

BI-IDEALS IN k -REGULAR AND INTRA k -REGULAR SEMIRINGS

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Abstract

Here we introduce the k -bi-ideals in semirings and the intra k -regular semirings. An intra k -regular semiring S is a semiring whose additive reduct is a semilattice and for each $a \in S$ there exists $x \in S$ such that $a + xa^2x = xa^2x$. Also it is a semiring in which every k -ideal is semiprime. Our aim in this article is to characterize both the k -regular semirings and intra k -regular semirings using of k -bi-ideals.

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1. INTRODUCTION

In 1952, R.A. Good and D.R. Hughes [4] first defined the notion of bi-ideals of a semigroup. It is also a special case of the (m,n) -ideal introduced by S. Lajos [11]. Lajos characterized both regular and intra regular semigroups by bi-ideals [12] and by generalized bi-ideals [13]. Different classes of semigroups has been characterized using bi-ideals by many authors in [3, 6, 7, 8, 9, 14, 15, 16, 21].

In this article we introduce the notion of k -bi-ideals in a semiring and characterize the k -regular semirings by k -bi-ideals. Bourne [2] introduced the k -regular semirings as a generalization of regular rings. Later these semirings has been studied by Sen, Weinert, Bhuniya, Adhikari [1, 18, 19, 20]. For any semigroup F , the set $P_f(F)$ of all finite subsets of F is a semiring whose additive reduct is a semilattice, where addition and multiplication are defined by the set union and usual product of subsets of a semigroup respectively. The semiring $P_f(F)$ is a k -regular semiring if and only if F is regular [20]. Here we show that B is a k -bi-ideal of the semiring $P_f(F)$ if and only if $B = P_f(A)$ for some bi-ideal A of F . Thus it is of interest to characterize the k -regular semirings using k -bi-ideals.

Some elementary results together with prerequisites have been discussed in Section 2.

Section 3 is devoted to characterize the k -regular semirings by k -bi-ideals.

In section 4, we introduce intra k -regular semirings and characterize these semirings by k -bi-ideals. The semiring $P_f(F)$ is intra k -regular if and only if F is an intra regular semigroup. Also a semiring S is intra k -regular if and only if every k -ideal of S is semiprime. Several equivalent characterizations for the semirings which are both k -regular and intra k -regular has been given here in terms of k -bi-ideals.

2. PRELIMINARIES

A semiring $(S, +, \cdot)$ is an algebra with two binary operations $+$ and \cdot such that both the *additive reduct* $(S, +)$ and the *multiplicative reduct* (S, \cdot) are

semigroups and such that the following distributive laws hold:

$$x(y + z) = xy + xz \text{ and } (x + y)z = xz + yz.$$

A band is a semigroup in which every element is an idempotent. A commutative band is called a semilattice. Throughout this paper, unless otherwise stated, S is always a semiring whose additive reduct is a semilattice.

A nonempty subset L of a semiring S is called a *left ideal* of S if $L + L \subseteq L$ and $SL \subseteq L$. The *right ideals* are defined dually. A subset I of S is called an *ideal* of S if it is both a left and a right ideal of S . A nonempty subset A is called an *interior ideal* of S if $A + A \subseteq A$ and $SAS \subseteq A$. Henriksen [5] defined an *ideal (left, right) I* of a semiring S to be a k -ideal (left, right) if for $a, x \in S$,

$$a, a + x \in I \Rightarrow x \in I.$$

We define interior k -ideal similarly.

Later on the notion of k -subset of a semiring evolved. A nonempty subset A of S is called a k -subset of S if for $x \in S$,

$$a \in A, x + a \in A \text{ implies that } x \in A.$$

Let A be a non empty subset of S . Since the intersection of any family of k -subsets of S is a k -subset (provided the intersection is nonempty), the smallest k -subset of S which contains A exists. This smallest subset of S can be thought as the k -subset generated by A . This k -subset of S is called the k -closure of A in S and will be denoted by \overline{A} . For a nonempty subset A of S ,

$$\overline{A} = \{x \in S \mid \exists a, b \in A \text{ such that } x + a = b\}.$$

If A and B be two subsets of S such that $A \subseteq B$ then it follows that $\overline{A} \subseteq \overline{B}$. A nonempty subset A of S is a k -subset of S if and only if $\overline{A} = A$. Thus an ideal(left, right) K of S is a k -ideal(left, right) if and only if $\overline{K} = K$.

Definition 2.1. A subsemiring A is called a k -bi-ideal of S if $ASA \subseteq A$ and $\overline{A} = A$.

A nonempty subset A is called a generalized k -bi-ideal of S if $A + A \subseteq A$, $ASA \subseteq A$ and $\overline{A} = A$.

