# THE MONOID OF GENERALIZED HYPERSUBSTITUTIONS OF TYPE $\tau=(n)$ 

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#### Abstract

A (usual) hypersubstitution of type $\tau$ is a function which takes each operation symbol of the type to a term of the type, of the same arity. The set of all hypersubstitutions of a fixed type $\tau$ forms a monoid under composition, and semigroup properties of this monoid have been studied by a number of authors. In particular, idempotent and regular elements, and the Green's relations, have been studied for type ( $n$ ) by S.L. Wismath.

A generalized hypersubstitution of type $\tau=(n)$ is a mapping $\sigma$ which takes the $n$-ary operation symbol $f$ to a term $\sigma(f)$ which does not necessarily preserve the arity. Any such $\sigma$ can be inductively extended to a map $\hat{\sigma}$ on the set of all terms of type $\tau=(n)$, and any two such extensions can be composed in a natural way. Thus, the set $H y p_{G}(n)$ of all generalized hypersubstitutions of type $\tau=(n)$ forms a monoid. In this paper we study the semigroup properties of $\operatorname{Hyp}_{G}(n)$.


[^0]In particular, we characterize the idempotent and regular generalized hypersubstitutions, and describe some classes under Green's relations of this monoid.
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## 1. Introduction

Identities are used to classify algebras into collections called varieties. Hyperidentities are used to classify varieties into collections called hypervarieties. The concepts of hyperidentities and hypervarieties were introduced by W. Taylor in 1981 [7]. Hyperidentities in a variety $V$ are identities which have the property that, after replacing the operation symbols which occur in these identities by any terms of the same arity, the resulting equation is still satisfied in the variety. The main tool to study hyperidentities is the concept of a hypersubstitution, which was introduced by K. Denecke, D. Lau, R. Pöschel and D. Schweigert in 1991 [1]. Let $\tau=\left(n_{i}\right)_{i \in I}$ be a type and let $W_{\tau}(X)$ be the set of all terms of type $\tau$ built up by operation symbols from $\left\{f_{i} \mid i \in I\right\}$ where $f_{i}$ is $n_{i}$-ary and variables from a countably infinite alphabet $X:=\left\{x_{1}, x_{2}, \ldots\right\}$. A hypersubstitution of type $\tau$ is a mapping $\sigma:\left\{f_{i} \mid i \in I\right\} \rightarrow W_{\tau}(X)$ which maps $n_{i}$-ary operation symbols to $n_{i}$-ary terms. Let $\operatorname{Hyp}(\tau)$ be the set of all hypersubstitutions of type $\tau$. For every $\sigma \in \operatorname{Hyp}(\tau)$ induces a mapping $\hat{\sigma}: W_{\tau}(X) \rightarrow W_{\tau}(X)$ as follows: for any $t \in W_{\tau}(X), \hat{\sigma}[t]$ is defined inductively by
(i) $\hat{\sigma}[x]:=x \in X$,
(ii) $\hat{\sigma}\left[f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right]:=\sigma\left(f_{i}\right)\left(\hat{\sigma}\left[t_{1}\right], \ldots, \hat{\sigma}\left[t_{n_{i}}\right]\right)$, for any $n_{i}$-ary operation symbol $f_{i}$.

It turns out that $\left(\operatorname{Hyp}(\tau) ; \circ_{h}, \sigma_{i d}\right)$ is a monoid where $\sigma_{1} \circ_{h} \sigma_{2}:=\hat{\sigma}_{1} \circ \sigma_{2}$ and $\sigma_{i d}\left(f_{i}\right)=f_{i}\left(x_{1}, \ldots, x_{n_{i}}\right)$ is the identity element.

In 2000, S. Leeratanavalee and K. Denecke generalized the concept of a hypersubstitution to a generalized hypersubstitution [2]. S. Leeratanavalee and K. Denecke used generalized hypersubstitutions as the tools to study
strong hyperidentities and used strong hyperidentities to classify varieties into collections called strong hypervarieties. Varieties whose identities are closed under arbitrary application of generalized hypersubstitutions are called strongly solid.

A generalized hypersubstitution of type $\tau$, or for short simply a generalized hypersubstitution, is a mapping $\sigma$ which maps each $n_{i}$-ary operation symbol of type $\tau$ to a term of this type in $W_{\tau}(X)$ which does not necessarily preserve the arity. We denoted the set of all generalized hypersubstitutions of type $\tau$ by $H y p_{G}(\tau)$. First, we define inductively the concept of generalized superposition of terms $S^{m}: W_{\tau}(X)^{m+1} \rightarrow W_{\tau}(X)$ by the following steps:
(i) If $t=x_{j}, 1 \leq j \leq m$, then $S^{m}\left(x_{j}, t_{1}, \ldots, t_{m}\right):=t_{j}$.
(ii) If $t=x_{j}, m<j \in \mathbb{N}$, then $S^{m}\left(x_{j}, t_{1}, \ldots, t_{m}\right):=x_{j}$.
(iii) If $t=f_{i}\left(s_{1}, \ldots, s_{n_{i}}\right)$, then

$$
S^{m}\left(t, t_{1}, \ldots, t_{m}\right):=f_{i}\left(S^{m}\left(s_{1}, t_{1}, \ldots, t_{m}\right), \ldots, S^{m}\left(s_{n_{i}}, t_{1}, \ldots, t_{m}\right)\right) .
$$

To define a binary operation on $\operatorname{Hyp}_{G}(\tau)$, we extend a generalized hypersubstitution $\sigma$ to a mapping $\hat{\sigma}: W_{\tau}(X) \rightarrow W_{\tau}(X)$ inductively defined as follows:
(i) $\hat{\sigma}[x]:=x \in X$,
(ii) $\hat{\sigma}\left[f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right]:=S^{n_{i}}\left(\sigma\left(f_{i}\right), \hat{\sigma}\left[t_{1}\right], \ldots, \hat{\sigma}\left[t_{n_{i}}\right]\right)$, for any $n_{i}$-ary operation symbol $f_{i}$.

Then we define a binary operation ${ }^{\circ}{ }_{G}$ on $H y p_{G}(\tau)$ by $\sigma_{1}{ }^{\circ}{ }_{G} \sigma_{2}:=\hat{\sigma}_{1} \circ \sigma_{2}$ where $\circ$ denotes the usual composition of mappings and $\sigma_{1}, \sigma_{2} \in H y p_{G}(\tau)$. Let $\sigma_{i d}$ be the hypersubstitution which maps each $n_{i}$-ary operation symbol $f_{i}$ to the term $f_{i}\left(x_{1}, \ldots, x_{n_{i}}\right)$. S. Leeratanavalee and K. Denecke proved the following propositions.

Proposition 1.1 ([2]). For arbitrary terms $t, t_{1}, \ldots, t_{n} \in W_{\tau}(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_{1}, \sigma_{2}$ we have
(i) $S^{n}\left(\hat{\sigma}[t], \hat{\sigma}\left[t_{1}\right], \ldots, \hat{\sigma}\left[t_{n}\right]\right)=\hat{\sigma}\left[S^{n}\left(t, t_{1}, \ldots, t_{n}\right)\right]$,
(ii) $\left(\hat{\sigma}_{1} \circ \sigma_{2}\right)^{\wedge}=\hat{\sigma}_{1} \circ \hat{\sigma}_{2}$.

Proposition $1.2([2]) . \operatorname{Hyp}_{G}(\tau)=\left(\operatorname{Hyp}_{G}(\tau) ; \circ_{G}, \sigma_{i d}\right)$ is a monoid, with $\sigma_{i d}$ as the identity element, and the set of all hypersubstitutions of type $\tau$ forms a submonoid of $\underline{H y p_{G}(\tau)}$.

Many properties of the monoid of hypersubstitutions of type $\tau=(n)$ were described by S.L. Wismath [8]. In this paper we extend the results from [8] to the case of $\operatorname{Hyp}_{G}(n)$.

## 2. Projection and dual generalized hypersubstitutions of type $\tau=(n)$

We assume that from now on the type $\tau=(n)$, for some $n \in \mathbb{N}$, i.e. we have only one $n$-ary operation symbol, say $f$. By $\sigma_{t}$ we denote the generalized hypersubstitution which maps $f$ to the term $t$ in $W_{(n)}(X)$. A generalized hypersubstitution $\sigma_{t}$ is called a projection generalized hypersubstitution if $t$ is a variable [3]. We denoted the set of all projection generalized hypersubstitutions of type $\tau=(n)$ by $P_{G}(n)$, i.e. $P_{G}(n)=\left\{\sigma_{x_{i}} \mid x_{i} \in X\right\}$.

Lemma 2.1. For any $\sigma_{t} \in \operatorname{Hyp}_{G}(n)$ and $\sigma_{x_{i}} \in P_{G}(n)$, we have
(i) $\sigma_{t}{ }^{\circ}{ }_{G} \sigma_{x_{i}}=\sigma_{x_{i}}$,
(ii) $\sigma_{x_{i}}{ }^{\circ}{ }_{G} \sigma_{t} \in P_{G}(n)$.

