ON FINITE FUNCTIONS WITH NON-TRIVIAL ARITY GAP

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Abstract

Given an n-ary k-valued function f, $gap(f)$ denotes the minimal number of essential variables in f which become fictive when identifying any two distinct essential variables in f.

We particularly solve a problem concerning the explicit determination of n-ary k-valued functions f with $2 \leq gap(f) \leq n \leq k$. Our methods yield new combinatorial results about the number of such functions.

Keywords: essential variable, identification minor, essential arity gap.

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1. Introduction

Given a function f, the essential variables in f are defined as variables which occur in f and weigh with the values of that function. The number of essential variables is an important measure of complexity for discrete functions. Some deep results characterizing such variables and sets of essential variables are known ([2, 3, 8, 9, 10]). Similar problems about terms in universal algebra are discussed in [7, 11, 12] and about tree automata in [13]. So, any opportunity to reduce the number of essential variables in discrete functions is an important procedure in theoretical and applied computer science and modeling. There are two ways to decrease this number - by replacing some variables in function with constants or with other variables (i.e. with identification of variables).

M. Couceiro and E. Lehtonen classify finite valued functions on a finite set A in terms of their arity gap ([6]). The aim of the present paper is the representation and description of such functions. We shall use the well known fact that each n-ary function $f : A^n \to A$ can be represented as sums of conjunctions.

In Section 2 we introduce the basic definitions and some preliminary results concerning k-valued functions.

In Section 3 we study the essential arity gap of functions. Here a complete description of the functions in k-valued logic depending essentially on all of its n variables whose essential arity gap is equal to p with $2 < p \leq n \leq k$ is obtained.

In Section 4 we consider the class $G_{2,k}^n$ of n-ary k-valued functions which have essential arity gap equal to 2 when $k > 2$. This class is presented as union of two subclasses which are investigated.

In Section 5 we discuss a special class of the ternary k-valued functions which have essential arity gap equal to 2. The number of functions in this class is found.

2. Preliminaries

Let $k \geq 2$ be a natural number. Denote by $K = \{0, 1, \ldots, k - 1\}$ the set (ring) of remainders modulo k. A function (operation) on K is a mapping $f : K^n \to K$ where n is a natural number, called *the arity* of f. The set of the all such functions is denoted by P_k^n. Operations from P_2^n are called

Boolean functions.

Let $X_n = \{x_1, \ldots, x_n\}$ be the set of n variables and $f \in P_k^n$ be a k-valued function.

Definition 2.1. A variable x_i is called *essential* in f, or f *essentially depends* on x_i, if there exist values $a_1, \ldots, a_n, b \in K$, such that

$$f(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \neq f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n).$$

The set of all essential variables in a function f is denoted by $Ess(f)$ and the number of its essential variables is denoted by $ess(f) = |Ess(f)|$. The variables from X_n which are not essential in $f \in P_k^n$ are called *fictive* and the set of fictive variables in f is denoted by $Fic(f)$.

Let x_i and x_j be two distinct essential variables in f. We say that the function g is obtained from $f \in P_k^n$ by *the identification of the variable x_i with x_j*, if

$$g(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, a_j, a_{i+1}, \ldots, a_n),$$

for all $(a_1, \ldots, a_n) \in K^n$.

Briefly, when g is obtained from f, by identification of the variable x_i with x_j, we will write $g = f_{i \leftarrow j}$ and g is called *the identification minor of f* and $Min(f)$ denotes the set of all identification minors of f.

We shall allow formation of identification minors when x_i or x_j are not essential in f, also. Such minors of f are called *trivial* and they do not belong to $Min(f)$. For instance, if x_i does not occur in f, then $f_{i \leftarrow j} := f$.

Remark 2.1. Let i and j be two natural numbers with $1 \leq j, i \leq n$, $i \neq j$. Then we have:

   (i) $ess(f_{i \leftarrow j}) \leq ess(f)$, because $x_i \notin Ess(f_{i \leftarrow j})$, even though it might be essential in f.

  (ii) If x_j is an essential variable in f then $Ess(f_{i \leftarrow j}) \subseteq Ess(f)$.

Definition 2.2. Let $f \in P_k^n$ be an n-ary k-valued function. Then *the essential arity gap* (briefly *arity gap* or *gap*) of f is defined as follows

$$gap(f) := ess(f) - \max_{g \in Min(f)} ess(g).$$

We say that the function f has non-trivial arity gap if $gap(f) \geq 2$.

We let $G_{p,k}^m$ denote the set of all functions in P_k^n which essentially depend on m variables whose arity gap is equal to p i.e. $G_{p,k}^m = \{f \in P_k^n \mid ess(f) = m \ \& \ gap(f) = p\}$, with $m \leq n$.

The set of all n-ary k-valued functions which essentially depend on m variables is denoted by $P_{m,k}^n$, i.e. $P_{m,k}^n = \{f \in P_k^n \mid ess(f) = m\}$, $m \leq n$.

An upper bound of $gap(f)$ for Boolean functions is found by K. Chimev, A. Salomaa and O. Lupanov [4, 10, 9]. It is proved that $gap(f) \leq 2$, when $f \in P_2^n$, $n \geq 2$.

This result is generalized for arbitrary finite valued functions in [5]. It is shown that $gap(f) \leq k$ for all $f \in P_k^n$.

In [5] the Boolean functions whose arity gap is equal to 2 are described. In [14] the class $G_{2,2}^n$ is investigated, also. Several combinatorial results concerning the number of the functions in this class are obtained.

The case $2 \leq p \leq k < n$ is fully described in [15] where it is proved that $gap(f) \leq 2$ and if $f \in G_{2,k}^n$ then f is a totally symmetric function.

So, in the present paper we shall pay attention to the case $2 < k$ and $n \leq k$.

We show that if $f \in G_{p,k}^n$, $2 < p \leq n \leq k$ then $f = h \oplus g$ where $ess(h) = n - p$ and $g \in G_{n,k}^n$ which arises in this way, solving the following problem:

For each $1 \leq p \leq k$, determine explicitly the functions $f \in P_k^n$ whose arity gap is equal to p, $p \geq 2$ ([5], p. 6, Problem 1).

Let $m \in N$, $0 \leq m \leq k^n - 1$ be an integer. It is well known that for every $k, n \in N$, $k \geq 2$ there is an unique finite sequence $(\alpha_1, \ldots, \alpha_n) \in K^n$ such that

$$(1) \qquad\qquad m = \alpha_1 k^{n-1} + \alpha_2 k^{n-2} + \ldots + \alpha_n.$$

The equation (1) is known as the representation of m in $k-$ary positional numerical system. One briefly writes $m = \overline{\alpha_1 \alpha_2 \ldots \alpha_n}$ instead (1).

Given a variable x and $\alpha \in K$, x^α is an important function defined by:

$$x^\alpha = \begin{cases} 1 & if \quad x = \alpha \\ 0 & if \quad x \neq \alpha. \end{cases}$$

In this paper we shall use *sums of conjunctions (SC)* for representation of functions in P_k^n. This is the most natural representation of the functions in finite algebras. It is based on so called operation tables of the functions.

Theorem 2.1. *Each function $f \in P_k^n$ can be uniquely represented in SC-form as follows*

$$(2) \qquad f = a_0.x_1^0 \dots x_n^0 \oplus \dots \oplus a_m.x_1^{\alpha_1} \dots x_n^{\alpha_n} \oplus \dots \oplus a_{k^n-1}.x_1^{k-1} \dots x_n^{k-1}$$

with $m = \overline{\alpha_1 \dots \alpha_n}$, $a_m \in K$, where "$\oplus$" and "." are the operations addition and multiplication modulo k in the ring K.

A fact we shall use repeatedly is this: A variable x_i is fictive (inessential) in the function $f \in P_k^n$, if and only if

$$f(x_1, \dots, x_n) = x_i^0.f_1 \ \oplus \ x_i^1.f_2 \dots \oplus x_i^{k-1}.f_k,$$

with $f_1 = f_2 = \dots = f_k$ and $x_i \notin Ess(f_j)$, where f_j are $n-1$-ary k-valued functions with set of variables $\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}$ for $j = 1, 2, \dots, k$.

Consequently, if $f, g \in P_k^n$, $x_i \in Fic(f)$ and $x_i \in Fic(g)$, then $x_i \in Fic(f \oplus g)$.

Let us note that if $f_{i \leftarrow j} = g_{i \leftarrow j}$ and $\alpha_i = \alpha_j$ then $f(\alpha_1, \dots, \alpha_j, \dots, \alpha_i, \dots, \alpha_n) = g(\alpha_1, \dots, \alpha_j, \dots, \alpha_i, \dots, \alpha_n)$.

