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THE MONOID OF GENERALIZED HYPERSUBSTITUTIONS OF TYPE $\tau = (n)$

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Abstract

A (usual) hypersubstitution of type τ is a function which takes each operation symbol of the type to a term of the type, of the same arity. The set of all hypersubstitutions of a fixed type τ forms a monoid under composition, and semigroup properties of this monoid have been studied by a number of authors. In particular, idempotent and regular elements, and the Green's relations, have been studied for type (n) by S.L. Wismath.

A generalized hypersubstitution of type $\tau = (n)$ is a mapping σ which takes the n-ary operation symbol f to a term $\sigma(f)$ which does not necessarily preserve the arity. Any such σ can be inductively extended to a map $\hat{\sigma}$ on the set of all terms of type $\tau = (n)$, and any two such extensions can be composed in a natural way. Thus, the set $Hyp_G(n)$ of all generalized hypersubstitutions of type $\tau = (n)$ forms a monoid. In this paper we study the semigroup properties of $Hyp_G(n)$.

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In particular, we characterize the idempotent and regular generalized hypersubstitutions, and describe some classes under Green's relations of this monoid.

Keywords: monoid, regular elements, idempotent elements, Green's relations, generalized hypersubstitution.

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1. Introduction

Identities are used to classify algebras into collections called *varieties.* Hyperidentities are used to classify varieties into collections called *hypervarieties.* The concepts of hyperidentities and hypervarieties were introduced by W. Taylor in 1981 [7]. Hyperidentities in a variety V are identities which have the property that, after replacing the operation symbols which occur in these identities by any terms of the same arity, the resulting equation is still satisfied in the variety. The main tool to study hyperidentities is the concept of a hypersubstitution, which was introduced by K. Denecke, D. Lau, R. Pöschel and D. Schweigert in 1991 [1]. Let $\tau = (n_i)_{i \in I}$ be a type and let $W_\tau(X)$ be the set of all terms of type τ built up by operation symbols from $\{f_i | i \in I\}$ where f_i is n_i-ary and variables from a countably infinite alphabet $X := \{x_1, x_2, \ldots\}$. A hypersubstitution of type τ is a mapping $\sigma : \{f_i | i \in I\} \to W_\tau(X)$ which maps n_i-ary operation symbols to n_i-ary terms. Let $Hyp(\tau)$ be the set of all hypersubstitutions of type τ. For every $\sigma \in Hyp(\tau)$ induces a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ as follows: for any $t \in W_\tau(X)$, $\hat{\sigma}[t]$ is defined inductively by

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := \sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i.

It turns out that $(Hyp(\tau); \circ_h, \sigma_{id})$ is a monoid where $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ and $\sigma_{id}(f_i) = f_i(x_1, \ldots, x_{n_i})$ is the identity element.

In 2000, S. Leeratanavalee and K. Denecke generalized the concept of a hypersubstitution to a generalized hypersubstitution [2]. S. Leeratanavalee and K. Denecke used generalized hypersubstitutions as the tools to study strong hyperidentities and used strong hyperidentities to classify varieties into collections called *strong hypervarieties*. Varieties whose identities are closed under arbitrary application of generalized hypersubstitutions are called *strongly solid*.

A generalized hypersubstitution of type τ, or for short simply a generalized hypersubstitution, is a mapping σ which maps each n_i-ary operation symbol of type τ to a term of this type in $W_\tau(X)$ which does not necessarily preserve the arity. We denoted the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$. First, we define inductively the concept of *generalized superposition of terms* $S^m : W_\tau(X)^{m+1} \to W_\tau(X)$ by the following steps:

(i) If $t = x_j, 1 \leq j \leq m$, then $S^m(x_j, t_1, \ldots, t_m) := t_j$.

(ii) If $t = x_j, m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \ldots, t_m) := x_j$.

(iii) If $t = f_i(s_1, \ldots, s_{n_i})$, then
$S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$.

