $\sigma_{2}$ of type $\tau$ are called a $V$-generalized equivalent if $\sigma_{1}\left(f_{i}\right) \approx \sigma_{2}\left(f_{i}\right)$ are identities in $V$ for all $i \in I$. In this case we write $\sigma_{1} \sim_{V G} \sigma_{2}$.

Theorem 2.4 ([5]). Let $V$ be a variety of algebras of type $\tau$, and let $\sigma_{1}, \sigma_{2} \in$ $H y p_{G}(\tau)$. Then the following statements are equivalent:
(i) $\sigma_{1} \sim_{V G} \sigma_{2}$.
(ii) For all $t \in W_{\tau}(X)$, the equations $\hat{\sigma}_{1}[t] \approx \hat{\sigma}_{2}[t]$ are identities in $V$.
(iii) For all $\underline{A} \in V, \sigma_{1}[\underline{A}]=\sigma_{2}[\underline{A}]$ where $\sigma_{k}[\underline{A}]=\left(A ;\left(\sigma_{k}\left(f_{i}\right)^{A}\right)_{i \in I}\right) ; k=1,2$.

Proposition 2.5 ([5]). Let $V$ be a variety of algebras of type $\tau$. Then the following statements hold:
(i) For all $\sigma_{1}, \sigma_{2} \in H y p_{G}(\tau)$, if $\sigma_{1} \sim_{V G} \sigma_{2}$ then $\sigma_{1}$ is a $V$-proper generalized hypersubstitution iff $\sigma_{2}$ is a $V$-proper generalized hypersubstitution.
(ii) For all $s, t \in W_{\tau}(X)$ and for all $\sigma_{1}, \sigma_{2} \in H y p_{G}(\tau)$, if $\sigma_{1} \sim_{V G} \sigma_{2}$ then $\hat{\sigma}_{1}[s] \approx \hat{\sigma}_{1}[t]$ is an identity in $V$ iff $\hat{\sigma}_{2}[s] \approx \hat{\sigma}_{2}[t]$ is an identity in $V$.

The relation $\sim_{V G}$ is an equivalence relation on $H y p_{G}(\tau)$, but it is not neccessary a congruence relation. We factorize $H y p_{G}(\tau)$ by $\sim_{V G}$ and consider the submonoid $P_{G}(V)$ of $H y p_{G}(\tau)$ is the union of equivalence classes of the relation $\sim_{V G}$. This is also true for a submonoid $\underline{M}$ of $\underline{H y p_{G}(\tau)}$ and the relation $\sim_{V G_{\mid M}}$.

Lemma 2.6 ([5]). Let $\underline{M}$ be a submonoid of $\operatorname{Hyp}_{G}(\tau)$ and let $V$ be a variety of type $\tau$. Then the monoid $P_{G} \cap M$ is the $\overline{\text { union of all equivalence classes }}$ of the restricted relation $\sim \sim_{\left.V\right|_{M}}$.

Definition 2.7 ([5]). Let $\underline{M}$ be a monoid of generalized hypersubstitutions of type $\tau$, and let $V$ be a variety of type $\tau$. Let $\phi$ be a choice function which choosed from $M$ one generalized hypersubstitution from each equivalence class of the relation $\sim_{V G_{\left.\right|_{M}}}$, and let $N_{\phi}^{M}(V)$ be the set of generalized hypersubstitutions which are chosen. Thus $N_{\phi}^{M}(V)$ is a set of distinguished generalized hypersubstitutions from $M$, which we might call $V$-normal form generalized hypersubstitutions. We will say that the variety $V$ is $N_{\phi}^{M}(V)$ strongly solid if for every identity $s \approx t \in I d V$ and for every generalized hypersubstitution $\sigma \in N_{\phi}^{M}(V), \hat{\sigma}[s] \approx \hat{\sigma}[t] \in I d V$.

Theorem 2.8 ([5]). Let $\underline{M}$ be a monoid of generalized hypersubstitutions of type $\tau$ and let $V$ be a variety of type $\tau$. For any choice function $\phi, V$ is $M$-strongly solid if and only if $V$ is $N_{\phi}^{M}(V)$-strongly solid.

## 3. Pre-strongly solid varieties of semigroups

The concept of pre-solid varieties was introduced by K. Denecke and S.L. Wismath [2]. In 2007, S. Leeratanavalee and S. Phatchat generalized the concept of pre-solid varieties to pre-strongly solid varieties [5]. Firstly, we recall the definitions of a pre-generalized hypersubstitution and a pre-strong hyperidentity. Let us fix a type $\tau=(2)$. So we have only one binary operation symbol, say $f$. From now on, the generalized hypersubstitution $\sigma$ which maps $f$ to the term $t$ is denoted by $\sigma_{t}$.

Definition 3.1. A generalized hypersubstitution $\sigma \in \operatorname{Hyp}_{G}(2)$ is called a pre-generalized hypersubstitution if $\sigma \in \operatorname{Hyp}_{G}(2) \backslash\left\{\sigma_{x_{1}}, \sigma_{x_{2}}\right\}$ where $\sigma_{x_{1}}$ and $\sigma_{x_{2}}$ denoted the generalized hypersubstitutions which map $f$ to $x_{1}$ and to $x_{2}$, respectively. We denote the set of all pre-generalized hypersubstitutions of type $\tau=(2)$ by $\operatorname{Pre}_{G}(2)$.

The reason to delete the generalized hypersubstitutions $\sigma_{x_{1}}$ and $\sigma_{x_{2}}$ from $H y p_{G}(2)$ is if we apply the generalized hypersubstitution $\sigma_{x_{1}}$ or $\sigma_{x_{2}}$ on the both sides of the commutative law $x_{1} x_{2} \approx x_{2} x_{1}$ we obtain the equation $x_{1} \approx x_{2}$ which satisfied only in a one-element semigroup.

Definition 3.2. An identity $t \approx t^{\prime}$ is called a pre-strong hyperidentity in a variety $V$ if $\hat{\sigma}[t] \approx \hat{\sigma}\left[t^{\prime}\right] \in I d V$ for all $\sigma \in \operatorname{Pre}_{G}(2)$.

A variety $V$ is called a pre-strongly solid variety if every identity in $V$ is a pre-strong hyperidentity of $V$.