Lemma 2.1. *Let f and g be two k-valued functions, depending essentially on n, $n \geq k + 1$ variables. If $f_{i \leftarrow j} = g_{i \leftarrow j}$ for all i, j with $1 \leq j, i \leq n$ and $i \neq j$, then $f = g$.*

Proof. Let $(\alpha_1, \alpha_2, \dots, \alpha_n) \in K^n$ be an arbitrary n-tuple of integers from K. Since $n \geq k + 1$ it follows that there exist two natural numbers i, j with $1 \leq j < i \leq n$ and $\alpha_i = \alpha_j$. Then $f_{i \leftarrow j} = g_{i \leftarrow j}$ implies

$$f(\alpha_1, \dots, \alpha_j, \dots, \alpha_i, \dots, \alpha_n) = g(\alpha_1, \dots, \alpha_j, \dots, \alpha_i, \dots, \alpha_n).$$

Consequently $f = g$. ∎

Example 2.1. Let us consider the functions $f = x_1^0 x_2^0 x_3^0 \oplus x_1^1 x_2^0 x_3^2$ and $g = x_1^0 x_2^0 x_3^0 \oplus x_1^0 x_2^1 x_3^2$ from P_3^3. It is not too hard to show that $f_{i \leftarrow j} = g_{i \leftarrow j} = x_j^0 x_m^0$, where $m \in \{1, 2, 3\} \setminus \{i, j\}$ for all i, j with $1 \leq j, i \leq 3$ and $i \neq j$. On the other hand we have $f \neq g$. This example shows that $n \geq k+1$ is an essential condition in Lemma 2.1.

Lemma 2.2. *If $x_i \notin Ess(f)$ and $f \in P_k^n$ then $f = f_{i \leftarrow j}$, for all $j \in \{1 \ldots n\}$, $i \neq j$.*

Proof. Without loss of generality assume that $j = 1$ and $i = 2$. Then we have

$$f(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = f(\alpha_1, \beta_2, \alpha_3, \ldots, \alpha_n),$$

for all $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n, \beta_2 \in K$.
Let $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n \in K$ be arbitrary n integers from K. Then we have

$$f(\gamma_1, \gamma_1, \gamma_3, \ldots, \gamma_n) = f(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n).$$

Consequently $f = f_{2 \leftarrow 1}$.   ■

Lemma 2.2 implies that if $x_i \notin Ess(f)$ then $Ess(f) = Ess(f_{i \leftarrow j})$, for all $j \in \{1 \ldots n\}$ with $i \neq j$.

3. Essential arity gap of k-valued functions

We are going to study the n-ary k-valued functions whose arity gap is equal to n. The set of all strings over K with length m, $m \geq 1$ will be denoted by K^m.

Given two natural numbers $k, n \geq 2$, Eq_k^n denotes the set of all strings over $K = \{0, 1, \ldots, k-1\}$ with length n which have at least two equal letters i.e.

$$Eq_k^n := \{\alpha_1 \ldots \alpha_n \in K^n \mid \alpha_i = \alpha_j, \text{ for some } i, j \leq n, \ i \neq j\}.$$

Lemma 3.1. *If $f \in G_{n,k}^n$ and $2 \leq n \leq k$, then $f(\alpha_1, \ldots, \alpha_n) = f(0, \ldots, 0)$ for all $\alpha_1 \ldots \alpha_n \in Eq_k^n$.*

Proof. Let $\alpha_1 \alpha_2 \ldots \alpha_n$ be an arbitrary string from Eq_k^n. Without loss of generality let us assume that $\alpha_1 = \alpha_2$. Since $f \in G_{n,k}^n$ it follows that the function $f_{2 \leftarrow 1} = f(x_1, x_1, x_3, \ldots, x_n)$ does not essentially depend on any of its variables $x_1, x_3, \ldots, x_n$. So, we have $f(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = f(\alpha_1, \alpha_1, \alpha_3, \ldots, \alpha_n) = f(0, \ldots, 0)$. ∎

Theorem 3.1. *Let $f \in P_k^n$, be a function which depends essentially on all of its n variables and $2 \leq n \leq k$. Then $f \in G_{n,k}^n$ if and only if it can be represented as follows*

$$(3) \qquad f = \left[\bigoplus_{\beta_1 \ldots \beta_n \notin Eq_k^n} a_r . x_1^{\beta_1} \ldots x_n^{\beta_n} \right] \oplus a_0 . \left[\bigoplus_{\alpha_1 \ldots \alpha_n \in Eq_k^n} x_1^{\alpha_1} \ldots x_n^{\alpha_n} \right],$$

where $r = \overline{\beta_1 \ldots \beta_n}$ and at least two among the coefficients

$$\{a_0\} \cup \{a_r \mid r = \overline{\beta_1 \ldots \beta_n}, \ \& \ \beta_1 \ldots \beta_n \notin Eq_k^n\}$$

are distinct.

Proof. "$\Rightarrow$" Let $f \in G_{n,k}^n$ be represented in its SC-form as follows

$$f = \bigoplus_{m=0}^{k^n - 1} a_m . x_1^{\beta_1} \ldots x_n^{\beta_n} \quad \text{where} \quad m = \overline{\beta_1 \ldots \beta_n}.$$

By Lemma 3.1 we have $f(\alpha_1, \ldots, \alpha_n) = f(0, \ldots, 0) = a_0$ for $\alpha_1 \ldots \alpha_n \in Eq_k^n$. This shows that f has to be in the form (3.1). Moreover, at least two among the coefficients $\{a_0\} \cup \{a_r \mid r = \overline{\beta_1 \ldots \beta_n} \ \& \ \beta_1 \ldots \beta_n \in Eq_k^n\}$ are distinct.

"$\Leftarrow$" Let f be represented in the form (3). Then f depends essentially on all of its variables since at least two among the coefficients $\{a_0\} \cup \{a_r \mid r = \overline{\beta_1 \ldots \beta_n}, \ \& \ \beta_1 \ldots \beta_n \notin Eq_k^n\}$, are distinct. We have to prove that for all i, j with $1 \leq j, i \leq n$ and $i \neq j$ the functions

$$f_{i \leftarrow j} = a_0 . \bigoplus_{\beta_i = \beta_j} x_1^{\beta_1} \ldots x_j^{\beta_j} \ldots x_i^{\beta_i} \ldots x_n^{\beta_n}$$

do not depend on any of their variables.

Without loss of generality let us prove this for $j = 1$ and $i = 2$. Then

$$f_{2\leftarrow 1} = a_0.\left(x_1^0 \oplus x_1^1 \oplus \ldots \oplus x_1^{k-1}\right) \left(\bigoplus_{\beta_3\ldots\beta_n \in K^{n-2}} x_3^{\beta_3}\ldots x_n^{\beta_n}\right) = a_0.1.1 = a_0.$$

Hence $ess(f_{2\leftarrow 1}) = 0$. This completes the proof of the theorem. ∎

Corollary 3.1. *For every* n, k, *with* $2 \le n \le k$ *the functions*

(4) $$p_k(x_1,\ldots,x_n) = \bigoplus_{\alpha_1\ldots\alpha_k \notin Eq_k^n} a_m.x_1^{\alpha_1}\ldots x_n^{\alpha_n},$$

with $a_m \in K$, *have the essential arity gap equals to* n, *excluding when all* a_m's *are equal to* 0 *and* $p_k(x_i = x_j) = 0$ *for all* $1 \le i, j \le n$ *with* $i \ne j$.

Theorem 3.2. *If* $2 \le n \le k$ *then*

$$|G_{n,k}^n| = k^{\left(\binom{k}{n}.n!+1\right)} - k.$$

Proof. The number of coefficients a_i in (3) is equal to $\binom{k}{n}.n! + 1$ and they can be chosen in $k^{[\binom{k}{n}.n!+1]}$ ways. There are k "forbidden" cases, when a_0 and a_r's in (3) are the same. ∎

Lemma 3.2. *Let* $f \in P_k^n$ *be a* k-*valued function. If* $x_i \notin Ess(f_{u\leftarrow v})$, *with* $1 \le i, u, v \le n$, $u \ne v$ *and* $i \notin \{u, v\}$ *then* $f_{u\leftarrow v} = [f_{i\leftarrow j}]_{u\leftarrow v}$, *for all* j, $j \in \{1\ldots n\}$, $j \ne i$.

Proof. Suppose with no loss of generality that $u = 2$, $v = 1$ and $i = 3$. Then $f_{2\leftarrow 1} = f(x_1, x_1, x_3, x_4, \ldots, x_n)$ and from $x_3 \notin Ess(f_{2\leftarrow 1})$ we have

$$f(x_1, x_1, \alpha, x_4, \ldots, x_n) = f(x_1, x_1, \beta, x_4, \ldots, x_n)$$

for all $\alpha, \beta \in K$.

Let $j \in \{1, \ldots, n\}$ and $j \neq 3$. Then we have

$$[f_{3 \leftarrow j}]_{2 \leftarrow 1} = f(x_1, x_1, x_j, x_4, \ldots, x_n) = f(x_1, x_1, x_3, x_4, \ldots, x_n) = f_{2 \leftarrow 1}.$$

∎

Lemma 3.3. *Let $f \in P_k^n$ be a k-valued function. If $x_v \notin Ess(f_{u \leftarrow v})$ for some $u, v \leq n$, then $f_{u \leftarrow v} = [f_{v \leftarrow j}]_{u \leftarrow j} = [f_{u \leftarrow j}]_{v \leftarrow j}$, for all $j, j \in \{1 \ldots n\}$, $j \neq u, v$.*

Proof. Without loss of generality assume that $u = 2$ and $v = 1$.

Then $f_{2 \leftarrow 1} = f(x_1, x_1, x_3, \ldots, x_n)$ and from $x_1 \notin Ess(f_{2 \leftarrow 1})$ we obtain

$$f_{2 \leftarrow 1} = f(\alpha, \alpha, x_3, \ldots, x_n) = f(\beta, \beta, x_3, \ldots, x_n)$$

for all $\alpha, \beta \in K$.