To define a binary operation on $Hyp_G(\tau)$, we extend a generalized hypersubstitution σ to a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ inductively defined as follows:

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i.

Then we define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$. S. Leeratanavalee and K. Denecke proved the following propositions.

Proposition 1.1 ([2]). *For arbitrary terms $t, t_1, \ldots, t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$ we have*

(i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \ldots, t_n)]$,

(ii) $(\hat{\sigma}_1 \circ \sigma_2)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2$.

Proposition 1.2 ([2]). $\underline{Hyp_G(\tau)} = (Hyp_G(\tau); \circ_G, \sigma_{id})$ *is a monoid, with* σ_{id} *as the identity element, and the set of all hypersubstitutions of type* τ *forms a submonoid of* $\underline{Hyp_G(\tau)}$.

Many properties of the monoid of hypersubstitutions of type $\tau = (n)$ were described by S.L. Wismath [8]. In this paper we extend the results from [8] to the case of $Hyp_G(n)$.

2. Projection and dual generalized hypersubstitutions of type $\tau = (n)$

We assume that from now on the type $\tau = (n)$, for some $n \in I\!N$, i.e. we have only one n-ary operation symbol, say f. By σ_t we denote the generalized hypersubstitution which maps f to the term t in $W_{(n)}(X)$. A generalized hypersubstitution σ_t is called a *projection generalized hypersubstitution* if t is a variable [3]. We denoted the set of all projection generalized hypersubstitutions of type $\tau = (n)$ by $P_G(n)$, i.e. $P_G(n) = \{\sigma_{x_i} | x_i \in X\}$.

Lemma 2.1. *For any* $\sigma_t \in Hyp_G(n)$ *and* $\sigma_{x_i} \in P_G(n)$, *we have*

(i) $\sigma_t \circ_G \sigma_{x_i} = \sigma_{x_i}$,

(ii) $\sigma_{x_i} \circ_G \sigma_t \in P_G(n)$.

Proof. (i) We have $(\sigma_t \circ_G \sigma_{x_i})(f) = (\hat{\sigma}_t \circ \sigma_{x_i})(f) = \hat{\sigma}_t[\sigma_{x_i}(f)] = \hat{\sigma}_t[x_i] = x_i = \sigma_{x_i}(f)$. So $\sigma_t \circ_G \sigma_{x_i} = \sigma_{x_i}$.

(ii) We will proceed by induction on the complexity of the term t. If $t \in X$, then by (i) we get $\sigma_{x_i} \circ_G \sigma_t = \sigma_t \in P_G(n)$. Assume that $t = f(u_1, \ldots, u_n)$ and $\sigma_{x_i} \circ_G \sigma_{u_1}, \ldots, \sigma_{x_i} \circ_G \sigma_{u_n} \in P_G(n)$. Thus $\hat{\sigma}_{x_i}[u_1], \ldots, \hat{\sigma}_{x_i}[u_n] \in X$. We have $(\sigma_{x_i} \circ_G \sigma_t)(f) = (\sigma_{x_i} \circ_G \sigma_{f(u_1,\ldots,u_n)})(f) = S^n(x_i, \hat{\sigma}_{x_i}[u_1], \ldots, \hat{\sigma}_{x_i}[u_n])$. If $x_i \in X_n$ where $X_n = \{x_1, \ldots, x_n\}$, then $(\sigma_{x_i} \circ_G \sigma_t)(f) = \hat{\sigma}_{x_i}[u_i] \in X$. If $i > n$, then $(\sigma_{x_i} \circ_G \sigma_t)(f) = x_i \in X$. So $\sigma_{x_i} \circ_G \sigma_t \in P_G(n)$.

Corollary 2.2.