Proof. (i) We have $\left(\sigma_{t} \circ_{G} \sigma_{x_{i}}\right)(f)=\left(\hat{\sigma}_{t} \circ \sigma_{x_{i}}\right)(f)=\hat{\sigma}_{t}\left[\sigma_{x_{i}}(f)\right]=\hat{\sigma}_{t}\left[x_{i}\right]=$ $x_{i}=\sigma_{x_{i}}(f)$. So $\sigma_{t}{ }^{\circ}{ }_{G} \sigma_{x_{i}}=\sigma_{x_{i}}$.
(ii) We will proceed by induction on the complexity of the term $t$. If $t \in$ $X$, then by $(i)$ we get $\sigma_{x_{i}}{ }^{\circ}{ }_{G} \sigma_{t}=\sigma_{t} \in P_{G}(n)$. Assume that $t=f\left(u_{1}, \ldots, u_{n}\right)$ and $\sigma_{x_{i}} \circ_{G} \sigma_{u_{1}}, \ldots, \sigma_{x_{i}} \circ_{G} \sigma_{u_{n}} \in P_{G}(n)$. Thus $\hat{\sigma}_{x_{i}}\left[u_{1}\right], \ldots, \hat{\sigma}_{x_{i}}\left[u_{n}\right] \in X$. We have $\left(\sigma_{x_{i}} \circ_{G} \sigma_{t}\right)(f)=\left(\sigma_{x_{i}}{ }^{\circ} G \sigma_{f\left(u_{1}, \ldots, u_{n}\right)}\right)(f)=S^{n}\left(x_{i}, \hat{\sigma}_{x_{i}}\left[u_{1}\right], \ldots, \hat{\sigma}_{x_{i}}\left[u_{n}\right]\right)$. If $x_{i} \in X_{n}$ where $X_{n}=\left\{x_{1}, \ldots, x_{n}\right\}$, then $\left(\sigma_{x_{i}} \circ_{G} \sigma_{t}\right)(f)=\hat{\sigma}_{x_{i}}\left[u_{i}\right] \in X$. If $i>n$, then $\left(\sigma_{x_{i}}{ }^{\circ}{ }_{G} \sigma_{t}\right)(f)=x_{i} \in X$. So $\sigma_{x_{i}}{ }^{\circ}{ }_{G} \sigma_{t} \in P_{G}(n)$.

## Corollary 2.2.

(i) $P_{G}(n) \cup\left\{\sigma_{i d}\right\}$ is a submonoid of $\operatorname{Hyp}_{G}(n)$ and $P_{G}(n)$ is the smallest two-sided ideal of $\operatorname{Hyp}_{G}(n)$, called the kernel of $\operatorname{Hyp}_{G}(n)$. Thus, $H y p_{G}(n)$ is not simple.
(ii) $P_{G}(n)$ is the set of all right-zero elements of $\operatorname{Hyp}_{G}(n)$, so that $P_{G}(n)$ itself is a right-zero band.
(iii) $H y p_{G}(n)$ contains no left-zero elements.

Proof. These follow immediately from Lemma 2.1.
Another special kind of generalized hypersubstitutions in $\operatorname{Hyp}_{G}(n)$ are dual generalized hypersubstitutions, which are defined using permutations of the set $J:=\{1, \ldots, n\}$. For any such permutation $\pi$, we let $\sigma_{\pi}=$ $\sigma_{f\left(x_{\pi(1)}, \ldots, x_{\pi(n)}\right)}$. We let $D_{G}$ be the set of all such dual generalized hypersubstitutions.

## Lemma 2.3.

(i) For any two permutations $\pi$ and $\rho, \sigma_{\rho}{ }^{\circ}{ }_{G} \sigma_{\pi}=\sigma_{\pi \circ \rho}$.
(ii) For any permutation $\pi$ with the inverse permutation $\pi^{-1}$, the generalized hypersubstitutions $\sigma_{\pi}$ and $\sigma_{\pi^{-1}}$ are inverse of each other.

Proof. (i) We have $\left(\sigma_{\rho} \circ_{G} \sigma_{\pi}\right)(f)=\hat{\sigma}_{\rho}\left[f\left(x_{\pi(1)}, \ldots, x_{\pi(n)}\right)\right]=S^{n}\left(f\left(x_{\rho(1)}, \ldots\right.\right.$, $\left.\left.x_{\rho(n)}\right), x_{\pi(1)}, \ldots, x_{\pi(n)}\right)=f\left(x_{\pi(\rho(1))}, \ldots, x_{\pi(\rho(n))}\right)=\sigma_{\pi \circ \rho}(f)$.
(ii) This follows from (i).

Lemma 2.4. If $\sigma \circ_{G} \rho \in D_{G}$, then both $\sigma$ and $\rho$ are in $D_{G}$.

Proof. Let $\sigma(f)=f\left(u_{1}, \ldots, u_{n}\right)$ and $\rho(f)=f\left(v_{1}, \ldots, v_{n}\right)$. Since $\sigma \circ_{G} \rho \in D_{G}$, thus there exists a permutation $\pi$ such that $\left(\sigma \circ_{G} \rho\right)(f)=$ $f\left(x_{\pi(1)}, \ldots, x_{\pi(n)}\right)$. So $f\left(x_{\pi(1)}, \ldots, x_{\pi(n)}\right)=\left(\sigma \circ_{G} \rho\right)(f)=S^{n}\left(f\left(u_{1}, \ldots, u_{n}\right)\right.$, $\left.\hat{\sigma}\left[v_{1}\right], \ldots, \hat{\sigma}\left[v_{n}\right]\right)$. Since $\pi$ is a permutation, thus this forces all the $u_{i}$ 's to be distinct variables in $X_{n}$, and all the $v_{i}$ 's to be distinct variables in $X_{n}$. It follows that both $\sigma$ and $\rho$ are in $D_{G}$.

Corollary 2.5. $\underline{D_{G}}$ is a submonoid of $\operatorname{Hyp}_{G}(n)$ which forms a group, and no other elements of $H y p_{G}(n)$ have inverses in $\operatorname{Hyp}_{G}(n)$. Thus, $\underline{D_{G}}$ is the largest subgroup of $\underline{\operatorname{Hyp}_{G}(n)}$.

Lemma 2.6. Let $F$ be the set of generalized hypersubstitutions of the form $\sigma_{f\left(x_{i}, \ldots, x_{i}\right)}$ for $i \in \mathbb{N}$. Let $M=P_{G}(n) \cup D_{G} \cup F$. Then $\underline{M}$ is a submonoid of $\underline{H y p_{G}(n)}$.

Proof. It is straightforward to check that any product of two elements in $M$ is also in $M$.

## 3. Idempotent and regular elements in $H y p_{G}(n)$

All idempotent elements of the monoid of all generalized hypersubstitutions of type $\tau=(2)$ were studied by W. Puninagool and S. Leeratanavalee [6] and all regular elements of the monoid of all generalized hypersubstitutions of type $\tau=(2)$ were studied by W. Puninagool and S. Leeratanavalee [4]. In this section, we characterize the idempotent and regular generalized hypersubstitutions of type $\tau=(n)$.

We know from Corollary 2.2 (ii) that every projection generalized hypersubstitution is idempotent. We let $G(n):=\left\{\sigma_{t} \mid t \notin X, \operatorname{var}(t) \cap X_{n}=\emptyset\right\}$ where $\operatorname{var}(t)$ denotes the set of all variables occurring in $t$.

Lemma 3.1. If $\sigma_{t} \in G(n)$ and $\sigma_{s} \in H y p_{G}(n) \backslash P_{G}(n)$, then $\sigma_{t}{ }^{\circ}{ }_{G} \sigma_{s}=\sigma_{t}$, i.e. $G(n)$ itself is a left zero band.

Proof. Let $s=f\left(v_{1}, \ldots, v_{n}\right)$. We have $\left(\sigma_{t} \circ_{G} \sigma_{s}\right)(f)=S^{n}\left(t, \hat{\sigma}_{t}\left[v_{1}\right], \ldots\right.$, $\left.\hat{\sigma}_{t}\left[v_{n}\right]\right)=t$ since there is nothing to substitute in the term $t$. So $\sigma_{t}{ }^{\circ}{ }_{G} \sigma_{s}$ $=\sigma_{t}$.
Then we consider only the case $\sigma_{t} \in \operatorname{Hyp}_{G}(n) \backslash P_{G}(n)$ and $\operatorname{var}(t) \cap X_{n} \neq \emptyset$.
Theorem 3.2. Let $t=f\left(t_{1}, \ldots, t_{n}\right) \in W_{(n)}(X)$ and $\emptyset \neq \operatorname{var}(t) \cap X_{n}=$ $\left\{x_{i_{1}}, \ldots, x_{i_{m}}\right\}$. Then $\sigma_{t}$ is idempotent if and only $t_{i_{k}}=x_{i_{k}}$ for all $k \in$ $\{1, \ldots, m\}$.

Proof. Assume that $\sigma_{f\left(t_{1}, \ldots, t_{n}\right)}$ is idempotent. Then $S^{n}\left(f\left(t_{1}, \ldots, t_{n}\right)\right.$, $\left.\hat{\sigma}_{f\left(t_{1}, \ldots, t_{n}\right)}\left[t_{1}\right], \ldots, \hat{\sigma}_{f\left(t_{1}, \ldots, t_{n}\right)}\left[t_{n}\right]\right)=\sigma_{f\left(t_{1}, \ldots, t_{n}\right)}^{2}(f)=\sigma_{f\left(t_{1}, \ldots, t_{n}\right)}(f)=$ $f\left(t_{1}, \ldots, t_{n}\right)$. Suppose that there exists $k \in\{1, \ldots, m\}$ such that $t_{i_{k}} \neq x_{i_{k}}$. If $t_{i_{k}} \in X$, then $\hat{\sigma}_{f\left(t_{1}, \ldots, t_{n}\right)}\left[t_{i_{k}}\right]=t_{i_{k}} \neq x_{i_{k}}$. So $S^{n}\left(f\left(t_{1}, \ldots, t_{n}\right), \hat{\sigma}_{f\left(t_{1}, \ldots, t_{n}\right)}\right.$ $\left.\left[t_{1}\right], \ldots, \hat{\sigma}_{f\left(t_{1}, \ldots, t_{n}\right)}\left[t_{n}\right]\right) \stackrel{f}{\neq} f\left(t_{1}, \ldots, t_{n}\right)$ and we have a contradiction.