Example 2.2. we consider the distributive lattice (\mathbb{N}, \leq) where \mathbb{N} is the set of all natural numbers and \leq is the natural partial order on \mathbb{N} . Then $S = M_2(\mathbb{N})$, the set of all 2×2 matrices over \mathbb{N} is a semiring under the addition and multiplication defined in the following way:

For

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \in S,$$

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} + \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} = \begin{pmatrix} x_1 \vee y_1 & x_2 \vee y_2 \\ x_3 \vee y_3 & x_4 \vee y_4 \end{pmatrix}$$

and

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} = \begin{pmatrix} (x_1 \wedge y_1) \vee (x_2 \wedge y_3) & (x_1 \wedge y_2) \vee (x_2 \wedge y_4) \\ (x_3 \wedge y_1) \vee (x_4 \wedge y_3) & (x_3 \wedge y_2) \vee (x_4 \wedge y_4) \end{pmatrix}.$$

Let $B = \left\{ \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in M_2(\mathbb{N}) \mid x, y \leq a, z \leq b, t \leq c, \text{ where } a, b, c \in \mathbb{N} \text{ such that } a \leq b \leq c \right\}$.

Then for all $X, Y \in B$ we have $X + Y, XY \in B$. Also $BSB \subseteq B$. Thus B is a bi-ideal of S .

Let $\begin{pmatrix} u & v \\ w & z \end{pmatrix} \in B$ and $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in S$ be such that

$$\begin{pmatrix} u & v \\ w & z \end{pmatrix} + \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in B.$$

Then we have

$$\begin{aligned} x_1 \vee u &\leq a, \\ x_2 \vee v &\leq a, \\ x_3 \vee w &\leq b, \\ \text{and } x_4 \vee z &\leq c \end{aligned}$$

and so $x_1, x_2 \leq a, x_3 \leq b$ and $x_4 \leq c$, whence $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in B$. Thus B is a k -bi-ideal of S .

Let F be a semigroup and $P_f(F)$ be the set of all finite subsets of F . Define addition and multiplication on $P_f(F)$ by:

$$U + V = U \cup V$$

$$\text{and } UV = \{ab \mid a \in U, b \in V\}, \text{ for all } U, V \in P_f(F).$$

Then $(P_f(F), +, \cdot)$ is a semiring whose additive reduct is a semilattice. In the following theorem we investigate the relation between the k -bi-ideals of the semiring $P_f(F)$ and the bi-ideals of the semigroup F .

Theorem 2.3. *Let F be a semigroup. Then B is a k -bi-ideal of $P_f(F)$ if and only if $B = P_f(A)$ for some bi-ideal A of F .*

Proof. Let A be a bi-ideal of F and $B = P_f(A)$. Then B is a bi-ideal of $P_f(F)$. Now let $U \in P_f(F)$ and $V_1, V_2 \in B$ such that $U + V_1 = V_2$. Then we have

$$\begin{aligned} U \cup V_1 = V_2 &\Rightarrow U \subseteq V_2 \\ &\Rightarrow U \subseteq A \text{ and } U \text{ is finite} \\ &\Rightarrow U \in B. \end{aligned}$$

Thus B is a k -bi-ideal of $P_f(F)$.

Conversely, let B be a k -bi-ideal of $S = P_f(F)$. We consider $A = \cup_{U \in B} U$. Then $A \subseteq F$ and $B \subseteq P_f(A)$. Let $Y = \{y_1, y_2, y_3, \dots, y_n\} \in P_f(A)$. Then for each $i = 1, 2, \dots, n$, $y_i \in U$ for some $U \in B$.

Then we have

$$\{y_i\} + U = U \Rightarrow \{y_i\} \in B, \text{ since } B \text{ is a } k\text{-set of } S,$$

and so $Y = \{y_1\} + \{y_2\} + \{y_3\} + \dots + \{y_n\} \in B$. Hence $B = P_f(A)$. Also for $a, b \in A$, we have

$$\begin{aligned} \{a\}, \{b\} \in B &\Rightarrow \{a\}\{b\} \in B \\ &\Rightarrow \{ab\} \in B = P_f(A) \\ &\Rightarrow ab \in A, \end{aligned}$$

whence A is a subsemigroup of F . Again for $s \in F$ we have

$$\{asb\} = \{a\}\{s\}\{b\} \in BSB \subseteq B \Rightarrow asb \in A.$$

Thus A is a bi-ideal of F such that $B = P_f(A)$. ■

Let $a \in S$. We denote $B[a] = \{\sum_{i=1}^n x_i \mid x_i \in \{a\} \cup \{a^2\} \cup aSa\}$. Then $B[a]$ is a subsemiring of S . Also for any $s \in S$ and $b, c \in \{a\} \cup \{a^2\} \cup aSa$ we have $bsc \in aSa$ which implies that $B[a]SB[a] \subseteq B[a]$ and so $B[a]$ is a bi-ideal of S . In the following lemma we describe the principal k -bi-ideal of S , which is generated by a .