Let $j > 2$ and by symmetry, we may assume $j = 3$. Then we obtain

$$[f_{1 \leftarrow 3}]_{2 \leftarrow 3} = [f(x_3, x_2, x_3, x_4, \ldots, x_n)]_{2 \leftarrow 3} = f(x_3, x_3, x_3, x_4, \ldots, x_n)$$
$$= f(\alpha, \alpha, x_3, \ldots, x_n) = f_{2 \leftarrow 1}.$$

∎

We are going to describe the basic properties of the functions f whose arity gap is non-trivial i.e. $gap(f) = p$ with $2 \leq p < n \leq k$, in the rest of the paper.

Theorem 3.3. *Let $2 < p < n \leq k$. Then for each $f \in G_{p,k}^{n,}$ there is a function $h \in P_k^n$ with*

(i) $ess(h) = n - p$;

(ii) $f_{i \leftarrow j} = h$ *for all $1 \leq i, j \leq n$ with $i \neq j$ and $x_i \in Fic(h)$.*

Moreover, for all $1 \leq u, v \leq n$ with $v \neq u$ and $x_v \in Fic(f_{u \leftarrow v})$ holds $f_{i \leftarrow j} = f_{u \leftarrow v}$ for all $1 \leq i, j \leq n$ with $i \neq j$ and $x_i \in Fic(f_{u \leftarrow v})$ as well as $ess(f_{u \leftarrow v}) = n - p$.

Proof. Let $f \in G_{p,k}^n$ and $1 \leq i, j \leq n$ with $i \neq j$ and $ess(f_{i \leftarrow j}) = n - p$. Let us set $h = f_{i \leftarrow j}$.

First, we shall prove that there are $1 \leq r, s \leq n$ with $r \neq s$, $ess(f_{r \leftarrow s}) = n - p$ and $x_s \notin Ess(f_{r \rightarrow s})$.

If $x_j \notin Ess(f_{i \leftarrow j})$ we are done in this part of the proof.

Furthermore, let us assume that $x_j \in Ess(f_{i \leftarrow j})$. Since $n > p > 2$, i.e. $n > 3$ there are $1 \leq r, s \leq n$ with $r \neq s$ and $x_r, x_s \in Fic(h) \setminus \{x_i\}$. We shall prove that $ess(f_{r \leftarrow s}) = n - p$ and $x_s \notin Ess(f_{r \rightarrow s})$.

By Lemma 3.2, we have

$$f_{i \leftarrow j} = [f_{r \leftarrow s}]_{i \leftarrow j}.$$

This gives $n - p = ess(f_{i \leftarrow j}) \leq ess(f_{r \leftarrow s}) \leq n - p$ since $gap(f) = p$, i.e. $ess(f_{r \leftarrow s}) = n - p$. Further, let $x_m \in Ess(h) \setminus \{x_j\}$. Assume that $x_m \notin Ess(f_{r \rightarrow s})$. Then

$$f_{r \leftarrow s} = [f_{r \leftarrow s}]_{m \leftarrow i} \quad \text{and thus} \quad [f_{r \leftarrow s}]_{i \leftarrow j} = [[f_{r \leftarrow s}]_{m \leftarrow i}]_{i \leftarrow j}.$$

Moreover, $h = h_{r \leftarrow s}$ since $x_r \notin Ess(h)$. Because of $r \neq i$, we have

$$h_{r \leftarrow s} = [f_{r \leftarrow s}]_{i \leftarrow j} \quad \text{and thus} \quad h = [[f_{r \leftarrow s}]_{m \leftarrow i}]_{i \leftarrow j}.$$

Since $x_m \neq x_j$ we have $x_m \notin Ess([[f_{r \leftarrow s}]_{m \leftarrow i}]_{i \leftarrow j})$, i.e. $x_m \notin Ess(h)$, which is a contradiction. This shows that

$$Ess(h) \setminus \{x_j\} \subseteq Ess(f_{r \leftarrow s}) \quad \text{and thus} \quad Fic(f_{r \leftarrow s}) \subseteq \{x_j\} \cup Fic(h).$$

Now let us prove that $x_s \notin Ess(f_{r \leftarrow s})$. For suppose this were not true i.e. $x_s \in Ess(f_{r \leftarrow s})$. Because of $n - p = ess(f_{r \leftarrow s}) = ess(f_{i \leftarrow j})$, then $x_s \notin Fic(f_{r \leftarrow s})$ implies $Fic(f_{r \leftarrow s}) = (\{x_j\} \cup Fic(h)) \setminus \{x_s\}$. This provides $x_i, x_j \in Fic(f_{r \leftarrow s})$. Then we have $f_{r \leftarrow s} = [f_{r \leftarrow s}]_{i \leftarrow j}$. Moreover, $h = h_{r \leftarrow s}$. Since $r \neq i$, we have $[f_{r \leftarrow s}]_{i \leftarrow j} = h_{r \leftarrow s}$. Altogether, this provides $f_{r \leftarrow s} = h$. This shows that $Fic(h) = Fic(f_{r \leftarrow s})$, and in particular, $x_j \in Fic(h)$ which is a contradiction. Hence $x_s \notin Ess(f_{r \leftarrow s})$.

Second, we shall prove that $h = f_{u \leftarrow v}$ for all $1 \leq u, v \leq n$ with $u \neq v$ and $x_u \in Fic(h)$.

Let $1 \leq r, s \leq n$ with $r \neq s$ and $x_r \in Fic(h)$. Assume that $r \neq i$. By the same arguments as in the previous, we can show that $Ess(h) \subseteq Ess(f_{r \leftarrow s})$.

Because of $gap(f) = p$, this implies $Ess(h) = Ess(f_{r\leftarrow s})$. Let us assume that $s \neq i$. Since $x_r \notin Ess(h)$ and $x_i \notin Ess(h) = Ess(f_{r\leftarrow s})$, we have

$$h = h_{r\leftarrow s} \quad \text{and} \quad f_{r\leftarrow s} = [f_{r\leftarrow s}]_{i\leftarrow l}$$

for $l \in \{j, s\}$. Since $s \neq i$ and $r \neq i$, it is easy to see that $h_{r\leftarrow s} = [f_{r\leftarrow s}]_{i\leftarrow j}$ when $r \neq j$ and $h_{r\leftarrow s} = [f_{r\leftarrow s}]_{i\leftarrow s}$ when $r = j$. Altogether, this gives $f_{r\leftarrow s} = h$.

Next, let us assume that $s = i$. Since $x_j \notin Ess(h) = Ess(f_{r\leftarrow s})$, we have $f_{r\leftarrow s} = [f_{r\leftarrow s}]_{j\leftarrow s}$. Moreover, $x_j, x_r \notin Ess(h)$ implies $h = [h_{r\leftarrow s}]_{j\leftarrow s}$. Since $s = i$, we have

$$[f_{r\leftarrow s}]_{j\leftarrow s} = [h_{r\leftarrow s}]_{j\leftarrow s} \quad \text{and thus} \quad f_{r\leftarrow s} = h.$$

Finally, assume that $r = i$ and $s \neq i$. Because of $x_j \notin Ess(h)$, we have in particular $h = f_{j\leftarrow i}$ where $x_i \notin Ess(h) = Ess(f_{j\leftarrow i})$ (as we have shown in the previous). We might choose $\widetilde{h} := f_{j\leftarrow i}$ (instead of $h := f_{i\leftarrow j}$) and obtain $f_{j\leftarrow i} = f_{i\leftarrow s}$ by the previous considerations. Altogether, we have $h = f_{i\leftarrow s}$. ∎

Lemma 3.4. *Let $f \in G_{p,k}^n$. Then the following conditions hold:*

  (i) *If $2 < p < n$ then there exist $u, v \in \{1, \dots, n\}$ such that $f_{u\leftarrow v}$ depends essentially on $n - p$ variables and $x_v \in Ess(f_{u\leftarrow v})$;*

  (ii) *If $2 < p \leq n$ then there exist $u, v \in \{1, \dots, n\}$ such that $f_{u\leftarrow v}$ depends essentially on $n - p$ variables and $x_v \notin Ess(f_{u\leftarrow v})$.*

Proof. By Theorem 3.3, there are $u, v \in \{1, \dots, n\}$ such that $ess(f_{u\leftarrow v}) = n - p$ and $f_{u\leftarrow v} = f_{i\leftarrow j}$ for $1 \leq i, j \leq n$ with $i \neq j$ and $x_i \in Fic(f_{u\leftarrow v})$.

  (i) Since $p < n$, there is an $l \in \{1, \dots, n\}$ with $x_l \in Ess(f_{u\leftarrow v})$. Then $f_{u\leftarrow v} = f_{u\leftarrow l}$ since $x_u \in Fic(f_{u\leftarrow v})$. This shows $x_l \in Ess(f_{u\leftarrow l})$ where $ess(f_{u\leftarrow l}) = n - p$.