If $t_{i_{k}} \notin X$, then $\hat{\sigma}_{f\left(t_{1}, \ldots, t_{n}\right)}\left[t_{i_{k}}\right] \notin X$. We obtain $o p(t)=o p\left(S^{n}\left(f\left(t_{1}, \ldots, t_{n}\right)\right.\right.$, $\left.\left.\hat{\sigma}_{f\left(t_{1}, \ldots, t_{n}\right)}\left[t_{1}\right], \ldots, \hat{\sigma}_{f\left(t_{1}, \ldots, t_{n}\right)}\left[t_{n}\right]\right)\right)>o p(t)$ where $o p(t)$ denotes the number of all operation symbols occurring in $t$. This is a contradiction.

The proof of the converse direction is straightforward.
Now, we characterize the regular generalized hypersubstitutions of type $\tau=$ $(n)$. At first we want to recall the definition of the regular element.

Definition 3.3. An element $a$ of a semigroup $S$ is called regular if there exists $x \in S$ such that $a x a=a$. The semigroup $S$ is called regular if all its elements are regular.

Lemma 3.4. Let $t \in W_{(n)}(X)$ and $\emptyset \neq \operatorname{var}(t) \cap X_{n}=\left\{x_{i_{1}}, \ldots, x_{i_{m}}\right\}$ and let $a=f\left(a_{1}, \ldots, a_{n}\right) \in W_{(n)}(X)$. If $\hat{\sigma}_{t}[a]=t$, then $a_{l}=x_{l}$ for all $l=i_{1}, \ldots, i_{m}$.

Proof. Assume that $\hat{\sigma}_{t}[a]=t$. Then $t=\hat{\sigma}_{t}[a]=S^{n}\left(t, \hat{\sigma}_{t}\left[a_{1}\right], \ldots, \hat{\sigma}_{t}\left[a_{n}\right]\right)$. We will show that $a_{l}=x_{l}$ for all $l=i_{1}, \ldots, i_{m}$. Suppose that there exists $j^{\prime} \in\left\{i_{1}, \ldots, i_{m}\right\}$ such that $a_{j^{\prime}} \neq x_{j^{\prime}}$. If $a_{j^{\prime}}=x_{k} \in X$ where $x_{k} \neq x_{j^{\prime}}$, then $\hat{\sigma}_{t}\left[a_{j^{\prime}}\right]=x_{k}$. It follows that $t \neq S^{n}\left(t, \hat{\sigma}_{t}\left[a_{1}\right], \ldots, \hat{\sigma}_{t}\left[a_{n}\right]\right)$. This is a contradiction. If $a_{j^{\prime}} \notin X$, then $\hat{\sigma}_{t}\left[a_{j^{\prime}}\right] \notin X$. It follows that $o p(t)=o p\left(\hat{\sigma}_{t}[a]\right)=S^{n}\left(t, \hat{\sigma}_{t}\left[a_{1}\right], \ldots, \hat{\sigma}_{t}\left[a_{n}\right]\right)>o p(t)$ and we have a contradiction.

Theorem 3.5. Let $t=f\left(t_{1}, \ldots, t_{n}\right) \in W_{(n)}(X)$ and $\emptyset \neq \operatorname{var}(t) \cap X_{n}=$ $\left\{x_{i_{1}}, \ldots, x_{i_{m}}\right\}$. Then $\sigma_{t}$ is regular if and only if there exist $j_{1}, \ldots, j_{m} \in$ $\{1, \ldots, n\}$ such that $t_{j_{1}}=x_{i_{1}}, \ldots, t_{j_{m}}=x_{i_{m}}$.

Proof. Assume that $\sigma_{t}$ is regular. Then there exists $\sigma_{s} \in \operatorname{Hyp}_{G}(n)$ such that $\sigma_{t}{ }^{\circ}{ }_{G} \sigma_{s}{ }^{\circ}{ }_{G} \sigma_{t}=\sigma_{t}$. Since $t \notin X$, thus $s \notin X$. Then $s=$ $f\left(s_{1}, \ldots, s_{n}\right)$ for some $s_{1}, \ldots, s_{n} \in W_{(n)}(X)$. From $\sigma_{t}{ }^{\circ}{ }_{G} \sigma_{s}{ }^{\circ}{ }_{G} \sigma_{t}=\sigma_{t}$, thus $\hat{\sigma}_{t}\left[\hat{\sigma}_{s}[t]\right]=t$. By Lemma 3.4, $\hat{\sigma}_{s}[t]=f\left(u_{1}, \ldots, u_{n}\right)$ for some $u_{1}, \ldots, u_{n} \in$ $W_{(n)}(X)$ where $u_{i_{1}}=x_{i_{1}}, \ldots, u_{i_{m}}=x_{i_{m}}$. From $\hat{\sigma}_{s}[t]=f\left(u_{1}, \ldots, u_{n}\right)$, thus $S^{n}\left(f\left(s_{1}, \ldots, s_{n}\right), \hat{\sigma}_{s}\left[t_{1}\right], \ldots, \hat{\sigma}_{s}\left[t_{n}\right]\right)=f\left(u_{1}, \ldots, u_{n}\right)$. Since $u_{i_{1}}=x_{i_{1}}, \ldots, u_{i_{m}}$ $=x_{i_{m}}$ thus $s_{i_{1}}, \ldots, s_{i_{m}} \in X_{n}$. Let $s_{i_{1}}=x_{j_{1}}, \ldots, s_{i_{m}}=x_{j_{m}}$. Hence $t_{j_{1}}=x_{i_{1}}, \ldots, t_{j_{m}}=x_{i_{m}}$.

Conversely, assume the condition holds. Let $s=f\left(s_{1}, \ldots, s_{n}\right) \in W_{(n)}(X)$ where $s_{1}, \ldots, s_{n} \in W_{(n)}(X)$ such that $s_{i_{1}}=x_{j_{1}}, \ldots, s_{i_{m}}=x_{j_{m}}$.

Then $\left(\sigma_{t} \circ_{G} \sigma_{s} \circ_{G} \sigma_{t}\right)(f)=\hat{\sigma}_{t}\left[\hat{\sigma}_{s}[t]\right]=\hat{\sigma}_{t}\left[S^{n}\left(f\left(s_{1}, \ldots, s_{n}\right), \hat{\sigma}_{s}\left[t_{1}\right], \ldots\right.\right.$, $\left.\left.\hat{\sigma}_{s}\left[t_{n}\right]\right)\right]=\hat{\sigma}_{t}\left[f\left(k_{1}, \ldots, k_{n}\right)\right]\left(\right.$ where $\left.k_{i_{1}}=x_{i_{1}}, \ldots, k_{i_{m}}=x_{i_{m}}\right)=S^{n}\left(t, \hat{\sigma}_{t}\left[k_{1}\right]\right.$, $\left.\ldots, \hat{\sigma}_{t}\left[k_{n}\right]\right)=t$. Hence $\sigma_{t}{ }^{\circ} G_{G} \sigma_{s}{ }_{G} \sigma_{t}=\sigma_{t}$.

## 4. Term properties of the composition operation

We need to know more about the result of the composing two generalized hypersubstitutions in $\operatorname{Hyp}_{G}(n)$. In particular, we want to know how long the term corresponding to $\sigma_{s} \circ_{G} \sigma_{t}$ is and what variables it uses, compared to the lengths of the terms $s$ and $t$ and the variables they use. We begin with the necessary definitions.

Definition 4.1. Let $t \in W_{(n)}(X)$. We define the length of $t$ inductively by:
(i) The length of $t$ is 1 if $t$ is a variable.
(ii) If $t$ is a compound term $f\left(t_{1}, \ldots, t_{n}\right)$, then the length of $t$ is the sum of the lengths of the terms $t_{1}, \ldots, t_{n}$.
(iii) This length counts the total number of variable occurences in the term $t$, and will be denoted by $v b(t)$.

Definition 4.2 ([8]). Let $t \in W_{(n)}(X)$. We define some new terms, related to $t$, as follows. Recall that $J:=\{1, \ldots, n\}$.
(i) Let $\alpha$ be any function from $J$ to $J . C_{\alpha}[t]$ is the term formed from $t$ by replacing each occurrence in $t$ of a variable $x_{i} \in X_{n}$ by the variable $x_{\alpha(i)}$ i.e, $C_{\alpha}[t]=S^{n}\left(t, x_{\alpha(1)}, \ldots, x_{\alpha(n)}\right)$.
(ii) Let $\pi$ be any permutation of $J . \pi[t]$ is the term defined inductively by $\pi\left[x_{i}\right]=x_{i}$ for any variable $x_{i}$, and $\pi\left[f\left(u_{1}, \ldots, u_{n}\right)\right]=$ $f\left(\pi\left[u_{\pi(1)}\right], \ldots, \pi\left[u_{\pi(n)}\right]\right)$.

The previous length results for the type $\tau=(2)$ were found by W. Puninagool and S. Leeratanavalee in [6] and S.L. Wismath in [8]. The next two lemmas show how these results can be generalized to the type $\tau=(n)$.

Lemma 4.3. Let $n \in N$ with $n>1$. Let $\sigma_{f\left(u_{1}, \ldots, u_{n}\right)} \circ_{G} \sigma_{f\left(v_{1}, \ldots, v_{n}\right)}=\sigma_{w}$. Then $w$ is a longer term than $f\left(u_{1}, \ldots, u_{n}\right)$, unless the terms $f\left(u_{1}, \ldots, u_{n}\right)$ and $f\left(v_{1}, \ldots, v_{n}\right)$ satisfy the following condition (Q):
(Q) If a variable $x_{i} \in X_{n}$ is used anywhere in the term $f\left(u_{1}, \ldots, u_{n}\right)$, then the entry $v_{i}$ in $f\left(v_{1}, \ldots, v_{n}\right)$ is a variable.