Lemma 2.4. *Let S be a semiring and $a \in S$. Then the principal k -bi-ideal of S generated by a is given by:*

$$B_k(a) = \{u \in S \mid u + a + a^2 + asa = a + a^2 + asa, \text{ for some } s \in S\}.$$

Proof. Let $u, v \in B_k(a)$. Then there exist $s, t \in S$ such that $u + a + a^2 + asa = a + a^2 + asa$ and $v + a + a^2 + ata = a + a^2 + ata$. Then we have

$$\begin{aligned} u + a + a^2 + asa &= a + a^2 + asa \\ \Rightarrow uv + (a + a^2 + asa)v &= (a + a^2 + asa)v \\ \Rightarrow uv + (a + a^2 + asa)(v + a + a^2 + ata) &= (a + a^2 + asa)(v + a + a^2 + ata) \end{aligned}$$

$$\Rightarrow uv + (a + a^2 + asa)(a + a^2 + ata) = (a + a^2 + asa)(a + a^2 + ata)$$

$$\Rightarrow uv + a + a^2 + axa = a + a^2 + axa, \text{ since } (S, +) \text{ is a semilattice,}$$

where $x = a + a^2 + at + sa + a^2t + sa^2 + sa^2t$ and so $uv \in B_k(a)$. Also $u + v \in B_k(a)$. Thus $B_k(a)$ is a subsemiring of S .

Similarly $usv \in B_k(a)$ for all $u, v \in B_k(a)$ and $s \in S$. Thus $B_k(a)$ is a bi-ideal of S . In fact, $B_k(a) = \overline{B[a]}$, the k -closure of $B[a]$. Hence $B_k(a)$ is a k -bi-ideal of S .

Let B be a k -bi-ideal of S such that $a \in B$. Let $u \in B_k(a)$. Then there exists $s \in S$ such that $u + a + a^2 + asa = a + a^2 + asa$. Now $a \in B$ implies that $a + a^2 + asa \in B$ and so $u \in B$. Hence $B_k(a) \subseteq B$. Thus $B_k(a)$ is the least k -bi-ideal of S , which contains a . ■

The following lemma can be proved similarly.

Lemma 2.5. *Let S be a semiring and $a \in S$.*

1. *The principal left k -ideal of S generated by a is given by $L_k(a) = \{u \in S \mid u + a + sa = a + sa, \text{ for some } s \in S\}$.*
2. *The principal k -ideal of S generated by a is given by $J_k(a) = \{u \in S \mid u + a + sa + as + sas = a + sa + as + sas, \text{ for some } s \in S\}$.*

The principal right k -ideal $R_k(a)$ of S generated by a is given dually.

3. BI-IDEALS IN k -REGULAR SEMIRINGS

A semigroup S is called a regular semigroup if for each $a \in S$ there exists $x \in S$ such that $a = axa$. In [17], Von Neumann defined a ring R to be regular if the multiplicative reduct (R, \cdot) is a regular semigroup. Bourne [2] defined a semiring S to be regular if for each $a \in S$ there exist $x, y \in S$ such that $a + axa = aya$. If a semiring S happens to be a ring then the Von Neumann regularity and the Bourne regularity are equivalent.

This is not true in a semiring in general (for counter example we refer [18]). Adhikari, Sen and Weinert [1] renamed the Bourne regularity of a semiring as k -regularity to distinguish this from the notion of Von Neumann regularity.

Definition 3.1. A semiring S is called a k -regular semiring if for each $a \in S$ there exist $x, y \in S$ such that $a + axa = aya$.

Since $(S, +)$ is a semilattice, we have

$$\begin{aligned} a + axa = aya &\Rightarrow a + axa + (axa + aya) = aya + (axa + aya) \\ &\Rightarrow a + a(x + y)a = a(x + y)a. \end{aligned}$$

Thus, a semiring S is k -regular if and only if for all $a \in S$ there exists $x \in S$ such that

$$a + axa = axa.$$

Let S be a k -regular semiring and $a \in S$. Then there exists $x \in S$ such that $a + axa = axa$. Then we have

$$\begin{aligned} a + axa = axa &\Rightarrow a + ax(a + axa) = ax(a + axa) \\ &\Rightarrow a + axaxa = axaxa. \end{aligned}$$

Thus, a semiring S is k -regular if and only if for all $a \in S$ there exists $x \in S$ such that

$$(3.1) \quad a + axaxa = axaxa.$$

For examples and properties of k -regular semirings we refer [1, 18, 19, 20].

Now we give several equivalent characterizations of k -regularity in terms of k -bi-ideals.

Theorem 3.2. For a semiring S the following conditions are equivalent:

1. S is k -regular;
2. $B = \overline{BSB}$ for every k -bi-ideal B of S ;
3. $G = \overline{GSG}$ for every generalized k -bi-ideal G of S .

Proof. It is clear that (3) \Rightarrow (2); since each k -bi-ideal is a generalized k -bi-ideal. Hence we are to prove (1) \Rightarrow (3) and (2) \Rightarrow (1) only.