  (ii) It was already proved in Theorem 3.3. ∎

Theorem 3.4. *Let f be a k-valued function which depends essentially on the all of its n variables and $2 < p < n \leq k$. Then $f \in G_{p,k}^n$ if and only if there exist $n - p$ variables $y_{l_1}, \ldots, y_{l_{n-p}} \in X_n$ such that*

$$(5) \qquad\qquad\qquad\qquad f = h \oplus g,$$

where $Ess(h) = \{y_{l_1}, \ldots, y_{l_{n-p}}\}$ and $g \in G_{n,k}^n$. Moreover, $g_{i \leftarrow j} = 0$ for all $1 \leq i, j \leq n$ with $i \neq j$.

Proof. " $\Leftarrow$ " Let f be represented in the form (5), where h depends essentially on all of its $n - p$ variables and $g \in G_{n,k}^n$. With no loss of generality we might assume that h is an $(n - p)$-ary k-valued function.

Again, without loss of generality, let us assume that $y_{l_1}, \ldots, y_{l_{n-p}} = x_1, \ldots, x_{n-p}$.

Since $2 < p < n \leq k$ there is at least one variable $x_j \in X_n$ on which h does not depend essentially i.e. $n - p < j \leq n$. Then from Lemma 2.2 we have $Ess(h_{j \leftarrow i}) = Ess(h)$ for all i, $1 \leq i \leq n - p$. Since $g \in G_{n,k}^n$ it follows that $ess(g_{u \leftarrow v}) = 0$ for all u and v with $1 \leq u, v \leq n$ and $u \neq v$. Hence $ess(f_{u \leftarrow v}) = ess(h_{u \leftarrow v})$ for all u, v with $1 \leq u, v \leq n$ and $u \neq v$. On the other hand $ess(h_{u \leftarrow v}) = n - p$ when $u > n - p$. Consequently,

$$\max_{t \in Min(f)} ess(t) = n - p \quad \text{and hence} \quad gap(f) = p.$$

" $\Rightarrow$ " From Lemma 3.4 (ii) it follows that there are $u, v \in \{1, \ldots, n\}$ such that $f_{u \leftarrow v}$ depends essentially on $n - p$ variables and $x_v \notin Ess(f_{u \leftarrow v})$. With no loss of generality let us assume that $(v, u) = (n - 1, n)$ and let $h := f_{n \leftarrow n-1}$, where $Ess(h) = \{x_1, \ldots, x_{n-p}\}$.

Let $g \in P_k^n$ be the function defined by

$$(6) \qquad\qquad\qquad\qquad g := f \ominus h$$

i.e. g is the unique function in P_k^n such that $g \oplus h = f$.

We have to prove that $g \in G_{n,k}^n$ i.e. $Ess(g) = X_n$ and $g_{i \leftarrow j} = 0$ for all i, j with $1 \leq j, i \leq n$ and $i \neq j$.

First, $\{x_{n-p+1}, \ldots, x_n\} \subseteq Ess(g)$ because $Ess(f) = X_n$ and $\{x_{n-p+1}, \ldots, x_n\} \cap Ess(h) = \emptyset$.

Second, to prove that $\{x_1, \ldots, x_{n-p}\} \subseteq Ess(g)$ we shall suppose that this is not the case and without loss of generality assume that $x_1 \notin Ess(g)$. Then from Theorem 3.3 we obtain

$$f_{n\leftarrow 1} = h(x_1, x_2, \ldots, x_{n-p}) \oplus g_{n\leftarrow 1} = h(x_1, x_2, \ldots, x_{n-p}).$$

Since K is an additive group it follows that $g_{n\leftarrow 1} = 0$. Consequently $g_{1\leftarrow n} = 0$ and from Lemma 2.2 we obtain $g = g_{1\leftarrow n} = 0$ which is a contradiction. Hence $\{x_1, \ldots, x_{n-p}\} \subseteq Ess(g)$ and $Ess(g) = X_n$.

To prove that $ess(g_{i\leftarrow j}) = 0$ for all i, j with $1 \leq i, j \leq n$ and $i \neq j$, we shall consider several cases.

Case 1. Let $n - p < j, i < n$ and $i \neq j$. According to (6) we have $g_{n\leftarrow n-1} = f_{n\leftarrow n-1} \ominus h = 0$. From Theorem 3.3, it follows that $f_{i\leftarrow j} = f_{n\leftarrow n-1}$ and from Lemma 2.2 we have $h_{i\leftarrow j} = h$ which implies $g_{i\leftarrow j} = 0$ i.e. $ess(g_{i\leftarrow j}) = 0$ for all i, j with $n - p < j, i < n$ and $i \neq j$.

Case 2. Let $j \leq n - p < i$. We may assume that $j = 1$ and $i = n$. Since $x_n \notin Ess(h)$, we have $h = f_{n\leftarrow n-1} = f_{n\leftarrow 1}$ by Theorem 3.3. Moreover, from Lemma 2.2 we have $h = h_{n\leftarrow 1}$. Thus $g_{n\leftarrow 1} = f_{n\leftarrow 1} \ominus h_{n\leftarrow 1} = h \ominus h = 0$. On the other hand we have

$$h_{1\leftarrow n} = [f_{n\leftarrow n-1}]_{1\leftarrow n} = [f_{n\leftarrow 1}]_{1\leftarrow n} = f_{1\leftarrow n},$$

i.e. $h_{1\leftarrow n} = f_{1\leftarrow n}$ and thus $g_{1\leftarrow n} = f_{1\leftarrow n} \ominus h_{1\leftarrow n} = 0$. Hence $g_{1\leftarrow n} = g_{n\leftarrow 1} = 0$.

Case 3. Let $i, j \leq n - p$ and $i \neq j$. By symmetry we may assume that $j = 1$ and $i = 2$. We have to prove that $g_{2\leftarrow 1} = 0$.

First let us assume that there is some $u > n - p$ such that $x_u \notin Ess(f_{2\leftarrow 1})$. According to Case 2 we have

$$g_{u\leftarrow 1} = g_{u\leftarrow 2} = g_{1\leftarrow u} = g_{2\leftarrow u} = 0.$$

From Lemma 3.3 we obtain $g_{2\leftarrow u} = [g_{2\leftarrow 1}]_{u\leftarrow 1}$ and Lemma 2.2 implies $g_{2\leftarrow 1} = [g_{2\leftarrow 1}]_{u\leftarrow 1}$. Hence $g_{2\leftarrow u} = g_{2\leftarrow 1} = 0$.

Second, assume that $\{x_{n-p+1}, \ldots, x_n\} \subseteq Ess(f_{2\leftarrow 1})$. Since $p \geq 3$ then $f \in G^n_{p,k}$ implies $ess(f_{2\leftarrow 1}) < n - 2$.

Hence in the whole Case 3 the function $f_{2\leftarrow 1}$ can essentially depend on at most $n-3$ variables. Then there is a variable x_v, $3 \leq v \leq n-p$, which is not essential in $f_{2\leftarrow 1}$. Without loss of generality let us assume that $v = 3$. From Theorem 3.3 we have $f_{n-1\leftarrow 3} = h$ and hence $f_{3\leftarrow n-1} = h_{3\leftarrow n-1}$. This implies $[f_{3\leftarrow n-1}]_{2\leftarrow 1} = [h_{3\leftarrow n-1}]_{2\leftarrow 1}$. Hence $[f_{3\leftarrow n-1}]_{2\leftarrow 1} = [f_{2\leftarrow 1}]_{3\leftarrow n-1} = f_{2\leftarrow 1}$ and $[h_{3\leftarrow n-1}]_{2\leftarrow 1} = [[f_{2\leftarrow 1}]_{3\leftarrow n-1}]_{n\leftarrow n-1}$. From $f_{2\leftarrow 1} = [f_{2\leftarrow 1}]_{3\leftarrow n-1}$ it follows

$$[h_{3\leftarrow n-1}]_{2\leftarrow 1} = [f_{2\leftarrow 1}]_{n\leftarrow n-1} = [f_{n\leftarrow n-1}]_{2\leftarrow 1} = h_{2\leftarrow 1}.$$

Altogether, we have $f_{2\leftarrow 1} = h_{2\leftarrow 1}$, i.e. $g_{2\leftarrow 1} = 0$ and $ess(g_{2\leftarrow 1}) = 0$.

Consequently, $g \in G^n_{n,k}$ and $g_{i\leftarrow j} = 0$ for all $1 \leq i, j \leq n$ with $i \neq j$. ∎

Corollary 3.2. *Let $f \in G^n_{p,k}$. Then there is a partition of the set $Ess(f) = \{x_1, \ldots, x_n\}$ into the sets $V := Ess(h)$ and $W := Ess(f) \setminus V$, where h is the function defined in the proof of Theorem 3.4, such that*

$$(x_i, x_j) \in W^2 \Rightarrow \big(ess(f_{i\leftarrow j}) = n - p \ \& \ x_j \notin Ess(f_{i\leftarrow j})\big)$$

and

$$(x_i, x_j) \in W \times V \Rightarrow \big(ess(f_{i\leftarrow j}) = n - p \ \& \ x_j \in Ess(f_{i\leftarrow j})\big).$$

Theorem 3.5. *If $2 < p < n \leq k$, then*

$$|G^n_{p,k}| = \big(k^{\binom{k}{n}.n!} - 1\big).\sum_{j=p}^{n}(-1)^{j-p}\binom{j}{p}\binom{n}{j}.k^{k^{n-j}}.$$

Proof. Theorem 3.1 shows that the function g in (5) is unique which implies that the representation of f in (5) is unique, also. Let $f = h \oplus g \in G^n_{p,k}$ and $g_{i\leftarrow j} = 0$ for all $1 \leq i, j \leq n$ with $i \neq j$. Let $a \in K$ be a non-zero natural number from K i.e. $0 < a \leq k - 1$. Then clearly $t = g \oplus a \in G^n_{n,k}$ and $t_{i\leftarrow j} = a$ for all $1 \leq i, j \leq n$ with $i \neq j$. According to Theorem 3.2 the number of functions $g \in G^n_{n,k}$ with $g_{i\leftarrow j} = 0$ for all $1 \leq i, j \leq n$ when $i \neq j$ is equal to $|G^n_{n,k}|/k = \big(k^{\binom{k}{n}.n!} - 1\big)$.