Proof. If $\operatorname{var}\left(f\left(u_{1}, \ldots, u_{n}\right)\right) \cap X_{n}=\emptyset$, then $f\left(u_{1}, \ldots, u_{n}\right)$ and $f\left(v_{1}, \ldots, v_{n}\right)$ satisfy the condition (Q). Let $\operatorname{var}\left(f\left(u_{1}, \ldots, u_{n}\right)\right) \cap X_{n}=\left\{x_{i_{1}}, \ldots, x_{i_{k}}\right\}$. If $v_{i_{j}} \in X$ for all $j \in\{1, \ldots, k\}$, then $f\left(u_{1}, \ldots, u_{n}\right)$ and $f\left(v_{1}, \ldots, v_{n}\right)$ satisfy the condition (Q). If there exists $j \in\{1, \ldots, k\}$ where $v_{i_{j}} \notin X$, then $\hat{\sigma}_{f\left(u_{1}, \ldots, u_{n}\right)}\left[v_{i_{j}}\right] \notin X$. Since $n>1$ and $\hat{\sigma}_{f\left(u_{1}, \ldots, u_{n}\right)}\left[v_{i_{j}}\right] \notin X$, thus $v b\left(\hat{\sigma}_{f\left(u_{1}, \ldots, u_{n}\right)}\left[v_{i_{j}}\right]\right)>1$. So $v b(w)=v b\left(S^{n}\left(f\left(u_{1}, \ldots, u_{n}\right), \hat{\sigma}_{f\left(u_{1}, \ldots, u_{n}\right)}\right.\right.$ $\left.\left.\left[v_{1}\right], \ldots, \hat{\sigma}_{f\left(u_{1}, \ldots, u_{n}\right)}\left[v_{n}\right]\right)\right)>v b\left(f\left(u_{1}, \ldots, u_{n}\right)\right)$.

Lemma 4.4. Let $\sigma_{t} \in \operatorname{Hyp}_{G}(n)$ where $t \notin X$ and $x_{1}, \ldots, x_{n} \in \operatorname{var}(t)$. Then for any $s \in W_{(n)}(X), v b\left(\hat{\sigma}_{t}[s]\right) \geq v b(s)$.

Proof. We will proceed by induction on the complexity of the term $s$. If $s \in$ $X$, then $v b\left(\hat{\sigma}_{t}[s]\right)=v b(s)$. Assume that $s=f\left(u_{1}, \ldots, u_{n}\right)$ and $v b\left(\hat{\sigma}_{t}\left[u_{i}\right]\right) \geq$ $v b\left(u_{i}\right)$ for all $1 \leq i \leq n$. Then $v b\left(\hat{\sigma}_{t}[s]\right)=v b\left(S^{n}\left(t, \hat{\sigma}_{t}\left[u_{1}\right], \ldots, \hat{\sigma}_{t}\left[u_{n}\right]\right)\right) \geq$ $v b\left(f\left(u_{1}, \ldots, u_{n}\right)\right)$ since $x_{1}, \ldots, x_{n} \in \operatorname{var}(t)$ and $v b\left(\hat{\sigma}_{t}\left[u_{i}\right]\right) \geq v b\left(u_{i}\right)$ for all $1 \leq i \leq n$.

Lemma 4.5. Let $\sigma_{f\left(u_{1}, \ldots, u_{n}\right)}{ }^{\circ} \sigma_{f\left(v_{1}, \ldots, v_{n}\right)}=\sigma_{w}$ where $v b\left(f\left(u_{1}, \ldots, u_{n}\right)\right)>n$. If $x_{1}, \ldots, x_{n} \in \operatorname{var}\left(f\left(u_{1}, \ldots, u_{n}\right)\right)$, then $w$ is a longer term than $f\left(v_{1}, \ldots, v_{n}\right)$.

Proof. We write $\sigma=\sigma_{f\left(u_{1}, \ldots, u_{n}\right)}$. From $\sigma_{f\left(u_{1}, \ldots, u_{n}\right)} \circ_{G} \sigma_{f\left(v_{1}, \ldots, v_{n}\right)}=\sigma_{w}$, thus we get $w=S^{n}\left(f\left(u_{1}, \ldots, u_{n}\right), \hat{\sigma}\left[v_{1}\right], \ldots, \hat{\sigma}\left[v_{n}\right]\right)$. Since $x_{1}, \ldots, x_{n} \in$ $\operatorname{var}\left(f\left(u_{1}, \ldots, u_{n}\right)\right)$, thus $\hat{\sigma}\left[v_{i}\right]$ is used in $w$ for all $1 \leq i \leq n$. We will proceed by induction on the complexity of the term $f\left(v_{1}, \ldots, v_{n}\right)$. If $v_{1}, \ldots, v_{n} \in X$, then $v b(w)=v b\left(f\left(u_{1}, \ldots, u_{n}\right)\right)>n=v b\left(f\left(v_{1}, \ldots, v_{n}\right)\right)$. Assume that the claim holds for any term of length not less than $n$ but less than $k$, and $f\left(v_{1}, \ldots, v_{n}\right)$ has length $k$. Since $v b\left(f\left(v_{1}, \ldots, v_{n}\right)\right)=k>n$, thus there exists $i \in\{1, \ldots, n\}$ such that $v b\left(v_{i}\right) \geq n$. By induction, we get $v b\left(\hat{\sigma}\left[v_{i}\right]\right)>v b\left(v_{i}\right)$. By Lemma 4.4, any other $v_{j}$ has $v b\left(\hat{\sigma}\left[v_{j}\right]\right) \geq v b\left(v_{j}\right)$. Since all the $\hat{\sigma}\left[v_{i}\right]$ are used in $w$ for all $1 \leq i \leq n$, thus $w$ is longer than $f\left(v_{1}, \ldots, v_{n}\right)$.

Lemma 4.6. Let $\sigma_{s}, \sigma_{t} \in \operatorname{Hyp}_{G}(n)$.
(i) $\operatorname{var}\left(\left(\sigma_{s} \circ_{G} \sigma_{t}\right)(f)\right) \cap X_{n} \subseteq \operatorname{var}(t) \cap X_{n}$.
(ii) If $s$ uses only one variable, then the term for $\sigma_{s}{ }^{\circ}{ }_{G} \sigma_{t}$ uses only one variable (not necessarily the same variable as $s$ ).

Proof. We will proceed by induction on the complexity of the term $t$.
(i) If $t \in X$, then $\left(\sigma_{s} \circ_{G} \sigma_{t}\right)(f)=t$. So $\operatorname{var}\left(\left(\sigma_{s} \circ_{G} \sigma_{t}\right)(f)\right) \cap X_{n} \subseteq \operatorname{var}(t) \cap$ $X_{n}$. Assume that $t=f\left(t_{1}, \ldots, t_{n}\right)$ and $\operatorname{var}\left(\hat{\sigma}_{s}\left[t_{i}\right]\right) \cap X_{n} \subseteq \operatorname{var}\left(t_{i}\right) \cap X_{n}$ for all $1 \leq i \leq n$. So $\operatorname{var}\left(\left(\sigma_{s} \circ_{G} \sigma_{t}\right)(f)\right) \cap X_{n}=\operatorname{var}\left(S^{n}\left(s, \hat{\sigma}_{s}\left[t_{1}\right], \ldots, \hat{\sigma}_{s}\left[t_{n}\right]\right)\right) \cap X_{n} \subseteq$ $\cup_{i=1}^{n}\left(\operatorname{var}\left(\hat{\sigma}_{s}\left[t_{i}\right]\right)\right) \cap X_{n}=\cup_{i=1}^{n}\left(\operatorname{var}\left(\hat{\sigma}_{s}\left[t_{i}\right]\right) \cap X_{n}\right) \subseteq \cup_{i=1}^{n}\left(\operatorname{var}\left(t_{i}\right) \cap X_{n}\right)=$ $\cup_{i=1}^{n} \operatorname{var}\left(t_{i}\right) \cap X_{n}=\operatorname{var}(t) \cap X_{n}$.
(ii) If $t \in X$, then $\left(\sigma_{s}{ }^{\circ}{ }_{G} \sigma_{t}\right)(f)=t$. So the term for $\sigma_{s}{ }^{\circ}{ }_{G} \sigma_{t}$ uses only one variable. Assume that $t=f\left(t_{1}, \ldots, t_{n}\right)$ and $\hat{\sigma}_{s}\left[t_{i}\right]$ uses only one variable for all $1 \leq i \leq n$. So $\left(\sigma_{s} \circ_{G} \sigma_{t}\right)(f)=S^{n}\left(s, \hat{\sigma}_{s}\left[t_{1}\right], \ldots, \hat{\sigma}_{s}\left[t_{n}\right]\right)$. If $\operatorname{var}(s)=\left\{x_{i}\right\}$ for some $x_{i} \in X_{n}$, then $\operatorname{var}\left(\sigma_{s} \circ_{G} \sigma_{t}\right)(f)=\operatorname{var}\left(\hat{\sigma}_{s}\left[t_{i}\right]\right)$. If $\operatorname{var}(s)=\left\{x_{i}\right\}$ where $i>n$, then $\operatorname{var}\left(\sigma_{s} \circ_{G} \sigma_{t}\right)(f)=\operatorname{var}(s)$.

We conclude this section by extending the results from [8] to the case of $H y p_{G}(n)$ on properties of the composition operation with a lemma describing the special role of the terms $\pi[t]$ and $C_{\alpha[t]}$ from Definition 4.2.

Lemma 4.7. For $t \in W_{(n)}(X)$.
(i) Let $\pi$ be any permutation on $J$. Then $\sigma_{\pi}{ }^{\circ}{ }_{G} \sigma_{t}=\sigma_{\pi[t]}$.
(ii) Let $\alpha$ be any function on J. Define the generalized hypersubstitution $\sigma_{\alpha}$ by mapping the fundemental $f$ to the term $f\left(x_{\alpha(1)}, \ldots, x_{\alpha(n)}\right)$. Then $\sigma_{t}{ }^{\circ}{ }_{G} \sigma_{\alpha}=\sigma_{C_{\alpha[t]}}$.