(1) \Rightarrow (3): Let G be a generalized k -bi-ideal of S . Then $GSG \subseteq G$ implies that $\overline{GSG} \subseteq G$. Let $a \in G$. Since S is k -regular, there exists $x \in S$ such that $a + axa = axa$. Now $axa \in GSG$ implies that $a \in \overline{GSG}$ and so $G \subseteq \overline{GSG}$. Thus $G = \overline{GSG}$.

(2) \Rightarrow (1): Let $a \in S$. Consider the k -bi-ideal $B_k(a)$. Then $a \in B_k(a) = \overline{B_k(a)SB_k(a)}$ implies that there exist $b_1, b_2, b_3, b_4 \in B_k(a)$ and $s_1, s_2 \in S$ such that

$$\begin{aligned} a + b_1s_1b_2 &= b_3s_2b_4 \Rightarrow a + (b_1 + b_2 + b_3 + b_4)(s_1 + s_2)(b_1 + b_2 + b_3 + b_4) \\ &= (b_1 + b_2 + b_3 + b_4)(s_1 + s_2)(b_1 + b_2 + b_3 + b_4) \Rightarrow a + bsb = bsb, \end{aligned}$$

where $b = b_1 + b_2 + b_3 + b_4 \in B_k(a)$ and $s = s_1 + s_2 \in S$. Hence there exists $s \in S$ such that $b + a + a^2 + asa = a + a^2 + asa$. Then we have

$$\begin{aligned} a + bsb &= bsb \Rightarrow a + (b + a + a^2 + asa)s(b + a + a^2 + asa) \\ &= (b + a + a^2 + asa)s(b + a + a^2 + asa) \Rightarrow a + (a + a^2 + asa)s(a + a^2 + asa) \\ &= (a + a^2 + asa)s(a + a^2 + asa) \Rightarrow a + ata = ata \end{aligned}$$

for some $t \in S$. Thus S is k -regular. ■

Theorem 3.3. *For a semiring S , the following conditions are equivalent:*

1. S is k -regular;
2. $B \cap J = \overline{BJB}$ for every k -bi-ideal B and every k -ideal J of S ;
3. $B \cap I = \overline{BITB}$ for every k -bi-ideal B and every interior k -ideal I of S ;
4. $G \cap J = \overline{GJG}$ for every generalized k -bi-ideal G and every k -ideal J of S ;
5. $G \cap I = \overline{GIG}$ for every generalized k -bi-ideal G and every interior k -ideal I of S .

Proof. It is clear that $(5) \Rightarrow (4) \Rightarrow (2)$ and $(5) \Rightarrow (3) \Rightarrow (2)$, since each k -ideal is an interior k -ideal and each k -bi-ideal is generalized k -bi-ideal. Hence we are to prove $(1) \Rightarrow (5)$ and $(2) \Rightarrow (1)$ only.

(1) \Rightarrow (5): Let G be a generalized k -bi-ideal and I be an interior k -ideal of S . Then $GIG \subseteq G \cap I$ and so $\overline{GIG} \subseteq G \cap I$. Let $a \in G \cap I$. Then there exists $x \in S$ such that $a + axaxa = axaxa$, by (3.1). Now $axa \in SIS \subseteq I$ implies that $a(xax)a \in GIG$. This implies that $a \in \overline{GIG}$ and so $G \cap I \subseteq \overline{GIG}$. Thus $G \cap I = \overline{GIG}$.

(2) \Rightarrow (1): Let $a \in S$. Then $a \in B_k(a) \cap J_k(a) = \overline{B_k(a)J_k(a)B_k(a)}$. Therefore there exists $b_1, b_2, b_3, b_4 \in B_k(a)$ and $c_1, c_2 \in J_k(a)$ such that $a + b_1c_1b_2 = b_3c_2b_4$. Then similarly to $(2) \Rightarrow (1)$ of Theorem 3.2, it follows that S is k -regular. ■

Theorem 3.4. *For a semiring S the following conditions are equivalent:*

1. S is k -regular;
2. $B \cap L \subseteq \overline{BL}$ for every k -bi-ideal B and every left k -ideal L of S ;
3. $G \cap L \subseteq \overline{GL}$ for every generalized k -bi-ideal G and every left k -ideal L of S .

Proof. It is clear that $(3) \Rightarrow (2)$. Hence we are to prove $(1) \Rightarrow (3)$ and $(2) \Rightarrow (1)$ only.

(1) \Rightarrow (3): Let G be a generalized k -bi-ideal and L be a left k -ideal of S . Let $a \in G \cap L$. Then there exists $x \in S$ such that $a + axa = axa$. Now $a(xa) \in GL$ implies that $a \in \overline{GL}$. Thus $G \cap L \subseteq \overline{GL}$.