For a class $K$ of algebras of type $\tau$ and for a set $\sum$ of identities of this type we fix the following notations:
$I d K$ - the set of all identies of $K$,
$H I d K$ - the set of all hyperidenties of $K$,
$H_{\text {Pre }}^{G}$ Id $K$ - the set of all pre-strong hyperidenties of $K$,
$\operatorname{Mod} \sum=\left\{\underline{A} \in A l g(\tau) \mid \underline{A}\right.$ satisfies $\left.\sum\right\}$ - the variety defined by $\sum$,
$\operatorname{Hod} \sum=\left\{\underline{A} \in \operatorname{Alg}(\tau) \mid \underline{A}\right.$ hypersatisfies $\left.\sum\right\}$ - the hyperequational class defined by $\sum$,
$H_{\operatorname{Pre}}^{G}$ $\operatorname{Mod} \sum=\left\{\underline{A} \in \operatorname{Alg}(\tau) \mid \underline{A}\right.$ pre-strong hypersatisfies $\left.\sum\right\}$ - the pre-strong hyperequational class defined by $\sum$.

Proposition 3.3 ([5]). $\underline{\operatorname{Pre}}_{G}(2)$ is a submonoid of $\underline{H y p_{G}(2)}$.
Remark 3.4 ([5]). Every strongly solid variety of semigroups is a prestrongly solid variety.

Remark 3.5 ([5]). Every pre-strongly solid variety of semigroups is a presolid variety of semigroups.

Lemma 3.6 ([5]). The variety $Z:=\operatorname{Mod}\left\{x_{1} x_{2} \approx x_{3} x_{4}\right\}$ is the least nontrivial pre-strongly solid variety of semigroups.

Theorem 3.7 ([5]). The greatest non-trivial pre-strongly solid variety of semigroups which is not strongly solid is $Z:=\operatorname{Mod}\left\{x_{1} x_{2} \approx x_{3} x_{4}\right\}$.

Theorem 3.8 ([5]). The variety $V_{b i g}:=\operatorname{Mod}\left\{\left(x_{1} x_{2}\right) x_{3} \approx x_{1}\left(x_{2} x_{3}\right), x_{1}^{2} x_{2} \approx\right.$ $\left.x_{1} x_{2}^{2} \approx x_{1} x_{2}, x_{1} x_{2} x_{3} x_{4} \approx x_{1} x_{3} x_{2} x_{4}\right\}$ is the greatest pre-strongly solid variety of semigroups.

## 4. PRE-STRONGLY SOLID VARIETIES OF COMMUTATIVE SEMIGROUPS

Firstly, we recall the definition of a generalized hypersubstitution of type $\tau$ is a mapping $\sigma:\left\{f_{i} \mid i \in I\right\} \longrightarrow W_{\tau}(X)$ from the set of all $n_{i}$-ary operation symbols into the set of all terms built up by elements of the alphabet
$X:=\left\{x_{1}, x_{2}, \ldots\right\}$ and operation symbols from $\left\{f_{i} \mid i \in I\right\}$ which does not necessarily preserve the arity. We denote the set of all generalized hypersubstitutions of type $\tau$ by $\operatorname{Hyp}_{G}(\tau)$. A generalized superposition of terms $S^{m}: W_{\tau}(X)^{m+1} \longrightarrow W_{\tau}(X)$ is defined by the following steps:
for any term $t \in W_{\tau}(X)$,
(i) if $t=x_{j}, 1 \leq j \leq m$, then

$$
S^{m}\left(x_{j}, t_{1}, \ldots, t_{m}\right):=t_{j}
$$

(ii) if $t=x_{j}, m<j \in \mathbb{N}$, then

$$
S^{m}\left(x_{j}, t_{1}, \ldots, t_{m}\right):=x_{j}
$$

(iii) if $t=f_{i}\left(s_{1}, \ldots, s_{n_{i}}\right)$, then

$$
S^{m}\left(t, t_{1}, \ldots, t_{m}\right):=f_{i}\left(S^{m}\left(s_{1}, t_{1}, \ldots, t_{m}\right), \ldots, S^{m}\left(s_{n_{i}}, t_{1}, \ldots, t_{m}\right)\right)
$$

For every $\sigma \in \operatorname{Hyp}_{G}(\tau)$ induces a mapping $\hat{\sigma}: W_{\tau}(X) \longrightarrow W_{\tau}(X)$ by the following steps:
(i) $\hat{\sigma}[x]:=x \in X$,
(ii) $\hat{\sigma}\left[f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right]:=S^{n_{i}}\left(\sigma\left(f_{i}\right), \hat{\sigma}\left[t_{1}\right], \ldots, \hat{\sigma}\left[t_{n_{i}}\right]\right)$, for any $n_{i}$-ary operation symbol $f_{i}$ where $\hat{\sigma}\left[t_{j}\right], 1 \leq j \leq n_{i}$ are already defined.

In this section, we give an example of pre-strongly solid varieties of commutative semigroups and then determine the least and the greatest pre-strongly solid variety of commuutative semigroups.

Theorem 4.1. The variety $V_{1}:=\operatorname{Mod}\left\{\left(x_{1} x_{2}\right) x_{3} \approx x_{1}\left(x_{2} x_{3}\right), x_{1} x_{2} \approx x_{2} x_{1}\right.$, $\left.x_{1}^{2} x_{2} \approx x_{1} x_{2}^{2} \approx x_{1} x_{2}, x_{1}^{2} \approx x_{2}^{2}\right\}$ is a pre-strongly solid variety of commutative semigroups.

Proof. To show that the variety $V_{1}$ is a pre-strongly solid variety of commutative semigroups, we have to show that every identity satisfied in $V_{1}$ is a pre-strong hyperidentity of $V_{1}$. By using Theorem 2.8 , we can restrict our checking to the following pre-generalized hypersubstitutions $\sigma_{t}$ where $t \in\left\{x_{i} x_{j} \mid i, j \in \mathbb{N}\right\} \cup\left\{x_{i} x_{j} x_{k} \mid i \neq j \neq k\right\} \cup\left\{x_{i_{1}} x_{i_{2}} \ldots x_{i_{k}} \mid k, i_{1}, \ldots, i_{k} \in \mathbb{N}, k>3\right.$, and all of $i_{1}, \ldots, i_{k}$ are distinct $\}$.