Proof. (i) We will proceed by induction on the complexity of the term $t$. If $t \in X$, then by Lemma 2.1(i), $\sigma_{\pi}{ }^{\circ}{ }_{G} \sigma_{t}=\sigma_{t}=\sigma_{\pi[t]}$. Assume that $t=f\left(t_{1}, \ldots, t_{n}\right)$ and $\hat{\sigma}_{\pi}\left[t_{i}\right]=\pi\left[t_{i}\right]$ for all $1 \leq i \leq n$. So $\left(\sigma_{\pi} \circ_{G} \sigma_{t}\right)(f)=$ $S^{n}\left(f\left(x_{\pi(1)}, \ldots, x_{\pi(n)}\right), \hat{\sigma}_{\pi}\left[t_{1}\right], \ldots, \hat{\sigma}_{\pi}\left[t_{n}\right]\right)=f\left(\hat{\sigma}_{\pi}\left[t_{\pi(1)}\right], \ldots, \hat{\sigma}_{\pi}\left[t_{\pi(n)}\right]\right)$ $=f\left(\pi\left[t_{\pi(1)}\right], \cdots, \pi\left[t_{\pi(n)}\right]\right)=\pi\left[f\left(t_{1}, \cdots, t_{n}\right)\right]=\pi[t]$.
(ii) We have $\left(\sigma_{t}{ }^{\circ}{ }_{G} \sigma_{\alpha}\right)(f)=S^{n}\left(t, x_{\alpha(1)}, \ldots, x_{\alpha(n)}\right)=C_{\alpha}[t]$. So $\sigma_{t}{ }^{\circ}{ }_{G} \sigma_{\alpha}=$ $\sigma_{C_{\alpha[t]}}$.

## 5. Green's ralations on $H y p_{G}(n)$

Let $S$ be a semigroup and $1 \notin S$. We extend the binary operation from $S$ to $S \cup\{1\}$ by define $x 1=1 x=x$ for all $x \in S \cup\{1\}$. Then $S \cup\{1\}$ is a semigroup with identity 1 .

Let $S$ be a semigroup. Then we define,

$$
S^{1}= \begin{cases}S & \text { if } S \text { has an identity } \\ S \cup\{1\} & \text { otherwise }\end{cases}
$$

Let $S$ be a semigroup and $\emptyset \neq A \subseteq S$. We now set

$$
\begin{aligned}
& (A)_{l}=\cap\{L \mid L \text { is a left ideal of } S \text { containing } A\} \\
& (A)_{r}=\cap\{R \mid R \text { is a right ideal of } S \text { containing } A\} \\
& (A)_{i}=\cap\{I \mid I \text { is an ideal of } S \text { containing } A\} .
\end{aligned}
$$

Then $(A)_{l},(A)_{r}$ and $(A)_{i}$ are left ideal, right ideal and ideal of $S$, respectively. And we call $(A)_{l}\left((A)_{r},(A)_{i}\right)$ the left ideal (right ideal, ideal) of $S$ generated by $A$.

It is easy to see that

$$
\begin{aligned}
& (A)_{l}=S^{1} A=S A \cup A \\
& (A)_{r}=A S^{1}=A \cup S A \\
& (A)_{i}=S^{1} A S^{1}=S A S \cup S A \cup A S \cup A .
\end{aligned}
$$

For $a_{1}, a_{2}, \ldots, a_{n} \in S$, we write $\left(a_{1}, a_{2}, \ldots, a_{n}\right)_{l}$ instead of $\left(\left\{a_{1}, a_{2}, \ldots, a_{n}\right\}\right)_{l}$ and call it the left ideal of $S$ generated by $a_{1}, a_{2}, \ldots, a_{n}$. Similarly, we can define $\left(a_{1}, a_{2}, \ldots, a_{n}\right)_{r}$ and $\left(a_{1}, a_{2}, \ldots, a_{n}\right)_{i}$. If $A$ is a left ideal of $S$ and $A=(a)_{l}$ for some $a \in S$, we then call $A$ the principal left ideal generated by $a$. We can define principal right ideal and principal ideal in the same manner.

Let $S$ be a semigroup. We define the relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ on $S$ as follow:

$$
\begin{aligned}
a \mathcal{L} b & \Leftrightarrow(a)_{l}=(b)_{l} \\
a \mathcal{R} b & \Leftrightarrow(a)_{r}=(b)_{r} \\
\mathcal{H} & =\mathcal{L} \cap \mathcal{R} \\
\mathcal{D} & =\mathcal{L} \circ \mathcal{R} \\
a \mathcal{J} b & \Leftrightarrow(a)_{i}=(b)_{i}
\end{aligned}
$$

Then we have, for all $a, b \in S$

$$
\begin{aligned}
a \mathcal{L} b & \Leftrightarrow S a \cup\{a\}=S b \cup\{b\} \\
& \Leftrightarrow S^{1} a=S^{1} b \\
& \Leftrightarrow a=x b \text { and } b=y a \text { for some } x, y \in S^{1} \\
a \mathcal{R} b & \Leftrightarrow a S \cup\{a\}=b S \cup\{b\} \\
& \Leftrightarrow a S^{1}=b S^{1} \\
& \Leftrightarrow a=b x \text { and } b=a y \text { for some } x, y \in S^{1} \\
a \mathcal{H} b & \Leftrightarrow a \mathcal{L} b \text { and } a \mathcal{R} b \\
a \mathcal{D} b & \Leftrightarrow(a, c) \in \mathcal{L} \text { and }(c, b) \in \mathcal{R} \text { for some } c \in S \\
a \mathcal{J} b & \Leftrightarrow S a S \cup S a \cup a S \cup\{a\}=S b S \cup S b \cup b S \cup\{b\} \\
& \Leftrightarrow S^{1} a S^{1}=S^{1} b S^{1} \\
& \Leftrightarrow a=x b y \text { and } b=z a u \text { for some } x, y, z, u \in S^{1} .
\end{aligned}
$$

Remark 5.1. Let $S$ be a semigroup. Then the following statements hold.

1. $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ are equivalent relations.
2. $\mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$ and $\mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}$.

We call the relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ the Green's relations on $S$. For each $a \in S$, we denote $\mathcal{L}$-class, $\mathcal{R}$-class, $\mathcal{H}$-class, $\mathcal{D}$-class and $\mathcal{J}$-class containing $a$ by $L_{a}, R_{a}, H_{a}, D_{a}$ and $J_{a}$, respectively.

Green's relations on $\operatorname{Hyp}(n)$ have been studied by S.L. Wismath [8], and Green's relations on $\operatorname{Hyp}_{G}(2)$ were study by W. Puninagool and S. Leeratanavalee [5]. In this section, we describe some classes of the monoid of generalized hypersubstitutions of type $\tau=(n)$ with $n>1$.

Theorem 5.2. Any $\sigma_{x_{i}} \in P_{G}(n)$ is $\mathcal{L}$-related only to itself, but is $\mathcal{R}$-related, $\mathcal{D}$-related and $\mathcal{J}$-related to all elements of $P_{G}(n)$, and not related to any other generalized hypersubstitutions. Moerover, the set $P_{G}(n)$ forms a complete $\mathcal{R}$-, $\mathcal{D}$ - and $\mathcal{J}$-class.

Proof. By Lemma 2.1(i), for any $\sigma_{x_{i}} \in P_{G}(n), \sigma{ }^{\circ}{ }_{G} \sigma_{x_{i}}=\sigma_{x_{i}}$ for all $\sigma \in \operatorname{Hyp}_{G}(n)$. This shows that any $\sigma_{x_{i}} \in P_{G}(n)$ can be $\mathcal{L}$-related only to itself. Since $\sigma_{x_{i}}{ }^{\circ} G \sigma_{x_{j}}=\sigma_{x_{j}}$ for all $\sigma_{x_{i}}, \sigma_{x_{j}} \in P_{G}(n)$, so any two elements in $P_{G}(n)$ are $\mathcal{R}$-related. From $\mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}$, we see that any two elements in $P_{G}(n)$ are $\mathcal{D}$ - and $\mathcal{J}$-related. Moreover by Lemma 2.1, $\sigma_{s}{ }^{\circ}{ }_{G} \sigma_{x_{i}}{ }^{\circ} \sigma_{t} \in P_{G}(n)$ for all $\sigma_{s}, \sigma_{t} \in \operatorname{Hyp}_{G}(n)$, and $\sigma_{x_{i}} \in P_{G}(n)$. This implies if $\sigma \notin P_{G}(n)$, then $\sigma$ cannot be $\mathcal{J}$-related to every element in $P_{G}(n)$. So $P_{G}(n)$ is the $\mathcal{J}$-class of its elements. Since any two elements in $P_{G}(n)$ are $\mathcal{R}$ - and $\mathcal{D}$-related, $\mathcal{R} \subseteq \mathcal{J}, \mathcal{D} \subseteq \mathcal{J}$ and $P_{G}(n)$ is the $\mathcal{J}$-class of its elements, and thus $P_{G}(n)$ forms a complete $\mathcal{R}$-, $\mathcal{D}$-class.

Theorem 5.3. Any $\sigma_{t} \in G(n)$ is $\mathcal{R}$-related only to itself, but is $\mathcal{L}$-related, $\mathcal{D}$-related and $\mathcal{J}$-related to all elements of $G(n)$, and not related to any other generalized hypersubstitutions. Moreover, the set $G(n)$ forms a complete $\mathcal{L}$-, $\mathcal{D}$ - and $\mathcal{J}$-class.