(2) \Rightarrow (1): Let $a \in S$. Then $a \in B_k(a) \cap L_k(a) \subseteq \overline{B_k(a)L_k(a)}$. Then there exist $b_1, b_2 \in B_k(a)$ and $l_1, l_2 \in L_k(a)$ such that $a + b_1l_1 = b_2l_2$ which implies that $a + bl = bl$, where $b = b_1 + b_2 \in B_k(a)$ and $l = l_1 + l_2 \in L_k(a)$. Then there exist $x, y \in S$ such that $b + a + a^2 + axa = a + a^2 + axa$ and $l + a + ya = a + ya$. Then we have

$$\begin{aligned} a + bl = bl &\Rightarrow a + (b + a + a^2 + axa)(l + a + ya) \\ &= (b + a + a^2 + axa)(l + a + ya) \Rightarrow a + (a + a^2 + axa)(a + ya) \end{aligned}$$

$$\begin{aligned}
&= (a + a^2 + axa)(a + ya) \Rightarrow a + a^2 + asa = a^2 + asa, \quad \text{where } s \in S \\
&\Rightarrow a + a(a + a^2 + asa) + asa = a(a + a^2 + asa) + asa \Rightarrow a + ata = ata,
\end{aligned}$$

where $t \in S$ and hence S is k -regular. ■

Theorem 3.5. *For a semiring S , the following conditions are equivalent:*

1. S is k -regular;
2. $R \cap B \cap L \subseteq \overline{RBL}$ for every right k -ideal R , every k -bi-ideal B and every left k -ideal L of S ;
3. $R \cap G \cap L \subseteq \overline{RGL}$ for every right k -ideal R , every generalized k -bi-ideal G and every left k -ideal L of S .

Proof. It is clear that (3) \Rightarrow (2). Hence we are to prove (1) \Rightarrow (3) and (2) \Rightarrow (1) only.

(1) \Rightarrow (3): Let R , G and L be a right k -ideal, generalized k -bi-ideal and left k -ideal of S respectively. Let $a \in R \cap G \cap L$. Then there exists $x \in S$ such that $a + axaxa = axaxa$. Now $xax \in G$ implies that $a(xax)a \in RGL$ whence $a \in \overline{RGL}$. Thus $R \cap G \cap L \subseteq \overline{RGL}$.

(2) \Rightarrow (1): Let $a \in S$. Then $a \in R_k(a) \cap B_k(a) \cap L_k(a) \subseteq \overline{R_k(a)B_k(a)L_k(a)}$ implies that there exist $r_1, r_2 \in R_k(a)$, $b_1, b_2 \in B_k(a)$ and $l_1, l_2 \in L_k(a)$ such that $a + r_1b_1l_1 = r_2b_2l_2$ which implies that $a + rbl = rbl$, where $r = r_1 + r_2 \in R_k(a)$, $l = l_1 + l_2 \in L_k(a)$ and $b = b_1 + b_2 \in B_k(a)$. Then there exist $x, y, z \in S$ such that $r + a + ax = a + ax$, $b + a + a^2 + aya = a + a^2 + aya$ and $l + a + za = a + za$. Then we have

$$\begin{aligned}
&a + (r + a + ax)(b + a + a^2 + aya)(l + a + za) \\
&= (r + a + ax)(b + a + a^2 + aya)(l + a + za) \\
&\Rightarrow a + (a + ax)(a + a^2 + aya)(a + za) \\
&= (a + ax)(a + a^2 + aya)(a + za) \Rightarrow a + ata = ata,
\end{aligned}$$

where $t \in S$ and hence S is k -regular. ■

4. INTRA k -REGULAR SEMIRINGS

In this section we introduce intra- k -regular semirings and characterize these semirings using k -bi-ideals. Recall that a semigroup S is called intra-regular if $a \in Sa^2S$ for all $a \in S$. The intra k -regular semirings may be viewed as the semirings to which class the semiring $P_f(F)$ belongs when F is an intra regular semigroup.

Definition 4.1. A semiring S is called an intra k -regular semiring if for each $a \in S$,

$$a \in \overline{Sa^2S}.$$

It is easy to check that a semiring S is intra k -regular if and only if for each $a \in S$ there exists $x \in S$ such that

$$(4.1) \quad a + xa^2x = xa^2x.$$

Following proposition shows that intra k -regularity is a natural extension of the notion of intra-regularity in semigroups to the semirings whose additive reduct is a semilattice.

Proposition 4.2. *Let F be a semigroup. Then the semiring $P_f(F)$ is intra k -regular if and only if F is an intra-regular semigroup.*

Proof. Suppose F is an intra-regular semigroup. Let $A \in P_f(F)$. Then, for each $a \in A$, there exist $x_a, y_a \in F$ such that $a = x_a a^2 y_a$. Consider $X = \{x_a \mid a \in A\}$, $Y = \{y_a \mid a \in A\} \in P_f(F)$. Then we have

$$A \subseteq XA^2Y \Rightarrow A + XA^2Y = XA^2Y$$

and so $P_f(F)$ is intra k -regular.