It is well known that the number of all functions in P_k^n which depend essentially on exactly $n - p$ variables is equal to $|P_{n-p,k}^n| = \sum_{j=p}^{n} (-1)^{j-p} \binom{j}{p} \binom{n}{j} . k^{k^{n-j}}$.

Consequently,

$$|G_{p,k}^n| := (|G_{n,k}^n|/k) . (|P_{n-p,k}^n|)$$

$$= \left(k^{\binom{k}{n} . n!} - 1 \right) . \sum_{j=p}^{n} (-1)^{j-p} \binom{j}{p} \binom{n}{j} . k^{k^{n-j}} .$$

∎

4. The class $G_{2,k}^n$ with $4 \leq n \leq k$

There are two subclasses of the class $G_{2,k}^n$ with $4 \leq n \leq k$. First one, consists of functions satisfying conditions similar as the conditions in Theorem 3.3, Lemma 3.4 and Theorem 3.4 for $p = 2$. The second subclass consists of functions whose behavior is similar to the functions from $G_{2,k}^n$ with $n > k$ (Theorem 2.1 in [15]).

Lemma 4.1. *Let f be a k-valued function which depends essentially on all of its n, $n > 3$ variables and $gap(f) = 2$. Then there exist two distinct essential variables x_u, x_v such that $ess(f_{u \leftarrow v}) = n - 2$, and $x_v \notin Ess(f_{u \leftarrow v})$. Moreover, $ess(f_{u \leftarrow m}) = ess(f_{v \leftarrow m}) = n - 2$ for all m, $1 \leq m \leq n$ with $m \notin \{u, v\}$.*

Proof. Since $gap(f) = 2$, there are $1 \leq i, j \leq n$ with $i \neq j$ and $ess(f_{i \leftarrow j}) = n - 2$.

If $x_j \notin Ess(f_{i \leftarrow j})$ we are done.

Let us assume that $x_j \in Ess(f_{i \leftarrow j})$. Since $gap(f) = 2$, there is a $w \in \{1, \ldots, n\} \setminus \{i, j\}$ such that $x_w \notin Ess(f_{i \leftarrow j})$. From Lemma 3.2, we obtain

$$[f_{w \leftarrow j}]_{i \leftarrow j} = f_{i \leftarrow j} \quad \text{and} \quad n - 2 = ess(f_{i \leftarrow j}) \leq ess(f_{w \leftarrow j}).$$

Hence $ess(f_{i \leftarrow j}) = ess(f_{w \leftarrow j}) = n - 2$ since $gap(f) = 2$.

We shall prove that $x_j \notin Ess(f_{w \leftarrow j})$ or $x_i \notin Ess(f_{w \leftarrow i})$ which will complete the proof.

For suppose this were not true. Then $x_j \in Ess(f_{w \leftarrow j})$ and $x_i \in Ess(f_{w \leftarrow i})$. Assume that $x_i \in Ess(f_{w \leftarrow j})$. Then there is a $r \in \{1, \ldots, n\} \setminus \{i, j, w\}$ such that $x_r \notin Ess(f_{w \leftarrow j})$. Then $x_r \notin Ess([f_{w \leftarrow j}]_{i \leftarrow j})$, i.e. $x_r \notin Ess(f_{i \leftarrow j})$, which is a contradiction. Hence $x_i \notin Ess(f_{w \leftarrow j})$. By similar arguments, we obtain that $x_j \notin Ess(f_{w \leftarrow i})$.

Thus we have $Ess(f_{i \leftarrow j}) = X_n \setminus \{x_i, x_w\}$, $Ess(f_{w \leftarrow i}) = X_n \setminus \{x_j, x_w\}$ and $Ess(f_{w \leftarrow j}) = X_n \setminus \{x_i, x_w\}$.

Since $n > 3$ it follows that there is at least one essential variable x_s in f with $s \in \{1, \ldots, n\} \setminus \{i, j, w\}$.

Now, the equation $Ess(f_{w \leftarrow s}) = X_n \setminus \{x_w\}$ is implicit in [15] (Lemma 1.1 (5)). This equation contradicts $gap(f) = 2$.

Let $m \notin \{u, v\}$ be a natural number with $1 \leq m \leq n$. Lemma 3.2 implies $f_{u \leftarrow v} = [f_{u \leftarrow m}]_{v \leftarrow m}$ for all $1 \leq m \leq n$ with $m \notin \{u, v\}$. Hence $n - 2 = ess(f_{u \leftarrow v}) = ess([f_{u \leftarrow m}]_{v \leftarrow m}) \leq ess(f_{u \leftarrow m})$. Now, $gap(f) = 2$ shows that $ess(f_{u \leftarrow m}) = n - 2$ and by symmetry we obtain $ess(f_{v \leftarrow m}) = n - 2$. ∎

Let us denote by $G_{p,k}^{n,+}$ the set of all functions $f \in G_{p,k}^n$ for which there exist i and j with $1 \leq i, j \leq n$, $i \neq j$ such that $x_j \in Ess(f_{i \leftarrow j})$ and $ess(f_{i \leftarrow j}) = n - p$.

$G_{p,k}^{n,-}$ denotes the set of all functions $f \in G_{p,k}^n$ for which $x_v \notin Ess(f_{u \leftarrow v})$ for all $1 \leq u, v \leq n$ with $u \neq v$.

Proposition 4.1. *If $3 < n \leq k$ then $G_{2,k}^n = G_{2,k}^{n,+} \cup G_{2,k}^{n,-}$.*

Proof. Clearly, $G_{2,k}^{n,+} \cup G_{2,k}^{n,-} \subseteq G_{2,k}^n$. Let $f \in G_{2,k}^n$. Then Lemma 4.1 implies that there exist two distinct essential variables x_u, x_v such that $ess(f_{u \leftarrow v}) = n - 2$, and $x_v \notin Ess(f_{u \leftarrow v})$.

If $x_j \notin Ess(f_{i \leftarrow j})$ for all $1 \leq i, j \leq n$ with $i \neq j$ then $f \in G_{2,k}^{n,-}$.

Next, assume that there are $1 \leq i, j \leq n$ with $x_j \in Ess(f_{i \leftarrow j})$. We have to prove that $f \in G_{2,k}^{n,+}$ i.e. there exist r, s with $1 \leq r, s \leq n$, $x_r \in Ess(f_{s \leftarrow r})$ and $ess(f_{s \leftarrow r}) = n - 2$.

If $\{i, j\} \cap \{u, v\} \neq \emptyset$ we are done because of Lemma 4.1.

Let $\{i, j\} \cap \{u, v\} = \emptyset$. From Lemma 4.1 we have $ess(f_{u \leftarrow i}) = ess(f_{u \leftarrow j}) = ess(f_{v \leftarrow i}) = ess(f_{v \leftarrow j}) = n - 2$. If $x_j \in Ess(f_{u \leftarrow j})$ we are done as above. If $x_j \notin Ess(f_{i \leftarrow j})$ then we have $x_i \in Ess(f_{i \leftarrow j})$ and $ess(f_{u \leftarrow i}) = n - 2$. ∎

Corollary 4.1.

$$G_{p,k}^n = \begin{cases} G_{p,k}^{n,+} & if \quad 2 < p < n \le k \\[2mm] G_{p,k}^{n,-} & if \quad (2 \le n \le k \ \& \ p = n) \quad or \ (n > k) \\[2mm] G_{p,k}^{n,+} \cup G_{p,k}^{n,-} & if \quad 3 < n \le k \ \& \ p = 2. \end{cases}$$

Proof. This representation of the set $G_{p,k}^n$ follows by Theorem 3.4, Theorem 3.1, Proposition 4.1, Theorem 3.1 in [14] and Theorem 2.1 in [15]. ∎

4.1. The subclass $G_{2,k}^{n,+}$

Theorem 4.1 *Let $4 \le n \le k$ and $f \in P_k^n$. Then the following statements are equivalent:*

(i) *$f \in G_{2,k}^{n,+}$;*

(ii) *There is a function $h \in P_k^n$ with $ess(h) = n - 2$ and $f_{r\leftarrow s} = h$ for all $1 \le r, s \le n$ with $r \ne s$ and $x_r \in Fic(h)$.*

Proof. (ii) $\Rightarrow$ (i) is clear.