Proof. Let $\sigma_{t} \in G(n)$. Assume that $\sigma_{s} \in \operatorname{Hyp}_{G}(n)$ where $\sigma_{s} \mathcal{R} \sigma_{t}$. By Theorem 5.2, $s \notin X$. Then there exists $\sigma_{p} \in \operatorname{Hyp}_{G}(n)$ such that $\sigma_{s}=$ $\sigma_{t}{ }^{\circ}{ }_{G} \sigma_{p}$. Since $s \notin X$ and $\sigma_{s}=\sigma_{t}{ }^{\circ}{ }_{G} \sigma_{p}$, thus by Lemma 2.1(ii), $p \notin X$. Since $\sigma_{t} \in G(n)$ and $p \notin X$, thus by Lemma 3.1, $\sigma_{t}{ }^{\circ}{ }_{G} \sigma_{p}=\sigma_{t}$. So $\sigma_{s}=\sigma_{t}$. Thus $\sigma_{t}$ is $\mathcal{R}$-related only to itself.

Let $\sigma_{s}, \sigma_{t} \in G(n)$. By Lemma 3.1, $\sigma_{s} \circ_{G} \sigma_{t}=\sigma_{s}$ and $\sigma_{t} \circ_{G} \sigma_{s}=\sigma_{t}$. Thus $\sigma_{s} \mathcal{L} \sigma_{t}$. So any two elements in $G(n)$ are $\mathcal{L}$-related. Since $\mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$, thus any two elements in $G(n)$ are $\mathcal{D}$ - and $\mathcal{J}$-related. Assume that $\sigma_{t} \in G(n)$ and $\sigma_{s} \in \operatorname{Hyp}_{G}(n)$ where $\sigma_{s} \mathcal{J} \sigma_{t}$. By Theorem 5.2, $s \notin X$. Then there
exist $\sigma_{p}, \sigma_{q} \in \operatorname{Hyp}_{G}(n)$ such that $\sigma_{p}{ }^{\circ}{ }_{G} \sigma_{t}{ }^{\circ}{ }_{G} \sigma_{q}=\sigma_{s}$. Since $s \notin X$ and $\sigma_{p} \circ_{G} \sigma_{t} \circ_{G} \sigma_{q}=\sigma_{s}$, thus by Lemma 2.1, $p, q \notin X$. Since $\sigma_{t} \in G(n)$ and $q \notin X$, thus by Lemma 3.1, $\sigma_{t}{ }^{\circ} \sigma_{q}=\sigma_{t}$. Since $x_{1}, \ldots, x_{n} \notin \operatorname{var}(t)$, thus by Lemma 4.6 (i), $x_{1}, \ldots, x_{n}$ are not variables occurring in the term $\left(\sigma_{p} \circ_{G} \sigma_{t}\right)(f)=\left(\sigma_{p} \circ_{G} \sigma_{t} \circ_{G} \sigma_{q}\right)(f)$. Thus $x_{1}, \ldots, x_{n} \notin \operatorname{var}(s)$ and so $\sigma_{s} \in$ $G(n)$. So $G(n)$ is the $\mathcal{J}$-class of its elements. Since any two elements in $G(n)$ are $\mathcal{L}$ - and $\mathcal{D}$ - related, $\mathcal{L} \subseteq \mathcal{J}, \mathcal{D} \subseteq \mathcal{J}$ and $G(n)$ is the $\mathcal{J}$-class of its elements, and thus $G(n)$ forms a complete $\mathcal{L}$-, $\mathcal{D}$-class.

Lemma 5.4. Let $\sigma_{s}, \sigma_{t} \in \operatorname{Hyp}_{G}(n) \backslash\left(P_{G}(n) \cup G(n)\right)$. Then $\sigma_{s} \mathcal{R} \sigma_{t}$ if and only if $s=C_{\alpha[t]}$ for some bijection $\alpha$ on $J$.

Proof. Assume that $s=C_{\alpha[t]}$ for some bijection $\alpha$ on $J$. So $\sigma_{\alpha}$ and $\sigma_{\alpha^{-1}}$ are inverse generalized hypersubstitutions. So by Lemma 4.7(ii), $\sigma_{t}{ }^{\circ}{ }_{G} \sigma_{\alpha}=$ $\sigma_{C_{\alpha[t]}}=\sigma_{s}$ and $\sigma_{s} \circ_{G} \sigma_{\alpha^{-1}}=\sigma_{t}$. Thus $\sigma_{s} \mathcal{R} \sigma_{t}$. Conversely, assume that $\sigma_{s} \mathcal{R} \sigma_{t}$. Then there exist $p, q \in W_{(n)}(X) \backslash X$ such that $\sigma_{s}{ }^{\circ}{ }_{G} \sigma_{p}=\sigma_{t}$ and $\sigma_{t}{ }^{\circ}{ }_{G} \sigma_{q}=\sigma_{s}$. Let $p=f\left(p_{1}, \ldots, p_{n}\right)$ and $q=f\left(q_{1}, \ldots, q_{n}\right)$. So we have two equations

$$
\begin{align*}
& S^{n}\left(s, \hat{\sigma}_{s}\left[p_{1}\right], \ldots, \hat{\sigma}_{s}\left[p_{n}\right]\right)=t  \tag{1}\\
& S^{n}\left(t, \hat{\sigma}_{t}\left[q_{1}\right], \ldots, \hat{\sigma}_{t}\left[q_{n}\right]\right)=s . \tag{2}
\end{align*}
$$

Now, if neither of these equations satisfies the condition (Q) of Lemma 4.3, we would have the length of the term $t$ is longer than the length of the term $s$ and also the length of $s$ is longer than the length of $t$, which is clearly impossible. Thus, at least one of two equations must fit the condition (Q). But if one equation fits the condition (Q), Lemma 4.3 tells us that $s$ and $t$ have the same length, and therefore, the second equation also fits the condition (Q). By Lemma 4.3, if $x_{i} \in \operatorname{var}(t) \cap X_{n}$, then $q_{i} \in X$. If such $q_{i} \notin X_{n}$, then from (2) we get $q_{i} \in \operatorname{var}(s)$. So $S^{n}\left(s, \hat{\sigma}_{s}\left[p_{1}\right], \ldots, \hat{\sigma}_{s}\left[p_{n}\right]\right) \neq t$ which contradicts to (1). Thus such $q_{i} \in X_{n}$. Let $\alpha(i)=j$ if $x_{i} \in \operatorname{var}(t) \cap X_{n}$ and $q_{i}=x_{j}$. This defines a partial function on $J$. It is clear that $\alpha$ is injective. Extending this map to a bijection on $J$, which we shall also call $\alpha$. So $s=C_{\alpha[t]}$.

Lemma 5.5. Let $t \in W_{(n)}(X)$ and $\pi$ be a permutation on $J$. Then $\pi^{-1}$ $[\pi[t]]=t$.

Proof. We will proceed by induction on the complexity of the term $t$. If $t \in X$ then $\pi^{-1}[\pi[t]]=\pi^{-1}[t]=t$. Assume that $t=f\left(t_{1}, \ldots, t_{n}\right)$ and $\pi^{-1}\left[\pi\left[t_{i}\right]\right]=t_{i}$ for all $1 \leq i \leq n$. So $\pi^{-1}[\pi[t]]=\pi^{-1}\left[\pi\left[f\left(t_{1}, \ldots, t_{n}\right)\right]\right]=$ $\pi^{-1}\left[f\left(\pi\left[t_{\pi(1)}\right], \ldots, \pi\left[t_{\pi(n)}\right]\right)\right]=f\left(\pi^{-1}\left[\pi\left[t_{\pi\left(\pi^{-1}(1)\right)}\right]\right], \ldots, \pi^{-1}\left[\pi\left[t_{\left.\pi\left(\pi^{-1}(n)\right)\right]}\right]\right)=\right.$ $f\left(\pi^{-1}\left[\pi\left[t_{1}\right]\right], \ldots, \pi^{-1}\left[\pi\left[t_{n}\right]\right]\right)=f\left(t_{1}, \ldots, t_{n}\right)=t$.

Lemma 5.6. Let $\sigma_{t} \in \operatorname{Hyp}_{G}(n) \backslash P_{G}(n)$. Then, for any permutation $\pi$ on $J, \sigma_{t}$ is $\mathcal{L}$-related to the generalized hypersubstitution $\sigma_{\pi[t]}$.

Proof. We know from Lemma 4.7(i) that $\sigma_{\pi}{ }^{\circ}{ }_{G} \sigma_{t}=\sigma_{\pi[t]}$. From Lemma 4.7(i) and Lemma 5.4, $\sigma_{\pi^{-1}}{ }^{\circ}{ }_{G} \sigma_{\pi[t]}=\sigma_{\pi^{-1}[\pi[t]]}=\sigma_{t}$. So $\sigma_{t} \mathcal{L} \sigma_{\pi[t]}$.

Lemma 5.7. Two idempotents $\sigma_{s}$ and $\sigma_{t}$ in $\operatorname{Hyp}_{G}(n) \backslash P_{G}(n)$ are $\mathcal{L}$-related if and only if $\operatorname{var}(s) \cap X_{n}=\operatorname{var}(t) \cap X_{n}$.