Conversely, let $P_f(F)$ be an intra k -regular semiring. Let $a \in F$. Then $A = \{a\} \in P_f(F)$ and hence there exists $X \in P_f(F)$ such that

$$A + XA^2X = XA^2X \Rightarrow A \subseteq XA^2X.$$

This implies that there exist $x, y \in X$ such that $a = xa^2y$. Thus F is intra regular. ■

The following theorem shows that the k -ideals and the interior k -ideals coincide in an intra k -regular semiring.

Theorem 4.3. *Let S be an intra k -regular semiring and A be a non-empty subset of S . Then A is a k -ideal of S if and only if A is an interior k -ideal of S .*

Proof. Let A be an interior k -ideal of S . Let $u \in SA$. Then there exists $s \in S$ and $a \in A$ such that

$$u + sa = sa.$$

Since S is intra k -regular, there exists $x \in S$ such that $a + xa^2x = xa^2x$. Then we have

$$\begin{aligned} u + s(a + xa^2x) &= s(a + xa^2x) \\ \Rightarrow u + sxa^2x &= sxa^2x \end{aligned}$$

and so $u \in \overline{SaS} \subseteq \overline{A} = A$. Hence $SA \subseteq A$. Similarly $AS \subseteq A$. Thus A is a k -ideal of S .

Converse follows directly since each k -ideal of S is interior k -ideal. ■

Definition 4.4. Let S be a semiring. A subset P of S is called semiprime if for all $a \in S$,

$$a^2 \in P \text{ implies that } a \in P.$$

Following theorem characterizes the intra k -regular semirings as the semirings whose every k -ideal is semiprime.

Theorem 4.5. *For a semiring S , the following conditions are equivalent:*

1. S is intra k -regular;
2. every k -ideal of S is semiprime;
3. every interior k -ideal of S is semiprime.

Proof. Every k -ideal is an interior k -ideal, so it is clear that (3) \Rightarrow (2). Thus we have to prove only (1) \Rightarrow (3) and (2) \Rightarrow (1).

(1) \Rightarrow (3): Let A be an interior k -ideal of S . Suppose $a \in S$ such that $a^2 \in A$. Since S is intra k -regular, there exists $x \in S$ such that $a + xa^2x = xa^2x$. Then $xa^2x \in A$ implies that $a \in \overline{A} = A$. Hence every interior k -ideal of S is semiprime.

(2) \Rightarrow (1): Let $a \in S$. Then $I_k(a^4)$ is semiprime. Hence we have

$$\begin{aligned} a^4 \in I_k(a^4) &\Rightarrow a^2 \in I_k(a^4) \\ &\Rightarrow a \in I_k(a^4). \end{aligned}$$

Then there exists $x \in S$ such that $a + a^4 + xa^4 + a^4x + xa^4x = a^4 + xa^4 + a^4x + xa^4x$ which implies that $a + ya^2y = ya^2y$ for some $y \in S$. Thus S is intra k -regular. \blacksquare

In the following theorem we characterize the intra k -regular semirings by their left and right k -ideals.

Theorem 4.6. *Let S be a semiring. Then the following conditions are equivalent:*

1. S is intra k -regular;
2. $L \cap R \subseteq \overline{LR}$ for every left k -ideal L and every right k -ideal R of S .

Proof. (1) \Rightarrow (2): Let R and L be a right and a left k -ideal of S respectively. Let $a \in L \cap R$. Then there exists $x \in S$ such that $a + xa^2x = xa^2x$. Now $(xa)(ax) \in LR$ and so $a \in \overline{LR}$. Thus $L \cap R \subseteq \overline{LR}$.

(2) \Rightarrow (1): Let $a \in S$. Then $a \in L_k(a) \cap R_k(a) \subseteq \overline{L_k(a)R_k(a)}$ implies that there exist $u_1, u_2 \in L_k(a)$ and $v_1, v_2 \in R_k(a)$ such that

$$a + u_1v_1 = u_2v_2 \Rightarrow a + uv = uv,$$

where $u = u_1 + u_2 \in L_k(a)$ and $v = v_1 + v_2 \in R_k(a)$. Now there exist $s_1, s_2 \in S$ such that $u + a + s_1a = a + s_1a$ and $v + a + as_2 = a + as_2$.