(i) $P_G(n) \cup \{\sigma_{id}\}$ *is a submonoid of* $Hyp_G(n)$ *and* $P_G(n)$ *is the smallest two-sided ideal of* $Hyp_G(n)$, *called the kernel of* $Hyp_G(n)$. *Thus,* $Hyp_G(n)$ *is not simple.*

(ii) *$P_G(n)$ is the set of all right-zero elements of $Hyp_G(n)$, so that $P_G(n)$ itself is a right-zero band.*

(iii) *$Hyp_G(n)$ contains no left-zero elements.*

Proof. These follow immediately from Lemma 2.1.

Another special kind of generalized hypersubstitutions in $Hyp_G(n)$ are *dual generalized hypersubstitutions*, which are defined using permutations of the set $J := \{1, \ldots, n\}$. For any such permutation π, we let $\sigma_\pi = \sigma_{f(x_{\pi(1)},\ldots,x_{\pi(n)})}$. We let D_G be the set of all such dual generalized hypersubstitutions. ■

Lemma 2.3.

(i) *For any two permutations π and $\rho, \sigma_\rho \circ_G \sigma_\pi = \sigma_{\pi\circ\rho}$.*

(ii) *For any permutation π with the inverse permutation π^{-1}, the generalized hypersubstitutions σ_π and $\sigma_{\pi^{-1}}$ are inverse of each other.*

Proof. (i) We have $(\sigma_\rho \circ_G \sigma_\pi)(f) = \hat{\sigma}_\rho[f(x_{\pi(1)}, \ldots, x_{\pi(n)})] = S^n(f(x_{\rho(1)}, \ldots, x_{\rho(n)}), x_{\pi(1)}, \ldots, x_{\pi(n)}) = f(x_{\pi(\rho(1))}, \ldots, x_{\pi(\rho(n))}) = \sigma_{\pi\circ\rho}(f)$.

(ii) This follows from (i). ■

Lemma 2.4. *If $\sigma \circ_G \rho \in D_G$, then both σ and ρ are in D_G.*

Proof. Let $\sigma(f) = f(u_1, \ldots, u_n)$ and $\rho(f) = f(v_1, \ldots, v_n)$. Since $\sigma \circ_G \rho \in D_G$, thus there exists a permutation π such that $(\sigma \circ_G \rho)(f) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$. So $f(x_{\pi(1)}, \ldots, x_{\pi(n)}) = (\sigma \circ_G \rho)(f) = S^n(f(u_1, \ldots, u_n), \hat{\sigma}[v_1], \ldots, \hat{\sigma}[v_n])$. Since π is a permutation, thus this forces all the u_i's to be distinct variables in X_n, and all the v_i's to be distinct variables in X_n. It follows that both σ and ρ are in D_G. ■

Corollary 2.5. *$\underline{D_G}$ is a submonoid of $\underline{Hyp_G(n)}$ which forms a group, and no other elements of $Hyp_G(n)$ have inverses in $Hyp_G(n)$. Thus, $\underline{D_G}$ is the largest subgroup of $\underline{Hyp_G(n)}$.*

Lemma 2.6. *Let F be the set of generalized hypersubstitutions of the form $\sigma_{f(x_i,\ldots,x_i)}$ for $i \in \mathbb{N}$. Let $M = P_G(n) \cup D_G \cup F$. Then $\underline{M}$ is a submonoid of $\underline{Hyp_G(n)}$.*

Proof. It is straightforward to check that any product of two elements in M is also in M.

3. Idempotent and regular elements in $Hyp_G(n)$

All idempotent elements of the monoid of all generalized hypersubstitutions of type $\tau = (2)$ were studied by W. Puninagool and S. Leeratanavalee [6] and all regular elements of the monoid of all generalized hypersubstitutions of type $\tau = (2)$ were studied by W. Puninagool and S. Leeratanavalee [4]. In this section, we characterize the idempotent and regular generalized hypersubstitutions of type $\tau = (n)$.