(i) $\Rightarrow$ (ii). Let i and j be two distinct natural numbers fro which $1 \le i, j \le n$, $x_j \in Ess(f_{i\leftarrow j})$ and $ess(f_{i\leftarrow j}) = n - 2$. Following the proof of Lemma 4.1 we might conclude that there exists an essential variable x_w in f such that $x_w \notin Ess(f_{i\leftarrow j})$, $ess(f_{w\leftarrow i}) = ess(f_{w\leftarrow j}) = n - 2$, and $x_i \notin Ess(f_{w\leftarrow i})$ or $x_j \notin Ess(f_{w\leftarrow j})$. Without loss of generality let us assume that $x_i \notin Ess(f_{w\leftarrow i})$ i.e. $Ess(f_{w\leftarrow i}) = X_n \setminus \{x_w, x_i\}$.

From Lemma 4.1 we have $ess(f_{w\leftarrow r}) = ess(f_{i\leftarrow r}) = n - 2$ for all r, $1 \le r \le n$ with $r \notin \{w, i\}$.

First, we shall show that $X_n \setminus \{x_w, x_i, x_r\} \subset Ess(f_{w\leftarrow r})$ and $X_n \setminus \{x_w, x_i, x_r\} \subset Ess(f_{i\leftarrow r})$ for all r, $1 \le r \le n$ with $r \notin \{w, i\}$. Note that $X_n \setminus \{x_w, x_i, x_r\} \ne \emptyset$ because $n > 3$. Since $x_i \notin Ess(f_{w\leftarrow i})$ and from Lemma 3.3 it follows

$$f_{w\leftarrow i} = [f_{w\leftarrow r}]_{i\leftarrow r} = [f_{w\leftarrow i}]_{i\leftarrow r}.$$

Let x_s be an essential variable in f with $s \notin \{w, i, r\}$. Suppose that $x_s \notin Ess(f_{w \leftarrow r})$. Hence $x_s \notin Ess([f_{w \leftarrow i}]_{i \leftarrow r})$ i.e. $x_s \notin Ess(f_{w \leftarrow i})$, which contradicts $Ess(f_{w \leftarrow i}) = X_n \setminus \{x_w, x_i\}$.

Second, we shall prove that $x_r \in Ess(f_{w \leftarrow r})$ and $x_r \in Ess(f_{i \leftarrow r})$ for all r, $1 \leq r \leq n$ with $r \notin \{w, i\}$. Since $x_j \in Ess(f_{i \leftarrow j})$ it follows that $j \notin \{w, i\}$.

Consider the case $r = j$. We have known that $x_j \in Ess(f_{i \leftarrow j})$ and from $x_w \notin Ess(f_{i \leftarrow j})$ we obtain

$$f_{i \leftarrow j} = [f_{i \leftarrow j}]_{w \leftarrow j} = [f_{i \leftarrow w}]_{w \leftarrow j} = f_{i \leftarrow w}.$$

Hence $x_j \in Ess(f_{w \leftarrow j})$.

Assume that $r \notin \{w, i, j\}$. Suppose that $x_r \notin Ess(f_{i \leftarrow r})$ and from Lemma 3.3 we have $f_{i \leftarrow r} = [f_{i \leftarrow j}]_{r \leftarrow j}$. Since $x_w \notin Ess(f_{i \leftarrow j})$ we have $x_w \notin Ess([f_{i \leftarrow j}]_{r \leftarrow j})$ i.e. $x_w \notin Ess(f_{i \leftarrow r})$. Then $ess(f_{i \leftarrow r}) = n - 3$ because $x_r, x_i, x_w \notin Ess(f_{i \leftarrow r})$ which is a contradiction. Hence $x_r \in Ess(f_{i \leftarrow r})$. In a similar way it might be shown that $x_r \in Ess(f_{w \leftarrow r})$.

Hence $Ess(f_{i \leftarrow r}) = Ess(f_{w \leftarrow r}) = Ess(f_{w \leftarrow i})$ for all r, $1 \leq r \leq n$ with $r \notin \{w, i\}$.

Finally, let us set $h := f_{w \leftarrow i}$. Clearly $Ess(h) = X_n \setminus \{x_w, x_i\}$ and $ess(h) = n - 2$. Let r and s be two natural numbers such that $1 \leq r, s \leq n$, $r \neq s$ and $r \in \{w, i\}$. With no loss of generality let us assume that $r = w$. Since $x_i \notin Ess(f_{r \leftarrow i})$ then Lemma 3.3 implies

$$h = f_{r \leftarrow i} = [f_{r \leftarrow s}]_{i \leftarrow s} = [f_{i \leftarrow s}]_{r \leftarrow s} = f_{r \leftarrow s},$$

because $x_i \notin Ess(f_{r \leftarrow s})$.   ∎

Corollary 4.2. *If $f \in G_{2,k}^{n,+}$, $n > 3$ then there exist $x_u, x_v \in Ess(f)$ such that $f_{r \leftarrow s} = f_{u \leftarrow v} = h$ for all $r \in \{u, v\}$ and $s \in \{1, \ldots, n\}$, $s \neq r$, as well as $ess(f_{u \leftarrow v}) = n - 2$.*

Theorem 4.2. *Let f be a k-valued function which depends essentially on the all of its n variables, $n > 3$. Then the following sentences are equivalent:*

(i) $f \in G_{2,k}^{n,+}$;

(ii) *There exist $n-2$ variables $y_{l_1}, \ldots, y_{l_{n-2}} \in X_n$ such that*

$$f = h \oplus g,$$

where $Ess(h) = \{y_{l_1}, \ldots, y_{l_{n-2}}\}$ and $g \in G_{n,k}^n$. Moreover $g_{i \leftarrow j} = 0$ for all $1 \leq i, j \leq n$ with $i \neq j$.

Proof. We might prove the theorem in a similar way as Theorem 3.4 by using Theorem 4.1 instead of Theorem 3.3. ∎

Proposition 4.2. $|G_{2,k}^{n,+}| = \left(k^{\binom{k}{n} \cdot n!} - 1\right) \cdot \sum_{j=2}^{n} (-1)^j \binom{j}{2} \binom{n}{j} \cdot k^{k^{n-j}}.$

The proof can be done in a similar way as the proof of Theorem 3.5.

4.2. The subclass $G_{2,k}^{n,-}$

Thus Lemma 4.1 implies $G_{2,k}^{n,+} \cup G_{2,k}^{n,-} = G_{2,k}^n$. We are going to describe the class $G_{2,k}^{n,-}$ when $3 < n \leq k$.

The next theorem is proved for $n > k$ by R. Willard in [15].

Theorem 4.3. *Let f be a k-valued function which depends essentially on all of its n, $n > 3$ variables. If $f \in G_{2,k}^{n,-}$ then each identification minor of f is a symmetric function with respect to its essential variables.*

Proof. Since $G_{2,k}^{n,-} \subset G_{2,k}^n$ then Lemma 4.1 implies that there exist two distinct essential variables x_u, x_v such that $ess(f_{u \leftarrow v}) = n - 2$, and $x_v \notin Ess(f_{u \leftarrow v})$. With no loss of generality let us assume that $(v, u) = (1, 2)$ and it is enough to prove that $f_{2 \leftarrow 1}$ is a symmetric function with respect to the variables from the set $Ess(f_{2 \leftarrow 1})$. Since $Ess(f_{2 \leftarrow 1}) = \{x_3, \ldots, x_n\}$ there is an $n - 2$-ary function $h : K^{n-2} \to K$ such that $f_{2 \leftarrow 1} = h(x_3, \ldots, x_n)$. By symmetry, we have to prove that

$$h(x_3, x_4, x_5, \ldots, x_n) = h(x_4, x_3, x_5, \ldots, x_n).$$

In fact, we obtain

$$h(x_3, x_4, x_5 \ldots, x_n)$$

$$= f(x_1, x_1, x_3, x_4, x_5, \ldots, x_n)$$

$$= f(x_3, x_3, x_3, x_4, x_5, \ldots, x_n) \quad (\text{ since } x_1 \notin Ess(f_{2\leftarrow1}))$$

$$= f(x_4, x_3, x_4, x_4, x_5, \ldots, x_n) \quad (\text{ since } x_3 \notin Ess(f_{1\leftarrow3}))$$

$$= f(x_4, x_3, x_3, x_3, x_5, \ldots, x_n) \quad (\text{ since } x_4 \notin Ess(f_{3\leftarrow4}))$$

$$= f(x_4, x_4, x_4, x_3, x_5, \ldots, x_n) \quad (\text{ since } x_3 \notin Ess(f_{2\leftarrow3}))$$

$$= f(x_1, x_1, x_4, x_3, x_5, \ldots, x_n) \quad (\text{ since } x_1 \notin Ess(f_{2\leftarrow1}))$$

$$= h(x_4, x_3, x_5, \ldots, x_n).$$

∎

We can now give a representation of the functions in $G_{2,k}^{n,-}$ (see also Theorem 16 in [6])

Theorem 4.4. *Let f be an n-ary k-valued function with $3 < n \leq k$. Then the following sentences are equivalent:*

(i) $f \in G_{2,k}^{n,-}$;

(ii) $f = t \oplus g$ *where $g \in G_{n,k}^n$ and t is an n-ary totally symmetric function with $Ess(t_{i\leftarrow j}) = X_n \setminus \{x_i, x_j\}$ for all $i, j \in \{1, \ldots, n\}$, $i \neq j$. Moreover $g_{i\leftarrow j} = 0$ for all $1 \leq i, j \leq n$ with $i \neq j$.*