Then we have

$$\begin{aligned}
& a + (u + a + s_1a)(v + a + as_2) = (u + a + s_1a)(v + a + as_2) \\
\Rightarrow & a + (a + s_1a)(a + as_2) = (a + s_1a)(a + as_2) \\
\Rightarrow & a + a^2 + s_1a^2 + a^2s_2 + s_1a^2s_2 = a^2 + s_1a^2 + a^2s_2 + s_1a^2s_2 \\
\Rightarrow & a + a^2 + sa^2 + a^2s + sa^2s = a^2 + sa^2 + a^2s + sa^2s, \quad \text{where } s = s_1 + s_2.
\end{aligned}$$

This can be written as

$$\begin{aligned}
& a + a^2 + tat = a^2 + tat, \quad \text{for some } t \in S \\
\Rightarrow & a + a(a + a^2 + tat) + tat = a(a + a^2 + tat) + tat \\
\Rightarrow & a + a(a^2 + tat) + tat = a(a^2 + tat) + tat \\
\Rightarrow & a + rar = rar, \quad \text{for some } r \in S \\
\Rightarrow & a + r(a + a^2 + sa^2 + a^2s + sa^2s)r = r(a + a^2 + sa^2 + a^2s + sa^2s)r \\
\Rightarrow & a + r(a^2 + sa^2 + a^2s + sa^2s)r = r(a^2 + sa^2 + a^2s + sa^2s)r
\end{aligned}$$

and so $a + xa^2x = xa^2x$ for some $x \in S$. Thus S is an intra k -regular semiring. ■

We left it to check to the readers that a semiring S is both k -regular and intra k -regular if and only if for each $a \in S$ there exists $z \in S$ such that

$$(4.2) \quad a + aza^2za = aza^2za.$$

Now we prove a series of theorems characterizing the semirings which are both k -regular and intra k -regular using k -bi-ideals.

Theorem 4.7. *For a semiring S , the following conditions are equivalent:*

1. S is both k -regular and intra k -regular;
2. $B = \overline{B^2}$ for every k -bi-ideal B of S ;
3. $G = \overline{G^2}$ for every generalized k -bi-ideal G of S .

Proof. It is clear that (3) \Rightarrow (2). Hence we are to prove (1) \Rightarrow (3) and (2) \Rightarrow (1) only.

(1) \Rightarrow (3) : Let G be a generalized k -bi-ideal of S . Let $a \in G$. Since S is both regular and intra k -regular, there exists $x \in S$ such that

$$a + axa^2xa = axya^2xa.$$

Now $axa \in GSG \subseteq G$ shows that $axa^2xa \in G^2$ and so $a \in \overline{G^2}$. Thus $G \subseteq \overline{G^2}$.

Again let $b \in \overline{G^2}$. Then there exist $g_1, g_2, g_3, g_4 \in G$ such that $b + g_1g_2 = g_3g_4$. Also there exist $x \in S$ such that $b + bxb^2xb = bxb^2xb$. Then we have

$$\begin{aligned} b + (b + g_1g_2)xb^2x(b + g_1g_2) &= (b + g_1g_2)xb^2x(b + g_1g_2) \\ \Rightarrow b + g_3g_4xb^2xg_3g_4 &= g_3g_4xb^2xg_3g_4. \end{aligned}$$

and so $b \in \overline{GSG} \subseteq \overline{G} = G$. Hence $\overline{G^2} \subseteq G$. Thus $G = \overline{G^2}$.

(2) \Rightarrow (1) : Let $a \in S$. Then $a \in B_k(a) = \overline{B_k(a)^2}$ implies that there exists $b \in B_k(a)$ such that $a + b^2 = b^2$. Again $b \in B_k(a) = \overline{B_k(a)^2}$ implies that there exist $c \in B_k(a)$ such that $b + c^2 = c^2$. Then $a + c^4 = c^4$. Now there exists $x \in S$ such that $c + a + a^2 + axa = a + a^2 + axa$. Then, since $(S, +)$ is a semilattice, we have

$$\begin{aligned} a + c^4 &= c^4 \\ \Rightarrow a + (c + a + a^2 + axa)^4 &= (c + a + a^2 + axa)^4 \\ \Rightarrow a + (a + a^2 + axa)^4 &= (a + a^2 + axa)^4, \end{aligned}$$

which implies that there exist $s, t \in S$ such that

$$a + asa = asa \text{ and } a + ta^2t = ta^2t.$$

Thus S is both k -regular and intra k -regular. ■

Theorem 4.8. *For a semiring S , the following conditions are equivalent:*

1. S is k -regular and intra k -regular;
2. $B \cap C \subseteq \overline{BC}$ for every k -bi-ideals B and C of S ;
3. $B \cap G \subseteq \overline{BG}$ for every k -bi-ideal B and every generalized k -bi-ideal G of S ;
4. $G \cap B \subseteq \overline{GB}$ for every generalized k -bi-ideal G and every k -bi-ideal B of S ;
5. $G \cap H \subseteq \overline{GH}$ for every generalized k -bi-ideals G and H of S .

Proof. It is clear that (5) \Rightarrow (4) \Rightarrow (2) and (5) \Rightarrow (3) \Rightarrow (2). So we have to prove (1) \Rightarrow (5) and (2) \Rightarrow (1) only.