We know from Corollary 2.2 (ii) that every projection generalized hypersubstitution is idempotent. We let $G(n) := \{\sigma_t | t \notin X, var(t) \cap X_n = \emptyset\}$ where $var(t)$ denotes the set of all variables occurring in t.

Lemma 3.1. *If $\sigma_t \in G(n)$ and $\sigma_s \in Hyp_G(n) \setminus P_G(n)$, then $\sigma_t \circ_G \sigma_s = \sigma_t$, i.e. $G(n)$ itself is a left zero band.*

Proof. Let $s = f(v_1, \ldots, v_n)$. We have $(\sigma_t \circ_G \sigma_s)(f) = S^n(t, \hat{\sigma}_t[v_1], \ldots, \hat{\sigma}_t[v_n]) = t$ since there is nothing to substitute in the term t. So $\sigma_t \circ_G \sigma_s = \sigma_t$. ■

Then we consider only the case $\sigma_t \in Hyp_G(n) \setminus P_G(n)$ and $var(t) \cap X_n \neq \emptyset$.

Theorem 3.2. *Let $t = f(t_1, \ldots, t_n) \in W_{(n)}(X)$ and $\emptyset \neq var(t) \cap X_n = \{x_{i_1}, \ldots, x_{i_m}\}$. Then σ_t is idempotent if and only $t_{i_k} = x_{i_k}$ for all $k \in \{1, \ldots, m\}$.*

Proof. Assume that $\sigma_{f(t_1,\ldots,t_n)}$ is idempotent. Then $S^n(f(t_1, \ldots, t_n), \hat{\sigma}_{f(t_1,\ldots,t_n)}[t_1], \ldots, \hat{\sigma}_{f(t_1,\ldots,t_n)}[t_n]) = \sigma^2_{f(t_1,\ldots,t_n)}(f) = \sigma_{f(t_1,\ldots,t_n)}(f) = f(t_1, \ldots, t_n)$. Suppose that there exists $k \in \{1, \ldots, m\}$ such that $t_{i_k} \neq x_{i_k}$. If $t_{i_k} \in X$, then $\hat{\sigma}_{f(t_1,\ldots,t_n)}[t_{i_k}] = t_{i_k} \neq x_{i_k}$. So $S^n(f(t_1, \ldots, t_n), \hat{\sigma}_{f(t_1,\ldots,t_n)}[t_1], \ldots, \hat{\sigma}_{f(t_1,\ldots,t_n)}[t_n]) \neq f(t_1, \ldots, t_n)$ and we have a contradiction.

If $t_{i_k} \notin X$, then $\hat{\sigma}_{f(t_1,\ldots,t_n)}[t_{i_k}] \notin X$. We obtain $op(t) = op(S^n(f(t_1,\ldots,t_n),$ $\hat{\sigma}_{f(t_1,\ldots,t_n)}\ [t_1],\ldots,\hat{\sigma}_{f(t_1,\ldots,t_n)}[t_n])) > op(t)$ where $op(t)$ denotes the number of all operation symbols occurring in t. This is a contradiction. ■

The proof of the converse direction is straightforward.

Now, we characterize the regular generalized hypersubstitutions of type $\tau = (n)$. At first we want to recall the definition of the regular element.

Definition 3.3. An element a of a semigroup S is called *regular* if there exists $x \in S$ such that $axa = a$. The semigroup S is called *regular* if all its elements are regular.

Lemma 3.4. *Let* $t \in W_{(n)}(X)$ *and* $\emptyset \neq var(t) \cap X_n = \{x_{i_1},\ldots,x_{i_m}\}$ *and let* $a = f(a_1,\ldots,a_n) \in W_{(n)}(X)$. *If* $\hat{\sigma}_t[a] = t$, *then* $a_l = x_l$ *for all* $l = i_1,\ldots,i_m$.