If we apply $\sigma_{x_{i} x_{j}} ; i, j \in \mathbb{N}$ on the both sides of the associative law we have the following table.

| $i, j \in \mathbb{N}$ | $\hat{\sigma}_{x_{i} x_{j}}\left[\left(x_{1} x_{2}\right) x_{3}\right]=$ <br> $S^{2}\left(x_{i} x_{j}, S^{2}\left(x_{i} x_{j}, x_{1}, x_{2}\right), x_{3}\right)$ | $\hat{\sigma}_{x_{i} x_{j}}\left[x_{1}\left(x_{2} x_{3}\right)\right]=$ <br> $S^{2}\left(x_{i} x_{j}, x_{1}, S^{2}\left(x_{i} x_{j}, x_{2}, x_{3}\right)\right)$ |
| :---: | :---: | :---: |
| $i=j=1$ | $x_{1} x_{1} x_{1} x_{1}$ | $x_{1} x_{1}$ |
| $i=1, j=2$ | $x_{1} x_{2} x_{3}$ | $x_{1} x_{2} x_{3}$ |
| $i=1, j>2$ | $x_{1} x_{j} x_{j}$ | $x_{1} x_{j}$ |
| $i=j=2$ | $x_{3} x_{3}$ | $x_{3} x_{3} x_{3} x_{3}$ |
| $i=2, j>2$ | $x_{3} x_{j}$ | $x_{3} x_{j} x_{j}$ |
| $i, j>2$ | $x_{i} x_{j}$ | $x_{i} x_{j}$ |

Using the associative law, the commutative law and identities $x_{1}^{2} x_{2} \approx x_{1} x_{2}^{2} \approx x_{1} x_{2}, x_{1}^{2} \approx x_{2}^{2}$ we sides are equal.

If we apply $\sigma_{x_{i} x_{j}} ; i, j \in \mathbb{N}$ on the both sides of the commutative law we have the following tal

| $i, j \in \mathbb{N}$ | $\hat{\sigma}_{x_{i} x_{j}}\left[x_{1} x_{2}\right]=S^{2}\left(x_{i} x_{j}, x_{1}, x_{2}\right)$ | $\hat{\sigma}_{x_{i} x_{j}}\left[x_{2} x_{1}\right]=S^{2}\left(x_{i} x_{j}, x_{2}, x_{1}\right)$ |
| :---: | :---: | :---: |
| $i=j=1$ | $x_{1} x_{1}$ | $x_{2} x_{2}$ |
| $i=1, j=2$ | $x_{1} x_{2}$ | $x_{2} x_{1}$ |
| $i=1, j>2$ | $x_{1} x_{j}$ | $x_{2} x_{j}$ |
| $i=j=2$ | $x_{2} x_{2}$ | $x_{1} x_{1}$ |
| $i=2, j>2$ | $x_{2} x_{j}$ | $x_{2} x_{j}$ |
| $i, j>2$ | $x_{i} x_{j}$ | $x_{i} x_{j}$ |

Using the associative law, the commutative law and identities $x_{1}^{2} x_{2} \approx x_{1} x_{2}^{2} \approx x_{1} x_{2}, x_{1}^{2} \approx x_{2}^{2}$ we sides are equal.

If we apply $\sigma_{x_{i} x_{j}} ; i, j \in \mathbb{N}$ on the both sides of the identity $x_{1}^{2} \approx x_{2}^{2}$ we have the follo

| $i, j \in \mathbb{N}$ | $\hat{\sigma}_{x_{i} x_{j}}\left[x_{1} x_{1}\right]=S^{2}\left(x_{i} x_{j}, x_{1}, x_{1}\right)$ | $\hat{\sigma}_{x_{i} x_{j}}\left[x_{2} x_{2}\right]=S^{2}\left(x_{i} x_{j}\right.$ |
| :---: | :---: | :---: |
| $i=j=1$ | $x_{1} x_{1}$ | $x_{2} x_{2}$ |
| $i=1, j=2$ | $x_{1} x_{1}$ | $x_{2} x_{2}$ |
| $i=1, j>2$ | $x_{1} x_{j}$ | $x_{2} x_{j}$ |
| $i=j=2$ | $x_{1} x_{1}$ | $x_{2} x_{2}$ |
| $i=2, j>2$ | $x_{1} x_{j}$ | $x_{2} x_{j}$ |
| $i, j>2$ | $x_{i} x_{j}$ | $x_{i} x_{j}$ |

Using the associative law, the commutative law and identities $x_{1}^{2} x_{2} \approx x_{1} x_{2}^{2} \approx x_{1} x$ sides are equal.

If we apply $\sigma_{x_{i} x_{j}} ; i, j \in \mathbb{N}$ on the both sides of the identity $x_{1}^{2} x_{2} \approx x_{1} x_{2}^{2} \approx x_{1}$ table.

| $i, j \in \mathbb{N}$ | $\hat{\sigma}_{x_{i} x_{j}}\left[\left(x_{1} x_{1}\right) x_{2}\right]=$ <br> $S^{2}\left(x_{i} x_{j}, S^{2}\left(x_{i} x_{j}, x_{1}, x_{1}\right), x_{2}\right)$ | $\hat{\sigma}_{x_{i} x_{j}}\left[x_{1}\left(x_{2} x_{2}\right)\right]=$ <br> $S^{2}\left(x_{i} x_{j}, x_{1}, S^{2}\left(x_{i} x_{j}, x_{2}, x_{2}\right)\right)$ |
| :---: | :---: | :---: |
| $i=j=1$ | $x_{1} x_{1} x_{1} x_{1}$ | $x_{1} x_{1}$ |
| $i=1, j=2$ | $x_{1} x_{1} x_{2}$ | $x_{1} x_{2} x_{2}$ |
| $i=1, j>2$ | $x_{1} x_{j} x_{j}$ | $x_{1} x_{j}$ |
| $i=j=2$ | $x_{2} x_{2}$ | $x_{2} x_{2} x_{2} x_{2}$ |
| $i=2, j>2$ | $x_{2} x_{j}$ | $x_{2} x_{j} x_{j}$ |
| $i, j>2$ | $x_{i} x_{j}$ | $x_{i} x_{j}$ |

Using the associative law, the commutative law and identities $x_{1}^{2} x_{2} \approx x_{1} x_{2}^{2} \approx x_{1} x_{2}, x_{1}^{2} \approx x_{2}^{2}$ we sides are equal.