Proof. Assume that $\sigma_{s} \mathcal{L} \sigma_{t}$. Then there exist $u, v \in W_{(n)}(X)$ such that $\sigma_{u}{ }^{\circ}{ }_{G} \sigma_{t}=\sigma_{s}$ and $\sigma_{v}{ }^{\circ}{ }_{G} \sigma_{s}=\sigma_{t}$. By Lemma 4.6(i), var $(s) \cap X_{n} \subseteq \operatorname{var}(t) \cap X_{n}$ and $\operatorname{var}(t) \cap X_{n} \subseteq \operatorname{var}(s) \cap X_{n}$. So $\operatorname{var}(s) \cap X_{n}=\operatorname{var}(t) \cap X_{n}$. Conversely, we use the fact that for any two idempotents $e$ and $f$ in any semigroup, $e \mathcal{L} f$ if and only if $e f=e$ and $f e=f$. Since $\operatorname{var}(s) \cap X_{n}=\operatorname{var}(t) \cap X_{n}$, by Theorem 3.2 we can prove that $\sigma_{t}{ }^{\circ} G \sigma_{s}=\sigma_{t}$ and $\sigma_{s}{ }^{\circ}{ }_{G} \sigma_{t}=\sigma_{s}$.

Theorem 5.8. Let $\sigma_{t}$ be an idempotent in $\operatorname{Hyp}_{G}(n) \backslash\left(P_{G}(n) \cup G(n)\right)$. Then $L_{\sigma_{t}}=\left\{\sigma_{\pi[w]} \mid \pi\right.$ is a permutation of $J, w \notin X, \operatorname{var}(w) \cap X_{n}=\operatorname{var}(t) \cap X_{n}$ and $\sigma_{w}$ is an idempotent $\}$.

Proof. Let $\sigma_{\pi[w]} \in \operatorname{Hyp}_{G}(n)$ where $\pi$ is a permutation of $J, w \notin X, \operatorname{var}(w) \cap$ $X_{n}=\operatorname{var}(t) \cap X_{n}$ and $\sigma_{w}$ is an idempotent. By Lemma 5.7, $\sigma_{w} \mathcal{L} \sigma_{t}$. By Lemma 5.6, $\sigma_{w} \mathcal{L} \sigma_{\pi[w]}$. So $\sigma_{\pi[w]} \mathcal{L} \sigma_{t}$. Let $t=f\left(u_{1}, \ldots, u_{n}\right)$ and $s=$ $f\left(v_{1}, \ldots, v_{n}\right)$ with $\sigma_{s} \mathcal{L} \sigma_{t}$. Then there exists $f\left(b_{1}, \ldots, b_{n}\right) \in W_{(n)}(X)$ such that $\sigma_{f\left(b_{1}, \ldots, b_{n}\right)} \circ_{G} \sigma_{f\left(v_{1}, \ldots, v_{n}\right)}=\sigma_{f\left(u_{1}, \ldots, u_{n}\right)}$. We write $\sigma=\sigma_{f\left(b_{1}, \ldots, b_{n}\right)}$. From $\sigma_{f\left(b_{1}, \ldots, b_{n}\right)} \circ_{G} \sigma_{f\left(v_{1}, \ldots, v_{n}\right)}=\sigma_{f\left(u_{1}, \ldots, u_{n}\right)}$, we get $S^{n}\left(f\left(b_{1}, \ldots, b_{n}\right)\right.$, $\left.\hat{\sigma}\left[v_{1}\right], \ldots, \hat{\sigma}\left[v_{n}\right]\right)=f\left(u_{1}, \ldots, u_{n}\right)$. If $x_{i} \in \operatorname{var}(t) \cap X_{n}$, then $u_{i}=x_{i}$ since $\sigma_{t}$ is an idempotent. So $b_{i}=x_{j}$ for some $x_{j} \in X_{n}$. This implies $\hat{\sigma}\left[v_{j}\right]=x_{i}$ and then $v_{j}=x_{i}$. Let $\beta$ be a partial function on $J$ defined by $\beta(i)=j$ if $x_{i} \in \operatorname{var}(t) \cap X_{n}$ and $v_{j}=x_{i}$. If $\beta(i)=\beta(k)=j$, then $v_{j}=x_{i}=x_{k}$. So $i=k$ and $\beta$ is injective. So $\beta$ can be extended to a permutation $\alpha$ on $J$.

Let $w=f\left(p_{1}, \ldots, p_{n}\right)$ where $p_{i}=x_{i}$ if $x_{i} \in \operatorname{var}(t) \cap X_{n}$ and $p_{i}=\alpha\left[v_{\alpha(i)}\right]$ if $x_{i} \notin \operatorname{var}(t) \cap X_{n}$. We will show that $\operatorname{var}(w) \cap X_{n}=\operatorname{var}(t) \cap X_{n}, \sigma_{w}$ is an idempotent and $s=f\left(v_{1}, \ldots, v_{n}\right)=\pi[w]$ where $w=\alpha^{-1}$. We show first that $\operatorname{var}(w) \cap X_{n}=\operatorname{var}(t) \cap X_{n}$. Since $\sigma_{s} \mathcal{L} \sigma_{t}$, thus by Lemma 5.7, $\operatorname{var}(s) \cap X_{n}=\operatorname{var}(t) \cap X_{n}$. Let $x_{j} \in \operatorname{var}(w) \cap X_{n}$. Then $x_{j} \in \operatorname{var}\left(p_{i}\right)$ for some $i \in J$ and $x_{j} \in X_{n}$. If $p_{i}=x_{i}$ where $x_{i} \in \operatorname{var}(t) \cap X_{n}$, then $x_{j}=x_{i} \in \operatorname{var}(t)$. If $p_{i}=\alpha\left[v_{\alpha(i)}\right]$, then $x_{j} \in \operatorname{var}\left(p_{i}\right)=\operatorname{var}\left(\alpha\left[v_{\alpha(i)}\right]\right)=\operatorname{var}\left(v_{\alpha(i)}\right) \subseteq \operatorname{var}(s)$. But $\operatorname{var}(s) \cap X_{n}=\operatorname{var}(t) \cap X_{n}$, so $x_{j} \in \operatorname{var}(t)$. Let $x_{j} \in \operatorname{var}(t) \cap X_{n}$. Then $p_{j}=x_{j}$ and so $x_{j} \in \operatorname{var}(s) \cap X_{n}$. Next we show that $\sigma_{w}$ is an idempotent. Let $x_{i} \in \operatorname{var}(w) \cap X_{n}$. Then $x_{i} \in \operatorname{var}(t) \cap X_{n}$. So $p_{i}=x_{i}$. Thus $\sigma_{w}$ is an idempotent. Finally we show that $s=f\left(v_{1}, \ldots, v_{n}\right)=\pi[w]$ where $\pi=\alpha^{-1}$. To do this we will show that for all $1 \leq k \leq n, v_{k}=\pi\left[p_{\pi(k)}\right]$. Let $1 \leq k \leq n$. If there exists $i \in J$ such that $\beta(i)=k$, then $\alpha(i)=k$ and $\pi(k)=\alpha^{-1}(k)=$ $i$. So $p_{i}=x_{i}=v_{k}$. Thus $\pi\left[p_{\pi(k)}\right]=\pi\left[p_{i}\right]=\pi\left[x_{i}\right]=x_{i}=v_{k}$. If no such index $i$ exists, then $\pi\left[p_{\pi(k)}\right]=\pi\left[\alpha\left[v_{\alpha(\pi(k))}\right]\right]=\pi\left[\alpha\left[v_{\alpha\left(\alpha^{-1}(k)\right)}\right]\right]=\pi\left[\alpha\left[v_{k}\right]\right]=$ $\alpha^{-1}\left[\alpha\left[v_{k}\right]\right]=v_{k}$.

Corollary 5.9. Let $\sigma_{t}$ be an idempotent in $\operatorname{Hyp}_{G}(n) \backslash\left(P_{G}(n) \cup G(n)\right)$. Then $D_{\sigma_{t}}=\left\{\sigma_{w} \mid w=C_{\alpha[\pi[s]]}\right.$ for some $\alpha$ bijection on $J, \pi$ a permutation on $J, s \notin X$, and $\sigma_{s}$ an idempotent with $\left.\operatorname{var}(s) \cap X_{n}=\operatorname{var}(t) \cap X_{n}\right\}$.

Theorem 5.10. Let $\sigma_{t}$ be an idempotent in $\operatorname{Hyp}_{G}(n) \backslash\left(P_{G}(n) \cup G(n)\right)$. Then its $\mathcal{J}$-class is equal to its $\mathcal{D}$-class.

Proof. Let $t=f\left(u_{1}, \ldots, u_{n}\right)$ and let $c$ be the number of distinct variables in $X_{n}$ which occur in $t$. Let $s=f\left(v_{1}, \ldots, v_{n}\right)$ with $\sigma_{s} \mathcal{J} \sigma_{t}$. Then there exist $f\left(a_{1}, \ldots, a_{n}\right), f\left(b_{1}, \ldots, b_{n}\right), f\left(p_{1}, \ldots, p_{n}\right), f\left(r_{1}, \ldots, r_{n}\right) \in W_{(n)}(X)$ such that

$$
\begin{equation*}
\sigma_{f\left(a_{1}, \ldots, a_{n}\right)} \circ_{G} \sigma_{f\left(v_{1}, \ldots, v_{n}\right)} \circ_{G} \sigma_{f\left(b_{1}, \ldots, b_{n}\right)}=\sigma_{f\left(u_{1}, \ldots, u_{n}\right)} \tag{1}
\end{equation*}
$$

$$
\begin{equation*}
\sigma_{f\left(p_{1}, \ldots, p_{n}\right)} \circ_{G} \sigma_{f\left(u_{1}, \ldots, u_{n}\right)} \circ_{G} \sigma_{f\left(r_{1}, \ldots, r_{n}\right)}=\sigma_{f\left(v_{1}, \ldots, v_{n}\right)} \tag{2}
\end{equation*}
$$

Let $f\left(q_{1}, \ldots, q_{n}\right)$ be the term for $\sigma_{f\left(v_{1}, \ldots, v_{n}\right)} \circ_{G} \sigma_{f\left(b_{1}, \ldots, b_{n}\right)}$. We write $\sigma=$ $\sigma_{f\left(a_{1}, \ldots, a_{n}\right)}$. From (1), we get $S^{n}\left(f\left(a_{1}, \ldots, a_{n}\right), \hat{\sigma}\left[q_{1}\right], \ldots, \hat{\sigma}\left[q_{n}\right]\right)=f\left(u_{1}, \ldots, u_{n}\right)$.