Proof. Assume that $\hat{\sigma}_t[a] = t$. Then $t = \hat{\sigma}_t[a] = S^n(t,\hat{\sigma}_t[a_1],\ldots,\hat{\sigma}_t[a_n])$. We will show that $a_l = x_l$ for all $l = i_1,\ldots,i_m$. Suppose that there exists $j' \in \{i_1,\ldots,i_m\}$ such that $a_{j'} \neq x_{j'}$. If $a_{j'} = x_k \in X$ where $x_k \neq x_{j'}$, then $\hat{\sigma}_t[a_{j'}] = x_k$. It follows that $t \neq S^n(t,\hat{\sigma}_t[a_1],\ldots,\hat{\sigma}_t[a_n])$. This is a contradiction. If $a_{j'} \notin X$, then $\hat{\sigma}_t[a_{j'}] \notin X$. It follows that $op(t) = op(\hat{\sigma}_t[a]) = S^n(t,\hat{\sigma}_t[a_1],\ldots,\hat{\sigma}_t[a_n]) > op(t)$ and we have a contradiction. ■

Theorem 3.5. *Let* $t = f(t_1,\ldots,t_n) \in W_{(n)}(X)$ *and* $\emptyset \neq var(t) \cap X_n = \{x_{i_1},\ldots,x_{i_m}\}$. *Then* σ_t *is regular if and only if there exist* $j_1,\ldots,j_m \in \{1,\ldots,n\}$ *such that* $t_{j_1} = x_{i_1},\ldots,t_{j_m} = x_{i_m}$.

Proof. Assume that σ_t is regular. Then there exists $\sigma_s \in Hyp_G(n)$ such that $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$. Since $t \notin X$, thus $s \notin X$. Then $s = f(s_1,\ldots,s_n)$ for some $s_1,\ldots,s_n \in W_{(n)}(X)$. From $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$, thus $\hat{\sigma}_t[\hat{\sigma}_s[t]] = t$. By Lemma 3.4, $\hat{\sigma}_s[t] = f(u_1,\ldots,u_n)$ for some $u_1,\ldots,u_n \in W_{(n)}(X)$ where $u_{i_1} = x_{i_1},\ldots,u_{i_m} = x_{i_m}$. From $\hat{\sigma}_s[t] = f(u_1,\ldots,u_n)$, thus $S^n(f(s_1,\ldots,s_n),\hat{\sigma}_s[t_1],\ldots,\hat{\sigma}_s[t_n]) = f(u_1,\ldots,u_n)$. Since $u_{i_1} = x_{i_1},\ldots,u_{i_m} = x_{i_m}$ thus $s_{i_1},\ldots,s_{i_m} \in X_n$. Let $s_{i_1} = x_{j_1},\ldots,s_{i_m} = x_{j_m}$. Hence $t_{j_1} = x_{i_1},\ldots,t_{j_m} = x_{i_m}$. ■

Conversely, assume the condition holds. Let $s = f(s_1, \dots, s_n) \in W_{(n)}(X)$ where $s_1, \dots, s_n \in W_{(n)}(X)$ such that $s_{i_1} = x_{j_1}, \dots, s_{i_m} = x_{j_m}$.

Then $(\sigma_t \circ_G \sigma_s \circ_G \sigma_t)(f) = \hat{\sigma}_t[\hat{\sigma}_s[t]] = \hat{\sigma}_t[S^n(f(s_1, \dots, s_n), \hat{\sigma}_s[t_1], \dots, \hat{\sigma}_s[t_n])] = \hat{\sigma}_t[f(k_1, \dots, k_n)]$(where $k_{i_1} = x_{i_1}, \dots, k_{i_m} = x_{i_m}$) $= S^n(t, \hat{\sigma}_t[k_1], \dots, \hat{\sigma}_t[k_n]) = t$. Hence $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$.

4. Term properties of the composition operation

We need to know more about the result of the composing two generalized hypersubstitutions in $Hyp_G(n)$. In particular, we want to know how long the term corresponding to $\sigma_s \circ_G \sigma_t$ is and what variables it uses, compared to the lengths of the terms s and t and the variables they use. We begin with the necessary definitions.