Proof. "⇐" Clearly, $ess(t_{i\leftarrow j}) = n - 2$ and $ess(g_{i\leftarrow j}) = 0$ for all $i, j \in \{1, \ldots, n\}$, $i \neq j$. Hence $ess(f_{i\leftarrow j}) = n-2$ and $gap(f) = 2$. We have to prove that $Ess(f) = X_n$. By symmetry it is enough to show that $x_1 \in Ess(t \oplus g)$. Since $n > 3$ we have

$$f(x_1, x_2, x_2, x_4, \ldots, x_n) = f_{3\leftarrow2} = t_{3\leftarrow2} \oplus g_{3\leftarrow2}$$

and

$$Ess(f_{3\leftarrow2}) = Ess(t_{3\leftarrow2} \oplus g_{3\leftarrow2}) = \{x_1, x_4, \ldots, x_n\} = Ess(t_{3\leftarrow2}) \subseteq Ess(f).$$

"⇒" Let $f \in G_{2,k}^n$. Let us set $f = t \oplus g$, where

$$t = \bigoplus_{\alpha_1\alpha_2\ldots\alpha_n \in Eq_k^n} a_m . x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$$

and

$$g = \bigoplus_{\beta_1\beta_2\ldots\beta_n \notin Eq_k^n} a_r . x_1^{\beta_1} x_2^{\beta_2} \ldots x_n^{\beta_n},$$

with $m = \overline{\alpha_1\alpha_2\ldots\alpha_n}$ and $r = \overline{\beta_1\beta_2\ldots\beta_n}$.

Note that such representation of f can be obtained after a suitable reordering of the conjunctions in its SC-form.

Clearly, $g_{i \leftarrow j} = 0$ for all $i, j \in \{1, \ldots, n\}$, $i \neq j$.

From Theorem 4.3 it follows that $f_{i \leftarrow j}$ is totally symmetric and $Ess(f_{i \leftarrow j}) = X_n \setminus \{x_i, x_j\}$ for all $i, j \in \{1, \ldots, n\}$, $i \neq j$. Since $f_{i \leftarrow j} = t_{i \leftarrow j}$ from Theorem 4.3 we can conclude that $t_{i \leftarrow j}$ is totally symmetric. We have to show that t is totally symmetric, also. Since $Ess(t_{i \leftarrow j}) = X_n \setminus \{x_i, x_j\}$ there is an $n - 2$-ary function h such that $t_{i \leftarrow j} = h$, where h is a totally symmetric function (according to Theorem 4.3) which depends essentially on all of its variables. By $t(\alpha_1, \ldots, \alpha_n) = 0$ when $\alpha_1 \ldots \alpha_n \notin Eq_k^n$ it suffices to prove that

(7)
$$t(\alpha_1, \ldots, \alpha_{j-1}, \beta, \alpha_{j+1}, \ldots, \alpha_{i-1}, \beta, \alpha_{i+1}, \ldots, \alpha_n)$$
$$= h(\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n),$$

for all $i, j \in \{1, \ldots, n\}$, $i \neq j$.

First we shall prove (7) for $j = 1$ and $i = 3$. Since $x_1 \notin Ess(f_{3 \leftarrow 1})$ we obtain

$$t(\alpha_1, \alpha_2, \alpha_1, \ldots, \alpha_n) = t(\alpha_2, \alpha_2, \alpha_2, \ldots, \alpha_n) = h(\alpha_2, \ldots, \alpha_n)$$

as desired.

In a similar way we might show (7) for $j = 2$ and $i = 3$. As in the proof of Theorem 4.3 we may reorder the variables and show (7) for all $i, j \in \{1, \ldots, n\}$, $i \neq j$. ∎

Corollary 4.3. *Let $f \in G_{2,k}^n$, $3 < n \le k$ and $x_v \notin Ess(f_{u \leftarrow v})$ for all $v, u \in \{1, \ldots, n\}$, $u \ne v$. Then there exists an $n - 2$-ary totally symmetric function h such that*

$$
\begin{aligned}
f_{i \leftarrow j} &= f(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_n) \\
&= h(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),
\end{aligned}
$$

for all $i, j \in \{1, \ldots, n\}$, $i \ne j$.

Remark 4.1. We shall use some notation and results from [1] and [15].

Let $Sub(K)$ denote the set of all subsets of K. Define $oddsupp : K^n \to Sub(K)$ as follows

$$
oddsupp(\alpha_1, \ldots, \alpha_n) := \{\alpha_v : |\{u : 1 \le u \le n \ \& \ \alpha_u = \alpha_v\}| \text{ is odd}\}.
$$

A function f is determined by *oddsupp* if there exists a function $f^* : Sub(K) \to K$ such that $f = f^* \circ oddsupp$. It is easy to see that the functions which are determined by *oddsupp* are totally symmetric, too. Then from Theorem 16 [6] it follows that the function t defined in Theorem 4.4 is determined by *oddsupp*.

5.   The class $G_{2,k}^n$ with $2 \le n \le 3 \le k$

Let us note that the class $G_{2,k}^2$ is described in Section 3 by Theorem 3.1. Thus $f \in G_{2,k}^2$ if and only if

$$
f = \left[\bigoplus_{\alpha \ne \beta} a_r x_1^\alpha x_2^\beta \right] \oplus a_0 . \left(x_1^0 x_2^0 \oplus \ldots \oplus x_1^{k-1} x_2^{k-1} \right)
$$

where at least two among the coefficients $\{a_0\} \cup \{a_r \mid r = \overline{\alpha\beta}, \ \& \ \alpha \ne \beta\}$ are distinct.

Example 5.1. Let $f \in P_3^3$ be the following function

$$
f = x_1^0 x_2^0 x_3^0 \oplus x_1^0 x_2^0 x_3^1 \oplus x_1^0 x_2^0 x_3^2 \oplus x_1^0 x_2^1 x_3^0 \oplus x_1^0 x_2^2 x_3^0 \oplus x_1^1 x_2^0 x_3^0 \oplus x_1^2 x_2^0 x_3^0.
$$

It is easy to check that $f_{2\leftarrow 1} = f_{3\leftarrow 1} = x_1^0$, $f_{3\leftarrow 2} = x_2^0$ and hence $f \in G_{2,3}^3$.

Let $g \in P_3^3$ be the following function

$$g = x_1^0 x_2^0 x_3^0 \oplus x_1^0 x_2^1 x_3^1 \oplus x_1^0 x_2^2 x_3^2 \oplus x_1^1 x_2^0 x_3^1 \oplus x_1^1 x_2^1 x_3^0 \oplus x_1^2 x_2^0 x_3^2 \oplus x_1^2 x_2^2 x_3^0.$$

It is not difficult to see that $g_{2\leftarrow 1} = x_3^0$, $g_{3\leftarrow 1} = x_2^0$, $g_{3\leftarrow 2} = x_1^0$ and hence $g \in G_{2,3}^3$.

Let $h \in P_3^3$ be the following function

$$h = x_1^0 x_2^0 \oplus x_1^0 x_2^1 x_3^1 \oplus x_1^0 x_2^2 x_3^2 \oplus x_1^1 x_2^0 x_3^1 \oplus x_1^2 x_2^0 x_3^2.$$

It is clear that $h_{2\leftarrow 1} = h_{3\leftarrow 2} = x_1^0$, $h_{3\leftarrow 1} = x_2^0$ and hence $h \in G_{2,3}^3$.

Let $r \in P_3^3$ be the following function

$$r = x_2^0 \oplus 2.x_1^1 x_2^0 x_3^2 \oplus 2.x_1^2 x_2^0 x_3^1.$$

It is clear that $r_{3\leftarrow 1} = r_{3\leftarrow 2} = x_2^0$, $r_{2\leftarrow 1} = x_1^0$ and hence $r \in G_{2,3}^3$.

Clearly, the functions f, g and h do not satisfy Theorem 4.1 for $n = 3$, and the functions f, h, and r do not satisfy Theorem 4.4 for $n = 3$, but $g \in G_{2,3}^{3,-}$, and $r \in G_{2,3}^{3,+}$.

Example 5.1 shows that the case $(p, n) = (2, 3)$ is a special case in studying the functions with non-trivial arity gap.

In this section we shall pay attention to description of the class $G_{2,k}^3$ with $k > 2$.

Lemma 5.1. *If $f \in G_{2,k}^3$ then $ess(f_{i\leftarrow j}) = 1$ for all $i, j \in \{1, 2, 3\}$, $i \neq j$.*

Proof. Since $f \in G_{2,k}^3$ there is an identification minor of f which depends essentially on one variable and let us suppose $ess(f_{2\leftarrow 1}) = 1$. There are two possibilities.

(i) $Ess(f_{2\leftarrow1}) = \{x_3\}$. Then there are three constants $\alpha_1, \alpha_3, \beta_3 \in K$ such that $f(\alpha_1, \alpha_1, \alpha_3) \neq f(\alpha_1, \alpha_1, \beta_3)$.

(ii) $Ess(f_{2\leftarrow1}) = \{x_1\}$. Then there exist three constants $\mu_1, \mu_2, \nu_3 \in K$, such that $f(\mu_1, \mu_1, \nu_3) \neq f(\mu_2, \mu_2, \nu_3)$.