If we apply $\sigma_{x_{i} x_{j} x_{k}} ; i \neq j \neq k \in \mathbb{N}$ on the both sides of the commutative law we have the follo

| $i, j, k \in \mathbb{N}$ | $\hat{\sigma}_{x_{i} x_{j} x_{k}}\left[x_{1} x_{2}\right]=S^{2}\left(x_{i} x_{j} x_{k}, x_{1}, x_{2}\right)$ | $\hat{\sigma}_{x_{i} x_{j} x_{k}}\left[x_{2} x_{1}\right]=S^{2}\left(x_{i} x_{j} x_{k}, x_{2}, x\right.$ |
| :---: | :---: | :---: |
| $i=1, j=2, k>2$ | $x_{1} x_{2} x_{k}$ | $x_{2} x_{1} x_{k}$ |
| $i=1, j, k>2$ | $x_{1} x_{j} x_{k}$ | $x_{2} x_{j} x_{k}$ |
| $i=2, j, k>2$ | $x_{2} x_{j} x_{k}$ | $x_{2} x_{j} x_{k}$ |
| $i, j, k>2$ | $x_{i} x_{j} x_{k}$ | $x_{i} x_{j} x_{k}$ |

Using the associative law, the commutative law and identities $x_{1}^{2} x_{2} \approx x_{1} x_{2}^{2} \approx x_{1} x_{2}, x_{1}^{2} \approx x_{2}^{2}$ we sides are equal.
Using the associative law, the commutative law and identities $x_{1}^{2} x_{2} \approx x_{1} x_{2}^{2} \approx x_{1} x_{2}, x_{1}^{2} \approx x_{2}^{2}$ we sides are equal.

If we apply $\sigma_{x_{i} x_{j} x_{k}} ; i \neq j \neq k \in \mathbb{N}$ on the both sides of the associative law we have the followi

| $i, j, k \in \mathbb{N}$ | $\hat{\sigma}_{x_{i} x_{j} x_{k}}\left[\left(x_{1} x_{2}\right) x_{3}\right]=$ | $\hat{\sigma}_{x_{i} x_{j} x_{k}}\left[x_{1}\left(x_{2} x_{3}\right)\right]=$ |
| :---: | :---: | :---: |
|  | $S^{2}\left(x_{i} x_{j} x_{k}, S^{2}\left(x_{i} x_{j} x_{k}, x_{1}, x_{2}\right), x_{3}\right)$ | $S^{2}\left(x_{i} x_{j} x_{k}, x_{1}, S^{2}\left(x_{i} x_{j} x_{k}, x_{2}, x_{3}\right)\right.$ |
| $i=1, j=2, k>2$ | $x_{1} x_{2} x_{k} x_{3} x_{k}$ | $x_{1} x_{2} x_{3} x_{k} x_{k}$ |
| $i=1, j, k>2$ | $x_{1} x_{j} x_{k} x_{j} x_{k}$ | $x_{1} x_{j} x_{k}$ |
| $i=2, j, k>2$ | $x_{3} x_{j} x_{k}$ | $x_{3} x_{j} x_{k} x_{j} x_{k}$ |
| $i, j, k>2$ | $x_{i} x_{j} x_{k}$ | $x_{i} x_{j} x_{k}$ |

If we apply $\sigma_{x_{i} x_{j} x_{k}} ; i \neq j \neq k \in \mathbb{N}$ on the both sides of the identity $x_{1}^{2} \approx x_{2}^{2}$ we have

| $i, j, k \in \mathbb{N}$ | $\hat{\sigma}_{x_{i} x_{j} x_{k}}\left[x_{1} x_{1}\right]=S^{2}\left(x_{i} x_{j} x_{k}, x_{1}, x_{1}\right)$ | $\hat{\sigma}_{x_{i} x_{j} x_{k}}\left[x_{2} x_{2}\right]=S^{2}$ |
| :---: | :---: | :---: |
| $i=1, j=2, k>2$ | $x_{1} x_{2} x_{k}$ | $x_{2} x_{1} 2$ |
| $i=1, j, k>2$ | $x_{1} x_{j} x_{k}$ | $x_{2} x_{j} 2$ |
| $i=2, j, k>2$ | $x_{2} x_{j} x_{k}$ | $x_{1} x_{j} 2$ |
| $i, j, k>2$ | $x_{i} x_{j} x_{k}$ | $x_{i} x_{j} x$ |

Using the associative law, the commutative law and identities $x_{1}^{2} x_{2} \approx x_{1} x_{2}^{2} \approx x_{1} x$ sides are equal.

If we apply $\sigma_{x_{i} x_{j} x_{k}} ; i \neq j \neq k \in \mathbb{N}$ on the both sides of the identity $x_{1}^{2} x_{2} \approx x$ following table.

| $i, j, k \in \mathbb{N}$ | $\hat{\sigma}_{x_{i} x_{j} x_{k}}\left[\left(x_{1} x_{1}\right) x_{2}\right]=$ | $\hat{\sigma}_{x_{i} x_{j} x_{k}}\left[x_{1}\left(x_{2} x_{2}\right)\right]=$ | $\hat{\sigma}_{x_{i} x}$, |
| :---: | :---: | :---: | :---: |
|  | $S^{2}\left(x_{i} x_{j} x_{k}\right.$, | $S^{2}\left(x_{i} x_{j} x_{k}, x_{1}\right.$, | $S^{2}\left(x_{i}\right.$ |
|  | $\left.S^{2}\left(x_{i} x_{j} x_{k}, x_{1}, x_{1}\right), x_{2}\right)$ | $\left.S^{2}\left(x_{i} x_{j} x_{k}, x_{2}, x_{2}\right)\right)$ |  |
| $i=1, j=2, k>2$ | $x_{1} x_{1} x_{k} x_{2} x_{k}$ | $x_{1} x_{2} x_{2} x_{k} x_{k}$ |  |
| $i=1, j, k>2$ | $x_{1} x_{j} x_{k} x_{j} x_{k}$ | $x_{1} x_{j} x_{k}$ |  |
| $i=2, j, k>2$ | $x_{2} x_{j} x_{k}$ | $x_{2} x_{j} x_{k} x_{j} x_{k}$ |  |
| $i, j, k>2$ | $x_{i} x_{j} x_{k}$ | $x_{i} x_{j} x_{k}$ |  |

Using the associative law, the commutative law and identities $x_{1}^{2} x_{2} \approx x_{1} x_{2}^{2} \approx x_{1} x$ sides are equal.