Definition 4.1. Let $t \in W_{(n)}(X)$. We define the length of t inductively by:

(i) The length of t is 1 if t is a variable.

(ii) If t is a compound term $f(t_1, \dots, t_n)$, then the length of t is the sum of the lengths of the terms $t_1, \dots, t_n$.

(iii) This length counts the total number of variable occurences in the term t, and will be denoted by $vb(t)$.

Definition 4.2 ([8]). Let $t \in W_{(n)}(X)$. We define some new terms, related to t, as follows. Recall that $J := \{1, \dots, n\}$.

(i) Let α be any function from J to J. $C_\alpha[t]$ is the term formed from t by replacing each occurrence in t of a variable $x_i \in X_n$ by the variable $x_{\alpha(i)}$ i.e, $C_\alpha[t] = S^n(t, x_{\alpha(1)}, \dots, x_{\alpha(n)})$.

(ii) Let π be any permutation of J. $\pi[t]$ is the term defined inductively by $\pi[x_i] = x_i$ for any variable x_i, and $\pi[f(u_1, \dots, u_n)] = f(\pi[u_{\pi(1)}], \dots, \pi[u_{\pi(n)}])$.

The previous length results for the type $\tau = (2)$ were found by W. Puninagool and S. Leeratanavalee in [6] and S.L. Wismath in [8]. The next two lemmas show how these results can be generalized to the type $\tau = (n)$.

Lemma 4.3. *Let $n \in N$ with $n > 1$. Let $\sigma_{f(u_1,\ldots,u_n)} \circ_G \sigma_{f(v_1,\ldots,v_n)} = \sigma_w$. Then w is a longer term than $f(u_1,\ldots,u_n)$, unless the terms $f(u_1,\ldots,u_n)$ and $f(v_1,\ldots,v_n)$ satisfy the following condition* (Q):

(Q) *If a variable $x_i \in X_n$ is used anywhere in the term $f(u_1,\ldots,u_n)$, then the entry v_i in $f(v_1,\ldots,v_n)$ is a variable.*

Proof. If $var(f(u_1,\ldots,u_n)) \cap X_n = \emptyset$, then $f(u_1,\ldots,u_n)$ and $f(v_1,\ldots,v_n)$ satisfy the condition (Q). Let $var(f(u_1,\ldots,u_n)) \cap X_n = \{x_{i_1},\ldots,x_{i_k}\}$. If $v_{i_j} \in X$ for all $j \in \{1,\ldots,k\}$, then $f(u_1,\ldots,u_n)$ and $f(v_1,\ldots,v_n)$ satisfy the condition (Q). If there exists $j \in \{1,\ldots,k\}$ where $v_{i_j} \notin X$, then $\hat{\sigma}_{f(u_1,\ldots,u_n)}[v_{i_j}] \notin X$. Since $n > 1$ and $\hat{\sigma}_{f(u_1,\ldots,u_n)}[v_{i_j}] \notin X$, thus $vb(\hat{\sigma}_{f(u_1,\ldots,u_n)}[v_{i_j}]) > 1$. So $vb(w) = vb(S^n(f(u_1,\ldots,u_n),\hat{\sigma}_{f(u_1,\ldots,u_n)}[v_1],\ldots,\hat{\sigma}_{f(u_1,\ldots,u_n)}[v_n])) > vb(f(u_1,\ldots,u_n))$. ∎

Lemma 4.4. *Let $\sigma_t \in Hyp_G(n)$ where $t \notin X$ and $x_1,\ldots,x_n \in var(t)$. Then for any $s \in W_{(n)}(X), vb(\hat{\sigma}_t[s]) \geq vb(s)$.*