Now let us suppose the lemma is false and $ess(f_{3\leftarrow1}) = 0$. This implies that for all four constants $\delta_1, \varepsilon_1, \sigma_2, \psi_2 \in K$ we have $f(\delta_1, \sigma_2, \delta_1) = f(\varepsilon_1, \psi_2, \varepsilon_1)$ which contradicts the both inequalities, listed above in (i) and (ii). The similar arguments work if we suppose $ess(f_{3\leftarrow2}) = 0$. ∎

We are going to describe the properties of the functions from $G^3_{2,k}$. For this we need the following auxiliary functions.

$$s(x_1, x_2) := x_1^0 x_2^0 \oplus \ldots \oplus x_1^{k-1} x_2^{k-1}, \quad u^{(\alpha)}(x_1, x_2) := \bigoplus_{\beta \neq \alpha} x_1^\beta x_2^\beta,$$

$$(8) \qquad\qquad\qquad\qquad v^{(\alpha)}(x_1, x_2) := \bigoplus_{\beta \neq \alpha} x_1^\alpha x_2^\beta.$$

For example, let $k = 3$, then $s(x_1, x_2) = x_1^0 x_2^0 \oplus x_1^1 x_2^1 0 \oplus x_1^2 x_2^2$, $u^{(1)}(x_3, x_2) = x_3^0 x_2^0 \oplus x_3^2 x_2^2$ and $v^{(1)}(x_3, x_2) = x_3^1 x_2^0 \oplus x_3^1 x_2^2$.

Theorem 5.1. *If*

$$f = \bigoplus_{i=0}^{2} x_3^i . [\bigoplus_{j=0}^{8} a_j^{(i)} x_1^{\alpha_1} x_2^{\alpha_2}] \in G^3_{2,3},$$

where $j = \overline{\alpha_1 \alpha_2}$, then f has to be represented in one of the following special forms, up to permutation of variables:
(9)
$$f = \bigoplus_{i=0}^{2} a_0^{(i)} [x_3^i . s(x_1, x_2) \oplus x_2^i . u^{(i)}(x_1, x_3) \oplus x_1^i . u^{(i)}(x_2, x_3)] \oplus p_3(x_1, x_2, x_3),$$

$$(10) \quad f = \bigoplus_{i=0}^{2} a_0^{(i)} [x_1^i . x_2^i \oplus x_1^i . u^{(i)}(x_2, x_3) \oplus x_2^i . u^{(i)}(x_1, x_3)] \oplus p_3(x_1, x_2, x_3),$$

$$(11) \quad f = \bigoplus_{i=0}^{2} a_0^{(i)} [x_1^i . x_2^i \oplus x_2^i . v^{(i)}(x_3, x_1) \oplus x_2^i . u^{(i)}(x_1, x_3)] \oplus p_3(x_1, x_2, x_3),$$

$$(12) \quad f = \bigoplus_{i=0}^{2} a_0^{(i)} [x_1^i . x_2^i \oplus x_1^i . v^{(i)}(x_3, x_2) \oplus x_2^i . v^{(i)}(x_3, x_1)] \oplus p_3(x_1, x_2, x_3),$$

such that at least two among the coefficients $a_0^{(0)}, a_0^{(1)}, a_0^{(2)}$ are different and p_3 are arbitrary functions defined by (4).

Proof. According to Lemma 5.1, for $f \in G_{2,3}^3$ we need $ess(f_{2\leftarrow1}) = ess(f_{3\leftarrow1}) = ess(f_{3\leftarrow2}) = 1$. This is equivalent to the following conjunction of three disjunctions:

$$\left(x_1 \notin Ess(f_{2\leftarrow1}) \ \text{ or } \ x_3 \notin Ess(f_{2\leftarrow1})\right) \ \text{ and }$$

$$\left(x_1 \notin Ess(f_{3\leftarrow1}) \ \text{ or } \ x_2 \notin Ess(f_{3\leftarrow1})\right) \ \text{ and }$$

$$\left(x_2 \notin Ess(f_{3\leftarrow2}) \ \text{ or } \ x_1 \notin Ess(f_{3\leftarrow2})\right).$$

Thus we obtain the following linear systems of equations for the coefficients of the functions from $G_{2,3}^3$.

$$(13) \quad \begin{array}{ll} (a) & (a_0^{(0)}, a_0^{(1)}, a_0^{(2)}) = (a_4^{(0)}, a_4^{(1)}, a_4^{(2)}) = (a_8^{(0)}, a_8^{(1)}, a_8^{(2)}) \\[2mm] (b) & (a_0^{(0)}, a_4^{(0)}, a_8^{(0)}) = (a_0^{(1)}, a_4^{(1)}, a_8^{(1)}) = (a_0^{(2)}, a_4^{(2)}, a_8^{(2)}), \end{array}$$

$$(14) \quad \begin{array}{ll} (a) & (a_0^{(0)}, a_1^{(0)}, a_2^{(0)}) = (a_3^{(1)}, a_4^{(1)}, a_5^{(1)}) = (a_6^{(2)}, a_7^{(2)}, a_8^{(2)}) \\[2mm] (b) & (a_0^{(0)}, a_3^{(1)}, a_6^{(2)}) = (a_1^{(0)}, a_4^{(1)}, a_7^{(2)}) = (a_2^{(0)}, a_5^{(1)}, a_8^{(2)}), \end{array}$$

$$(a) \qquad (a_0^{(0)}, a_3^{(0)}, a_6^{(0)}) = (a_1^{(1)}, a_4^{(1)}, a_7^{(1)}) = (a_2^{(2)}, a_5^{(2)}, a_8^{(2)})$$

(15)

$$(b) \qquad (a_0^{(0)}, a_1^{(1)}, a_2^{(2)}) = (a_3^{(0)}, a_4^{(1)}, a_5^{(2)}) = (a_6^{(0)}, a_7^{(1)}, a_8^{(2)}).$$

Now any solution is determined by a word (string) over the alphabetic $\{a, b\}$ with length 3. For example let us consider the string aba. Then we use equations a from (13), b from (14) and a from (15).
Then we obtain

$$a_0^{(0)} = a_4^{(0)} = a_8^{(0)} = a_1^{(0)} = a_2^{(0)} = a_1^{(1)} = a_2^{(2)}$$

$$a_0^{(1)} = a_4^{(1)} = a_8^{(1)} = a_3^{(1)} = a_5^{(1)} = a_3^{(0)} = a_5^{(2)}$$

$$a_0^{(2)} = a_4^{(2)} = a_8^{(2)} = a_6^{(2)} = a_7^{(2)} = a_6^{(0)} = a_1^{(1)}.$$

Hence

$$f = \bigoplus_{i=0}^{2} a_0^{(i)} [x_1^i . x_3^i \oplus x_1^i . u^{(i)}(x_2, x_3) \oplus x_3^i . u^{(i)}(x_1, x_2)],$$

i.e. f is as the function presented by (10) where the variables x_2 and x_3 are permuted.

In this way we might generate $2^3 = 8$ linear systems for the coefficients.

The remaining seven cases of strings, might be checked in the same way. We will summarize the results. The functions in the form (9) are generated by the string aaa; (10) – by aba, baa and aab; (11) – by abb, bba and bab and (12) – by bbb. ∎

Proposition 5.1. $|G_{2,3}^3| \leq 139968$.

Proof. Clearly, each of the functions in equations (9)–(12) can be written as follows

$$f = \left[\bigoplus_{i=0}^{2} a_0^{(i)} . g^{(i)}(x_1, x_2, x_3) \right] \oplus p_3(x_1, x_2, x_3),$$

where p_3 are the functions defined in (4) when $k = 3$. The number of the functions $p_3(x_1, x_2, x_3)$ is equal to $3^{3!} = 729$.

The functions $g^{(i)}$ from the equations (9) and (12) are totally symmetric, but these from (10) and (11) are symmetric with respect to two of their arguments, only. Thus if the coefficients $a_0^{(i)}$ are fixed then there exist one function

$$\bigoplus_{i=0}^{2} a_0^{(i)}.g^{(i)}(x_1, x_2, x_3)$$

determined by (9), one – by (12), three functions are determined by (10) and three – by (11). So, there are 8 such functions. The triples $(a_0^{(0)}, a_0^{(1)}, a_0^{(2)})$ can be chosen in $3^3 - 3 = 24$ ways (the triples $(0,0,0)$, $(1,1,1)$ and $(2,2,2)$ are forbidden). Hence there are $8.24 = 192$ functions $\bigoplus_{i=0}^{2} a_0^{(i)}.g^{(i)}(x_1, x_2, x_3)$ which satisfied (9)–(12). Hence $|G_{2,3}^3| \leq 192.729 = 139968$. ∎

Proposition 5.2. $|G_{2,k}^3| \leq 8.k^{\binom{k}{3}3!}.(k^k - k)$.

Proof. Without any difficulties, excluding the more complex calculations, we might generalize results from $G_{2,3}^3$ to $G_{2,k}^3$ for arbitrary k, $k \geq 3$. In this case we obtain the same conjunction of three disjunctions to determine the functions from $G_{2,k}^3$. The difference is that in the equations (13)–(15) participate k tuples of k coefficients and a function belongs to $G_{2,k}^3$ if and only if it can be represented as in (9)–(12) as the sums are extended up to $k - 1$ instead of 2 in the case $G_{2,3}^3$. All other arguments work, here also. ∎

Acknowledgements

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