If we apply $\sigma_{t}$ where $t=x_{i_{1}} x_{i_{2}} \ldots x_{i_{k}}$ and $k, i_{1}, \ldots, i_{k} \in \mathbb{N}, k>3$ on the both sides of the associative law we have $\hat{\sigma}_{t}\left[\left(x_{1} x_{2}\right) x_{3}\right]=S^{2}\left(t, S^{2}\left(t, x_{1}, x_{2}\right), x_{3}\right)$ and $\hat{\sigma}_{t}\left[x_{1}\left(x_{2} x_{3}\right)\right]=S^{2}\left(t, x_{1}, S^{2}\left(t, x_{2}, x_{3}\right)\right)$.
(i) If there exists a unique $n \in\{1, \ldots, k\}$ such that $i_{n}=1$ and $i_{m}>2$ for all $m \neq n$, then

$$
\begin{aligned}
\hat{\sigma}_{t}\left[\left(x_{1} x_{2}\right) x_{3}\right] & =x_{i_{1}} \ldots x_{i_{n-1}} x_{i_{1}} \ldots x_{i_{n-1}} x_{1} x_{i_{n+1}} \ldots x_{i_{k}} x_{i_{n+1}} \ldots x_{i_{k}} \\
\hat{\sigma}_{t}\left[x_{1}\left(x_{2} x_{3}\right)\right] & =x_{i_{1}} \ldots x_{i_{n-1}} x_{1} x_{i_{n+1}} \ldots x_{i_{k}}
\end{aligned}
$$

(ii) If there exists a unique $n \in\{1, \ldots, k\}$ such that $i_{n}=2$ and $i_{m}>2$ for all $m \neq n$, then

$$
\begin{aligned}
\hat{\sigma}_{t}\left[\left(x_{1} x_{2}\right) x_{3}\right] & =x_{i_{1}} \ldots x_{i_{n-1}} x_{3} x_{i_{n+1}} \ldots x_{i_{k}} \\
\hat{\sigma}_{t}\left[x_{1}\left(x_{2} x_{3}\right)\right] & =x_{i_{1}} \ldots x_{i_{n-1}} x_{i_{1}} \ldots x_{i_{n-1}} x_{3} x_{i_{n+1}} \ldots x_{i_{k}} x_{i_{n+1}} \ldots x_{i_{k}}
\end{aligned}
$$

(iii) If there exists a unique $n \in\{1, \ldots, k\}$ such that $i_{n}=1$ and there exists a unique $l \in\{1, \ldots, k\}$ such that $i_{l}=2, i_{m}>2$ for all $m \neq n \neq l$ and $n<l$, then

$$
\begin{aligned}
& \hat{\sigma}_{t}\left[\left(x_{1} x_{2}\right) x_{3}\right] \\
& =x_{i_{1}} \ldots x_{i_{n-1}} x_{i_{1} \ldots} \ldots x_{i_{n-1}} x_{1} x_{i_{n+1}} \ldots x_{i_{l-1}} x_{2} x_{i_{l+1} \ldots} x_{i_{k}} x_{i_{n+1}} \ldots x_{i_{l-1}} x_{3} x_{i_{l+1}} \ldots x_{i_{k}} \\
& \quad \hat{\sigma}_{t}\left[x_{1}\left(x_{2} x_{3}\right)\right] \\
& \quad=x_{i_{1} \ldots} x_{i_{n-1}} x_{1} x_{i_{l-1}} x_{i_{1} \ldots x_{i_{n-1}}} x_{2} x_{i_{n+1} \ldots} x_{i_{l-1}} x_{3} x_{i_{l+1} \ldots x_{i_{k}}} x_{i_{l+1} \ldots x_{i_{k}}}
\end{aligned}
$$

(iv) If $i_{m}>2$ for all $m \in\{1,2, \ldots, k\}$, then

$$
\begin{aligned}
& \hat{\sigma}_{t}\left[\left(x_{1} x_{2}\right) x_{3}\right]=x_{i_{1}} \ldots x_{i_{k}} \\
& \hat{\sigma}_{t}\left[x_{1}\left(x_{2} x_{3}\right)\right]=x_{i_{1}} \ldots x_{i_{k}}
\end{aligned}
$$

Using the associative law, the commutative law and identities $x_{1}^{2} x_{2} \approx x_{1} x_{2}^{2} \approx$ $x_{1} x_{2}, x_{1}^{2} \approx x_{2}^{2}$ we have both sides are equal.

If we apply $\sigma_{t}$ where $t=x_{i_{1}} x_{i_{2}} \ldots x_{i_{k}}$ and $k, i_{1}, \ldots, i_{k} \in \mathbb{N}, k>3$ on the both sides of the commutative law we have $\hat{\sigma}_{t}\left[x_{1} x_{2}\right]=S^{2}\left(t, x_{1}, x_{2}\right)$ and $\hat{\sigma}_{t}\left[x_{2} x_{1}\right]=S^{2}\left(t, x_{2}, x_{1}\right)$.
(i) If there exists a unique $n \in\{1, \ldots, k\}$ such that $i_{n}=1$ and $i_{m}>2$ for all $m \neq n$, then

$$
\begin{aligned}
& \hat{\sigma}_{t}\left[x_{1} x_{2}\right]=x_{i_{1}} \ldots x_{i_{n-1}} x_{1} x_{i_{n+1}} \ldots x_{i_{k}} \\
& \hat{\sigma}_{t}\left[x_{2} x_{1}\right]=x_{i_{1}} \ldots x_{i_{n-1}} x_{2} x_{i_{n+1}} \ldots x_{i_{k}}
\end{aligned}
$$

(ii) If there exists a unique $n \in\{1, \ldots, k\}$ such that $i_{n}=2$ and $i_{m}>2$ for all $m \neq n$, then

$$
\begin{aligned}
& \hat{\sigma}_{t}\left[x_{1} x_{2}\right]=x_{i_{1}} \ldots x_{i_{n-1}} x_{2} x_{i_{n+1}} \ldots x_{i_{k}} . \\
& \hat{\sigma}_{t}\left[x_{2} x_{1}\right]=x_{i_{1}} \ldots x_{i_{n-1}} x_{1} x_{i_{n+1}} \ldots x_{i_{k}}
\end{aligned}
$$