Proof. We will proceed by induction on the complexity of the term s. If $s \in X$, then $vb(\hat{\sigma}_t[s]) = vb(s)$. Assume that $s = f(u_1,\ldots,u_n)$ and $vb(\hat{\sigma}_t[u_i]) \geq vb(u_i)$ for all $1 \leq i \leq n$. Then $vb(\hat{\sigma}_t[s]) = vb(S^n(t,\hat{\sigma}_t[u_1],\ldots,\hat{\sigma}_t[u_n])) \geq vb(f(u_1,\ldots,u_n))$ since $x_1,\ldots,x_n \in var(t)$ and $vb(\hat{\sigma}_t[u_i]) \geq vb(u_i)$ for all $1 \leq i \leq n$. ∎

Lemma 4.5. *Let $\sigma_{f(u_1,\ldots,u_n)} \circ_G \sigma_{f(v_1,\ldots,v_n)} = \sigma_w$ where $vb(f(u_1,\ldots,u_n)) > n$. If $x_1,\ldots,x_n \in var(f(u_1,\ldots,u_n))$, then w is a longer term than $f(v_1,\ldots,v_n)$.*

Proof. We write $\sigma = \sigma_{f(u_1,\ldots,u_n)}$. From $\sigma_{f(u_1,\ldots,u_n)} \circ_G \sigma_{f(v_1,\ldots,v_n)} = \sigma_w$, thus we get $w = S^n(f(u_1,\ldots,u_n),\hat{\sigma}[v_1],\ldots,\hat{\sigma}[v_n])$. Since $x_1,\ldots,x_n \in var(f(u_1,\ldots,u_n))$, thus $\hat{\sigma}[v_i]$ is used in w for all $1 \leq i \leq n$. We will proceed by induction on the complexity of the term $f(v_1,\ldots,v_n)$. If $v_1,\ldots,v_n \in X$, then $vb(w) = vb(f(u_1,\ldots,u_n)) > n = vb(f(v_1,\ldots,v_n))$. Assume that the claim holds for any term of length not less than n but less than k, and $f(v_1,\ldots,v_n)$ has length k. Since $vb(f(v_1,\ldots,v_n)) = k > n$, thus there exists $i \in \{1,\ldots,n\}$ such that $vb(v_i) \geq n$. By induction, we get $vb(\hat{\sigma}[v_i]) > vb(v_i)$. By Lemma 4.4, any other v_j has $vb(\hat{\sigma}[v_j]) \geq vb(v_j)$. Since all the $\hat{\sigma}[v_i]$ are used in w for all $1 \leq i \leq n$, thus w is longer than $f(v_1,\ldots,v_n)$. ∎

Lemma 4.6. *Let $\sigma_s, \sigma_t \in Hyp_G(n)$.*

(i) $var((\sigma_s \circ_G \sigma_t)(f)) \cap X_n \subseteq var(t) \cap X_n$.

(ii) *If s uses only one variable, then the term for $\sigma_s \circ_G \sigma_t$ uses only one variable (not necessarily the same variable as s).*

Proof. We will proceed by induction on the complexity of the term t.

(i) If $t \in X$, then $(\sigma_s \circ_G \sigma_t)(f) = t$. So $var((\sigma_s \circ_G \sigma_t)(f)) \cap X_n \subseteq var(t) \cap X_n$. Assume that $t = f(t_1, \ldots, t_n)$ and $var(\hat{\sigma}_s[t_i]) \cap X_n \subseteq var(t_i) \cap X_n$ for all $1 \leq i \leq n$. So $var((\sigma_s \circ_G \sigma_t)(f)) \cap X_n = var(S^n(s, \hat{\sigma}_s[t_1], \ldots, \hat{\sigma}_s[t_n])) \cap X_n \subseteq \cup_{i=1}^n (var(\hat{\sigma}_s[t_i])) \cap X_n = \cup_{i=1}^n (var(\hat{\sigma}_s[t_i]) \cap X_n) \subseteq \cup_{i=1}^n (var(t_i) \cap X_n) = \cup_{i=1}^n var(t_i) \cap X_n = var(t) \cap X_n$.