If $x_{k} \in \operatorname{var}(t) \cap X_{n}$, then $u_{k}=x_{k}$ since $\sigma_{t}$ is an idempotent. So $a_{k}=x_{j}$ for some $x_{j} \in X_{n}$. This implies $\hat{\sigma}\left[q_{j}\right]=x_{k}$ and then $q_{j}=x_{k}$. Let $\alpha$ be a function from $J(t)$ to $J$ defined by $\alpha(k)=j$ if $x_{k} \in \operatorname{var}(t) \cap X_{n}$ and $a_{k}=x_{j}$ where $J(t)=\left\{k \in J \mid x_{k} \in \operatorname{var}(t)\right\}$. So $\alpha$ can be extended to a permutation on $J$.

We write $\sigma_{1}=\sigma_{f\left(v_{1}, \ldots, v_{n}\right)}$. Since $f\left(q_{1}, \ldots, q_{n}\right)$ is the term for $\sigma_{f\left(v_{1}, \ldots, v_{n}\right)^{\circ} G}$ $\sigma_{f\left(b_{1}, \ldots, b_{n}\right)}$, thus $S^{n}\left(f\left(v_{1}, \ldots, v_{n}\right), \hat{\sigma}_{1}\left[b_{1}\right], \ldots, \hat{\sigma}_{1}\left[b_{n}\right]\right)=f\left(q_{1}, \ldots, q_{n}\right)$. For each $k \in J(t), q_{\alpha(k)}=x_{k}$. So $v_{\alpha_{k}}=x_{l}$ for some $x_{l} \in X_{n}$. So $\hat{\sigma}_{1}\left[b_{l}\right]=x_{k}$ and then $b_{l}=x_{k}$. Let $\beta: \alpha(J(t)) \rightarrow J$ defined by $\beta(\alpha(k))=l$ where $k \in J(t)$ and $v_{\alpha(k)}=x_{l}$. So $\beta$ can be extended to a permutation on $J$. Since $\alpha$ and $\beta$ are injective, thus at least $c$ distinct variables in $X_{n}$ occur as $v_{i}$ in entries of $s=f\left(v_{1} \ldots, v_{n}\right)$. We claim that the only variables in $X_{n}$ which occur in $s$ are those $c$ variables. Let $f\left(c_{1}, \ldots, c_{n}\right)$ be the term for $\sigma_{f\left(u_{1}, \ldots, u_{n}\right)}{ }^{\circ} \sigma_{f\left(r_{1}, \ldots, r_{n}\right)}$.

We write $\sigma_{2}=\sigma_{f\left(p_{1}, \ldots, p_{n}\right)}$. From (2), we get $S^{n}\left(f\left(p_{1}, \ldots, p_{n}\right), \hat{\sigma}_{2}\left[c_{1}\right], \ldots\right.$, $\left.\hat{\sigma}_{2}\left[c_{n}\right]\right)=f\left(v_{1}, \ldots, v_{n}\right)$. Since at least $c$ distinct variables in $X_{n}$ occur as $v_{i}$ in entries $s=f\left(v_{1} \ldots, v_{n}\right)$, thus at least $c$ distinct variables in $X_{n}$ occur as $p_{i}$ in entries $s=f\left(p_{1} \ldots, p_{n}\right)$ and then at least $c$ distinct variables in $X_{n}$ occur as $c_{i}$ in entries $f\left(c_{1} \ldots, c_{n}\right)$.

We write $\sigma_{3}=\sigma_{f\left(u_{1}, \ldots, u_{n}\right)}$. Since $f\left(c_{1}, \ldots, c_{n}\right)$ is the term for $\sigma_{f\left(u_{1}, \ldots, u_{n}\right)^{\circ} G}$ $\sigma_{f\left(r_{1}, \ldots, r_{n}\right)}$, thus $S^{n}\left(f\left(u_{1}, \ldots, u_{n}\right), \hat{\sigma}_{3}\left[r_{1}\right], \ldots, \hat{\sigma}_{3}\left[r_{n}\right]\right)=f\left(c_{1}, \ldots, c_{n}\right)$. But $f\left(u_{1}, \ldots, u_{n}\right)$ has only $c$ distinct variables in $X_{n}$. Thus all the $r_{j}^{\prime} s$ used in the composition in (2) are variables in $X_{n}$. So the number of distinct variables in $X_{n}$ which occur in $f\left(v_{1}, \ldots, v_{n}\right)$ is at most $c$. Thus the number of distinct variables in $X_{n}$ which occur in $f\left(v_{1}, \ldots, v_{n}\right)$ is $c$ and every variable in $X_{n}$ which occurs in it occurs as a $v_{i}$. Let $w_{1}=C_{(\beta \circ \alpha)^{-1}}\left[f\left(v_{1}, \ldots, v_{n}\right)\right]$. So $\operatorname{var}\left(w_{1}\right) \cap X_{n}=\operatorname{var}(t) \cap X_{n}$. From Lemma 5.4, we get $\sigma_{w_{1}} \mathcal{R} \sigma_{s}$. Let $w_{2}=\alpha\left[w_{1}\right]$. From Lemma 5.4, $\sigma_{w_{2}} \mathcal{L} \sigma_{w_{1}}$. We will show that $\sigma_{w_{2}}$ is an idempotent. Let $w_{1}=C_{(\beta \circ \alpha)^{-1}}\left[f\left(v_{1}, \ldots, v_{n}\right)\right]=f\left(d_{1}, \ldots, d_{n}\right)$. For each $x_{k} \in \operatorname{var}(t) \cap X_{n}, v_{\alpha(k)}=x_{\beta(\alpha(k))}$. So $d_{\alpha(k)}=x_{k}$. From $w_{2}=\alpha\left[w_{1}\right]$, we get $w_{2}=\alpha\left[f\left(d_{1}, \ldots, d_{n}\right)\right]=f\left(\alpha\left[d_{\alpha(1)}\right], \ldots, \alpha\left[d_{\alpha(n)}\right]\right)$ and $\operatorname{var}\left(w_{2}\right) \cap X_{n}=$ $\operatorname{var}(t) \cap X_{n}$. Let $x_{j} \in \operatorname{var}\left(w_{2}\right) \cap X_{n}$. Then $x_{j} \in \operatorname{var}(t) \cap X_{n}$. So $\alpha\left[d_{\alpha(j)}\right]=$ $\alpha\left[x_{j}\right]=x_{j}$. So $\sigma_{w_{2}}$ is an idempotent. By Lemma 5.7, $\sigma_{w_{2}} \mathcal{L} \sigma_{t}$. So $\sigma_{w_{1}} \mathcal{L} \sigma_{t}$. Thus $\sigma_{s} \mathcal{D} \sigma_{t}$.

Corollary 5.11. Let $\sigma_{s}, \sigma_{t}$ be idempotents in $\operatorname{Hyp}_{G}(n) \backslash\left(P_{G}(n) \cup G(n)\right)$. Then $\sigma_{s}$ and $\sigma_{t}$ are $\mathcal{J}$ - or $\mathcal{D}$-related if and only if the number of distinct variables in $X_{n}$ which occur in $s$ and $t$ are equal.

Proof. One direction follows immediately from Corollary 5.9. Conversely, let $s=f\left(u_{1}, \ldots, u_{n}\right), t=f\left(v_{1}, \ldots, v_{n}\right), \operatorname{var}(s) \cap X_{n}=\left\{x_{k_{1}}, \ldots, x_{k_{c}}\right\}$ and $\operatorname{var}(t) \cap X_{n}=\left\{x_{l_{1}}, \ldots, x_{l_{c}}\right\}$. Since $\sigma_{s}, \sigma_{t}$ are idempotents, thus $u_{k_{j}}=x_{k_{j}}$ and $v_{l_{j}}=x_{l_{j}}$ for all $1 \leq j \leq c$. Let $s^{\prime}=f\left(u_{1}^{\prime}, \ldots, u_{n}^{\prime}\right), t^{\prime}=f\left(v_{1}^{\prime}, \ldots, v_{n}^{\prime}\right)$ where $u_{k_{j}}^{\prime}=x_{k_{j}}$ and $v_{l_{j}}^{\prime}=x_{l_{j}}$ for all $1 \leq j \leq c$ and other $u_{j}^{\prime}=x_{k_{1}}, v_{j}^{\prime}=x_{l_{1}}$. By Lemma 5.7, $\sigma_{s} \mathcal{L} \sigma_{s^{\prime}}$ and $\sigma_{t} \mathcal{L} \sigma_{t^{\prime}}$. Let $\pi\left(l_{j}\right)=k_{j}$ for all $1 \leq j \leq c$. Then $\pi$ is injective. So $\pi$ can be extended to a permutation on $J$, which we will also call $\pi$. So $\pi\left[s^{\prime}\right]=f\left(p_{1}, \ldots, p_{n}\right)$ where $p_{l_{j}}=x_{k_{j}}$ for all $1 \leq j \leq c$ and other $p_{j}=x_{k_{1}}$. Let $\alpha\left(k_{j}\right)=l_{j}$ for all $1 \leq j \leq c$. So $\alpha$ can be extended to a bijection on $J$, which we will also call $\alpha$. So $C_{\alpha\left[\pi\left[s^{\prime}\right]\right]}=t^{\prime}$. Thus $\sigma_{s} \mathcal{J} \sigma_{s^{\prime}} \mathcal{J} \sigma_{C_{\alpha\left[\pi\left[s^{\prime}\right]\right]}}=t^{\prime} \mathcal{J} \sigma_{t}$.

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