(iii) If there exists a unique $n \in\{1, \ldots, k\}$ such that $i_{n}=1$ and there exists a unique $l \in\{1, \ldots, k\}$ such that $i_{l}=2, i_{m}>2$ for all $m \neq n \neq l$ and $n<l$, then

$$
\begin{aligned}
& \hat{\sigma}_{t}\left[x_{1} x_{2}\right]=x_{i_{1}} \ldots x_{i_{n-1}} x_{1} x_{i_{n+1}} \ldots x_{i_{l-1}} x_{2} x_{i_{l+1} \ldots} \ldots x_{i_{k}} \\
& \hat{\sigma}_{t}\left[x_{2} x_{1}\right]=x_{i_{1}} \ldots x_{i_{n-1}} x_{2} x_{i_{n+1}} \ldots x_{i_{l-1}} x_{1} x_{i_{l+1} \ldots} \ldots x_{i_{k}}
\end{aligned}
$$

(iv) If $i_{m}>2$ for all $m \in\{1,2, \ldots, k\}$, then

$$
\begin{aligned}
& \hat{\sigma}_{t}\left[x_{1} x_{2}\right]=x_{i_{1}} \ldots x_{i_{k}} \\
& \hat{\sigma}_{t}\left[x_{2} x_{1}\right]=x_{i_{1}} \ldots x_{i_{k}}
\end{aligned}
$$

Using the associative law, the commutative law and identities $x_{1}^{2} x_{2} \approx x_{1} x_{2}^{2} \approx$ $x_{1} x_{2}, x_{1}^{2} \approx x_{2}^{2}$ we have both sides are equal.

If we apply $\sigma_{t}$ where $t=x_{i_{1}} x_{i_{2}} \ldots x_{i_{k}}$ and $k, i_{1}, \ldots, i_{k} \in \mathbb{N}, k>3$ on the both sides of the identity $x_{1}^{2} \approx x_{2}^{2}$ we have $\hat{\sigma}_{t}\left[x_{1} x_{1}\right]=S^{2}\left(t, x_{1}, x_{1}\right)$ and $\hat{\sigma}_{t}\left[x_{2} x_{2}\right]=S^{2}\left(t, x_{2}, x_{2}\right)$.
(i) If there exists a unique $n \in\{1, \ldots, k\}$ such that $i_{n}=1$ and $i_{m}>2$ for all $m \neq n$, then

$$
\begin{aligned}
& \hat{\sigma}_{t}\left[x_{1} x_{1}\right]=x_{i_{1}} \ldots x_{i_{n-1}} x_{1} x_{i_{n+1}} \ldots x_{i_{k}} \\
& \hat{\sigma}_{t}\left[x_{2} x_{2}\right]=x_{i_{1}} \ldots x_{i_{n-1}} x_{2} x_{i_{n+1}} \ldots x_{i_{k}}
\end{aligned}
$$

(ii) If there exists a unique $n \in\{1, \ldots, k\}$ such that $i_{n}=2$ and $i_{m}>2$ for all $m \neq n$, then

$$
\begin{aligned}
& \hat{\sigma}_{t}\left[x_{1} x_{1}\right]=x_{i_{1}} \ldots x_{i_{n-1}} x_{1} x_{i_{n+1}} \ldots x_{i_{k}} . \\
& \hat{\sigma}_{t}\left[x_{2} x_{2}\right]=x_{i_{1}} \ldots x_{i_{n-1}} x_{2} x_{i_{n+1}} \ldots x_{i_{k}} .
\end{aligned}
$$

(iii) If there exists a unique $n \in\{1, \ldots, k\}$ such that $i_{n}=1$ and there exists a unique $l \in\{1, \ldots, k\}$ such that $i_{l}=2, i_{m}>2$ for all $m \neq n \neq l$ and $n<l$, then

$$
\begin{aligned}
& \hat{\sigma}_{t}\left[x_{1} x_{1}\right]=x_{i_{1}} \ldots x_{i_{n-1}} x_{1} x_{i_{n+1}} \ldots x_{i_{l-1}} x_{1} x_{i_{l+1} \ldots x_{i_{k}}} \\
& \hat{\sigma}_{t}\left[x_{2} x_{2}\right]=x_{i_{1}} \ldots x_{i_{n-1}} x_{2} x_{i_{n+1}} \ldots x_{i_{l-1}} x_{2} x_{i_{l+1}} \ldots x_{i_{k}}
\end{aligned}
$$

(iv) If $i_{m}>2$ for all $m \in\{1,2, \ldots, k\}$, then

$$
\begin{aligned}
& \hat{\sigma}_{t}\left[x_{1} x_{1}\right]=x_{i_{1}} \ldots x_{i_{k}} \\
& \hat{\sigma}_{t}\left[x_{2} x_{2}\right]=x_{i_{1}} \ldots x_{i_{k}}
\end{aligned}
$$

Using the associative law, the commutative law and identities $x_{1}^{2} x_{2} \approx x_{1} x_{2}^{2} \approx$ $x_{1} x_{2}, x_{1}^{2} \approx x_{2}^{2}$ we have both sides are equal.

If we apply $\sigma_{t}$ where $t=x_{i_{1}} x_{i_{2}} \ldots x_{i_{k}}$ and $k, i_{1}, \ldots, i_{k} \in \mathbb{N}, k>3$ on the both sides of the identity $x_{1}^{2} x_{2} \approx x_{1} x_{2}^{2} \approx x_{1} x_{2}$ we have $\hat{\sigma}_{t}\left[\left(x_{1} x_{1}\right) x_{2}\right]=$ $S^{2}\left(t, S^{2}\left(t, x_{1}, x_{1}\right), x_{2}\right)$ and $\hat{\sigma}_{t}\left[x_{1}\left(x_{2} x_{2}\right)\right]=S^{2}\left(t, x_{1}, S^{2}\left(t, x_{1}, x_{2}\right)\right)$ and $\hat{\sigma}_{t}\left[x_{1} x_{2}\right]$ $=S^{2}\left(t, x_{1}, x_{2}\right)$.
(i) If there exists a unique $n \in\{1, \ldots, k\}$ such that $i_{n}=1$ and $i_{m}>2$ for all $m \neq n$, then

$$
\begin{aligned}
& \hat{\sigma}_{t}\left[\left(x_{1} x_{1}\right) x_{2}\right]=x_{i_{1}} \ldots x_{i_{n-1}} x_{i_{1}} \ldots x_{i_{n-1}} x_{1} x_{i_{n+1}} \ldots x_{i_{k}} x_{i_{n+1}} \ldots x_{i_{k}} . \\
& \hat{\sigma}_{t}\left[x_{1}\left(x_{2} x_{2}\right)\right]=x_{i_{1} \ldots x_{i_{n-1}}} x_{1} x_{i_{n+1}} \ldots x_{i_{k}} . \\
& \hat{\sigma}_{t}\left[x_{1} x_{2}\right]=x_{i_{1}} \ldots x_{i_{n-1}} x_{1} x_{i_{n+1}} \ldots x_{i_{k}} .
\end{aligned}
$$