(1) \Rightarrow (5): Let G and H be two generalized k -bi-ideals of S and $a \in G \cap H$. Then there exists $x \in S$ such that

$$a + axa^2xa = axa^2xa.$$

Now $(axa)(axa) \in (GSG)(HSH) \subseteq GH$ implies that $a \in \overline{GH}$. Hence $G \cap H \subseteq \overline{GH}$.

(2) \Rightarrow (1): Let B be a k -bi-ideal of S . Then $B = B \cap B \subseteq \overline{B^2}$. Since B is a k -subsemiring, $\overline{B^2} \subseteq B$. Thus we have $B = \overline{B^2}$, whence S is both k -regular and intra k -regular, by Theorem 4.7. ■

Theorem 4.9. *For a semiring S , the following conditions are equivalent:*

1. S is k -regular and intra k -regular;
2. $B \cap L \subseteq \overline{BLB}$ for every k -bi-ideal B and every left k -ideal L of S ;

3. $B \cap R \subseteq \overline{BRB}$ for every k -bi-ideal B and every right k -ideal R of S ;
4. $B \cap C \subseteq \overline{BCB}$ for all k -bi-ideals B and C of S ;
5. $B \cap G \subseteq \overline{BGB}$ for every k -bi-ideal B and every generalized k -bi-ideal G of S ;
6. $G \cap L \subseteq \overline{GLG}$ for every generalized k -bi-ideal G and every left k -ideal L of S ;
7. $G \cap R \subseteq \overline{GRG}$ for every generalized k -bi-ideal G and every right k -ideal R of S ;
8. $G \cap B \subseteq \overline{GBG}$ for every generalized k -bi-ideal G and every k -bi-ideal B of S ;
9. $G \cap H \subseteq \overline{GHG}$ for all generalized k -bi-ideals G and H of S .

Proof. Every left k -ideal and right k -ideal is a k -bi-ideal. Hence it is clear that (9) \Rightarrow (8) \Rightarrow (7) \Rightarrow (3), (9) \Rightarrow (6) \Rightarrow (2) and (9) \Rightarrow (5) \Rightarrow (4) \Rightarrow (3). So it is sufficient to prove (1) \Rightarrow (9), (3) \Rightarrow (1) and (2) \Rightarrow (1) only.

(1) \Rightarrow (9): Let G and H be two generalized k -bi-ideals of S and $a \in G \cap H$. Then there exists $x \in S$ such that $a + axa^2xa = axa^2xa$. Then we have

$$\begin{aligned} a + axa^2xa = axa^2xa &\Rightarrow a + axa^2x(a + axa^2xa) = axa^2x(a + axa^2xa) \\ &\Rightarrow a + axa^2xaxa^2xa = axa^2xaxa^2xa. \end{aligned}$$

Now $axa^2xaxa^2xa = (axa)(axaxa)(axa) \in \overline{GHG}$ implies that $a \in \overline{GHG}$. Hence $G \cap H \subseteq \overline{GHG}$.

(2) \Rightarrow (1): Let $a \in S$. Consider $B = B_k(a)$ and $L = L_k(a)$. Then $a \in B \cap L \subseteq \overline{BLB}$ implies that there exist $b \in B$ and $l \in L$ such that $a + blb = blb$. Now there exist $s, t \in S$ such that $b + a + a^2 + asa = a + a^2 + asa$ and $l + a + ta = a + ta$. We denote $u = a + a^2 + asa$ and $v = a + ta$. Then we have

$$\begin{aligned} &a + (b + a + a^2 + asa)(l + v)(b + u) \\ &= (b + a + a^2 + asa)(l + v)(b + u) \Rightarrow a + (a + a^2 + asa)vu \end{aligned}$$

$$\begin{aligned}
&= (a + a^2 + asa)vu \Rightarrow a + avu + (a^2 + asa)vu \\
&= avu + (a^2 + asa)vu \Rightarrow a + (a + avu + (a^2 + asa)vu)vu + (a^2 + asa)vu \\
&= (a + avu + (a^2 + asa)vu)vu + (a^2 + asa)vu \Rightarrow a + (avu + (a^2 + asa)vu)vu \\
&\quad + (a^2 + asa)vu = (avu + (a^2 + asa)vu)vu + (a^2 + asa)vu \Rightarrow a + axau = axau,
\end{aligned}$$

where $x = avu + avut + (a^2 + asa)vu + (a^2 + asa)vut + (a^2 + asa) + (a^2 + asa)t$. Again this implies that

$$\begin{aligned}
a + axau = axau &\Rightarrow a + ax(a + axau)u = ax(a + axau)u \\
&\Rightarrow a + a(xa)^2u^2 = a(xa)^2u^2.
\end{aligned}$$

Proceeding in this way, we get $a + a(xa)^4u^4 = a(xa)^4u^4$. Then substituting the value of u , it can be rearranged in such a way that we have $y \in S$ such that

$$a + aya^2ya = aya^2ya.$$

Thus S is both k -regular and intra k -regular.

(3) \Rightarrow (1) : Similar to (2) \Rightarrow (1). ■

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