(ii) If there exists a unique $n \in\{1, \ldots, k\}$ such that $i_{n}=2$ and $i_{m}>2$ for all $m \neq n$, then

$$
\begin{aligned}
& \hat{\sigma}_{t}\left[\left(x_{1} x_{1}\right) x_{2}\right]=x_{i_{1}} \ldots x_{i_{n-1}} x_{2} x_{i_{n+1}} \ldots x_{i_{k}} . \\
& \hat{\sigma}_{t}\left[x_{1}\left(x_{2} x_{2}\right)\right]=x_{i_{1}} \ldots x_{i_{n-1}} x_{i_{1}} \ldots x_{i_{n-1}} x_{2} x_{i_{n+1}} \ldots x_{i_{k}} x_{i_{n+1}} \ldots x_{i_{k}} . \\
& \hat{\sigma}_{t}\left[x_{1} x_{2}\right]=x_{i_{1}} \ldots x_{i_{n-1}} x_{2} x_{i_{n+1}} \ldots x_{i_{k}} .
\end{aligned}
$$

(iii) If there exists a unique $n \in\{1, \ldots, k\}$ such that $i_{n}=1$ and there exists a unique $l \in\{1, \ldots, k\}$ such that $i_{l}=2, i_{m}>2$ for all $m \neq n \neq l$ and $n<l$, then

$$
\begin{aligned}
& \hat{\sigma}_{t}\left[\left(x_{1} x_{1}\right) x_{2}\right] \\
& =x_{i_{1} \ldots x_{i_{n-1}} x_{i_{1}} \ldots x_{i_{n-1}} x_{1} x_{i_{n+1}} \ldots x_{i_{l-1}} x_{1} x_{i_{l+1}} \ldots x_{i_{k}} x_{i_{n+1} \ldots} \ldots x_{i_{l-1}} x_{2} x_{i_{l+1}} \ldots x_{i_{k}} .} \quad \hat{\sigma}_{t}\left[x_{1}\left(x_{2} x_{2}\right)\right] \\
& \quad=x_{i_{1} \ldots x_{i_{n-1}}} x_{1} x_{i_{l-1}} x_{i_{1} \ldots x_{i_{n-1}}} x_{2} x_{i_{n+1} \ldots x_{i_{l-1}}} x_{2} x_{i_{l+1} \ldots} \ldots x_{i_{k}} x_{i_{l+1}} \ldots x_{i_{k}} . \\
& \quad \hat{\sigma}_{t}\left[x_{1} x_{2}\right]=x_{i_{1} \ldots x_{i_{n-1}}} x_{1} x_{i_{n+1}} \ldots x_{i_{l-1}} x_{2} x_{i_{l+1}} \ldots x_{i_{k}} .
\end{aligned}
$$

(iv) If $i_{m}>2$ for all $m \in\{1,2, \ldots, k\}$, then

$$
\begin{aligned}
& \hat{\sigma}_{t}\left[\left(x_{1} x_{1}\right) x_{2}\right]=x_{i_{1}} \ldots x_{i_{k}} \\
& \hat{\sigma}_{t}\left[x_{1}\left(x_{2} x_{2}\right)\right]=x_{i_{1}} \ldots x_{i_{k}} \\
& \hat{\sigma}_{t}\left[x_{1} x_{2}\right]=x_{i_{1}} \ldots x_{i_{k}}
\end{aligned}
$$

Using the associative law, the commutative law and identities $x_{1}^{2} x_{2} \approx x_{1} x_{2}^{2} \approx$ $x_{1} x_{2}, x_{1}^{2} \approx x_{2}^{2}$ we have both sides are equal.

Theorem 4.2. The variety $Z:=\operatorname{Mod}\left\{x_{1} x_{2} \approx x_{3} x_{4}\right\}$ is the least prestrongly solid variety of commutative semigroups.

Theorem 4.3. The variety $V_{2}:=\operatorname{Mod}\left\{\left(x_{1} x_{2}\right) x_{3} \approx x_{1}\left(x_{2} x_{3}\right), x_{1} x_{2} \approx x_{2} x_{1}\right.$, $\left.x_{1} x_{2} x_{3} \approx x_{1} x_{3}\right\}$ is the greatest pre-strongly solid variety of commutative semigroups.

Proof. The greatest pre-strongly solid variety of commutative semigroups is the class of all commutative semigroups for which the associative law and the commutative law are satisfied as pre-strong hyperidentities, i.e the class $H_{\text {Pre }_{G}} \operatorname{Mod}\left\{\left(x_{1} x_{2}\right) x_{3} \approx x_{1}\left(x_{2} x_{3}\right), x_{1} x_{2} \approx x_{2} x_{1}\right\}$. Applying $\sigma_{x_{1} x_{2}}, \sigma_{x_{1} x_{i}}, \sigma_{x_{i} x_{1}}$ $(i>2) \in \operatorname{Pre}_{G}$ on the associative law, $\sigma_{x_{1} x_{2}}$ gives $\left(x_{1} x_{2}\right) x_{3} \approx x_{1}\left(x_{2} x_{3}\right)$, $\sigma_{x_{1} x_{i}}$ gives $x_{1} x_{i}^{2} \approx x_{1} x_{i}, \sigma_{x_{i} x_{1}}$ gives $x_{i}^{2} x \approx x_{i} x$. If we substitute for $x_{i}$ a new variable $x_{2}$, then we have the identities $x_{1} x_{2}^{2} \approx x_{1} x_{2}, x_{2}^{2} x_{1} \approx x_{2} x_{1}$. That means $x_{1}^{2} x_{2} \approx x_{1} x_{2}^{2} \approx x_{1} x_{2} \in \operatorname{Id}\left(H_{\text {Pre }}^{G}\right.$ $\operatorname{Mod}\left\{\left(x_{1} x_{2}\right) x_{3} \approx x_{1}\left(x_{2} x_{3}\right), x_{1} x_{2} \approx\right.$ $\left.\left.x_{2} x_{1}\right\}\right)$. Applying $\sigma_{x_{1} x_{2}}, \sigma_{x_{1} x_{i}}(i>2)$ on the commutative law, $\sigma_{x_{1} x_{2}}$ gives $x_{1} x_{2} \approx x_{2} x_{1}, \sigma_{x_{1} x_{i}}$ gives $x_{i} x_{1} \approx x_{i} x_{2}$. Then $x_{i} x_{1} x_{2} \approx x_{i} x_{2} x_{2} \approx x_{i} x_{2}$, so $x_{i} x_{1} x_{2} \approx x_{i} x_{2}$. If we substitute $x_{i}$ by $x_{1}, x_{1}$ by $x_{2}$ and $x_{2}$ by $x_{3}$. Then we have $x_{1} x_{2} x_{3} \approx x_{1} x_{3}$. Thus $H_{\text {Pre }}^{G}$ $\operatorname{Mod}\left\{\left(x_{1} x_{2}\right) x_{3} \approx x_{1}\left(x_{2} x_{3}\right), x_{1} x_{2} \approx x_{2} x_{1}\right\}$ satisfies all identities of $V_{2}$, i.e $H_{\text {Pre }}^{G}$ $\operatorname{Mod}\left\{\left(x_{1} x_{2}\right) x_{3} \approx x_{1}\left(x_{2} x_{3}\right), x_{1} x_{2} \approx\right.$ $\left.x_{2} x_{1}\right\} \subseteq V_{2}$. To prove the converse inclusion we have to check the associative law , the commutative law and the rectangular law, i.e. $x_{1} x_{2} x_{3} \approx x_{1} x_{3}$ using all pre-generalized hypersubstitutions. We can restrict our checking to the following pre-generalized hypersubstitutions $\sigma_{x_{i} x_{j}}(i, j \in \mathbb{N})$.

If we apply $\sigma_{x_{i} x_{j}} ; i, j \in \mathbb{N}$ on the both sides of the associative law we have the following table.

| $i, j \in \mathbb{N}$ | $\hat{\sigma}_{x_{i} x_{j}}\left[\left(x_{1} x_{2}\right) x_{3}\right]=$ | $\hat{\sigma}_{x_{i} x_{j}}\left[x_{1}\left(x_{2} x_{3}\right)\right]=$ |
| :---: | :---: | :---: |
|  | $S^{2}\left(x_{i} x_{j}, S^{2}\left(x_{i} x_{j}, x_{1}, x_{2}\right), x_{3}\right)$ | $S^{2}\left(x_{i} x_{j}, x_{1}, S^{2}\left(x_{i} x_{j}, x_{2}, x_{3}\right)\right)$ |
| $i=j=1$ | $x_{1} x_{1} x_{1} x_{1}$ | $x_{1} x_{1}$ |
| $i=1, j=2$ | $x_{1} x_{2} x_{3}$ | $x_{1} x_{2} x_{3}$ |
| $i=j=2$ | $x_{3} x_{3}$ | $x_{3} x_{3} x_{3} x_{3}$ |
| $i=1, j>2$ | $x_{1} x_{j} x_{j}$ | $x_{1} x_{j}$ |
| $i=2, j>2$ | $x_{3} x_{j}$ | $x_{3} x_{j} x_{j}$ |
| $i, j>2$ | $x_{i} x_{j}$ | $x_{i} x_{j}$ |

Using the associative law, the commutative law and the identity $x_{1} x_{2} x_{3} \approx$ $x_{1} x_{3}$ we have both sides are equal.

If we apply $\sigma_{x_{i} x_{j}} ; i, j \in \mathbb{N}$ on the both sides of the commutative law we have the following table.

| $i, j \in \mathbb{N}$ | $\hat{\sigma}_{x_{i} x_{j}}\left[x_{1} x_{2}\right]=S^{2}\left(x_{i} x_{j}, x_{1}, x_{2}\right)$ | $\hat{\sigma}_{x_{i} x_{j}}\left[x_{2} x_{1}\right]=S^{2}\left(x_{i} x_{j}, x_{2}, x_{1}\right)$ |
| :---: | :---: | :---: |
| $i=j=1$ | $x_{1} x_{1}$ | $x_{2} x_{2}$ |
| $i=1, j=2$ | $x_{1} x_{2}$ | $x_{2} x_{1}$ |
| $i=j=2$ | $x_{2} x_{2}$ | $x_{1} x_{1}$ |
| $i=1, j>2$ | $x_{1} x_{j}$ | $x_{2} x_{j}$ |
| $i=2, j>2$ | $x_{2} x_{j}$ | $x_{1} x_{j}$ |
| $i, j>2$ | $x_{i} x_{j}$ | $x_{i} x_{j}$ |

Using the associative law, the commutative law and the identity $x_{1} x_{2} x_{3} \approx$ $x_{1} x_{3}$ we have both sides are equal.

If we apply $\sigma_{x_{i} x_{j}} ; i, j \in \mathbb{N}$ on the both sides of the identity $x_{1} x_{2} x_{3} \approx x_{1} x_{3}$ we have the following

| $i, j \in \mathbb{N}$ | $\hat{\sigma}_{x_{i} x_{j}}\left[\left(x_{1} x_{2}\right) x_{3}\right]=S^{2}\left(x_{i} x_{j}, S^{2}\left(x_{i} x_{j}, x_{1}, x_{2}\right), x_{3}\right)$ | $\hat{\sigma}_{x_{i} x_{j}}\left[x_{1} x_{3}\right]=S^{2}\left(x_{i} x_{j}, x_{1}, x^{2}\right.$ |
| :---: | :---: | :---: |
| $i=j=1$ | $x_{1} x_{1} x_{1} x_{1}$ | $x_{1} x_{1}$ |
| $i=1, j=2$ | $x_{1} x_{2} x_{3}$ | $x_{1} x_{3}$ |
| $i=j=2$ | $x_{3} x_{3}$ | $x_{3} x_{3}$ |
| $i=1, j>2$ | $x_{1} x_{j} x_{j}$ | $x_{1} x_{j}$ |
| $i=2, j>2$ | $x_{3} x_{j}$ | $x_{3} x_{j}$ |
| $i, j>2$ | $x_{i} x_{j}$ | $x_{i} x_{j}$ |

Using the associative law, the commutative law and the identity $x_{1} x_{2} x_{3} \approx x_{1} x_{3}$ we have both side

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