(ii) If $t \in X$, then $(\sigma_s \circ_G \sigma_t)(f) = t$. So the term for $\sigma_s \circ_G \sigma_t$ uses only one variable. Assume that $t = f(t_1, \ldots, t_n)$ and $\hat{\sigma}_s[t_i]$ uses only one variable for all $1 \leq i \leq n$. So $(\sigma_s \circ_G \sigma_t)(f) = S^n(s, \hat{\sigma}_s[t_1], \ldots, \hat{\sigma}_s[t_n])$. If $var(s) = \{x_i\}$ for some $x_i \in X_n$, then $var(\sigma_s \circ_G \sigma_t)(f) = var(\hat{\sigma}_s[t_i])$. If $var(s) = \{x_i\}$ where $i > n$, then $var(\sigma_s \circ_G \sigma_t)(f) = var(s)$. ■

We conclude this section by extending the results from [8] to the case of $Hyp_G(n)$ on properties of the composition operation with a lemma describing the special role of the terms $\pi[t]$ and $C_{\alpha[t]}$ from Definition 4.2.

Lemma 4.7. *For $t \in W_{(n)}(X)$.*

(i) *Let π be any permutation on J. Then $\sigma_\pi \circ_G \sigma_t = \sigma_{\pi[t]}$.*

(ii) *Let α be any function on J. Define the generalized hypersubstitution σ_α by mapping the fundemental f to the term $f(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$. Then $\sigma_t \circ_G \sigma_\alpha = \sigma_{C_{\alpha[t]}}$.*

Proof. (i) We will proceed by induction on the complexity of the term t. If $t \in X$, then by Lemma 2.1(i), $\sigma_\pi \circ_G \sigma_t = \sigma_t = \sigma_{\pi[t]}$. Assume that $t = f(t_1, \ldots, t_n)$ and $\hat{\sigma}_\pi[t_i] = \pi[t_i]$ for all $1 \leq i \leq n$. So $(\sigma_\pi \circ_G \sigma_t)(f) = S^n(f(x_{\pi(1)}, \ldots, x_{\pi(n)}), \hat{\sigma}_\pi[t_1], \ldots, \hat{\sigma}_\pi[t_n]) = f(\hat{\sigma}_\pi[t_{\pi(1)}], \ldots, \hat{\sigma}_\pi[t_{\pi(n)}]) = f(\pi[t_{\pi(1)}], \cdots, \pi[t_{\pi(n)}]) = \pi[f(t_1, \cdots, t_n)] = \pi[t]$.

(ii) We have $(\sigma_t \circ_G \sigma_\alpha)(f) = S^n(t, x_{\alpha(1)}, \ldots, x_{\alpha(n)}) = C_\alpha[t]$. So $\sigma_t \circ_G \sigma_\alpha = \sigma_{C_{\alpha[t]}}$. ■

5. Green's ralations on $Hyp_G(n)$

Let S be a semigroup and $1 \notin S$. We extend the binary operation from S to $S \cup \{1\}$ by define $x1 = 1x = x$ for all $x \in S \cup \{1\}$. Then $S \cup \{1\}$ is a semigroup with identity 1.

Let S be a semigroup. Then we define,

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity,} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

Let S be a semigroup and $\emptyset \neq A \subseteq S$. We now set

$$\begin{aligned} (A)_l &= \cap\{L|L \text{ is a left ideal of } S \text{ containing } A\} \\ (A)_r &= \cap\{R|R \text{ is a right ideal of } S \text{ containing } A\} \\ (A)_i &= \cap\{I|I \text{ is an ideal of } S \text{ containing } A\}. \end{aligned}$$

Then $(A)_l$,$(A)_r$ and $(A)_i$ are left ideal, right ideal and ideal of S, respectively. And we call $(A)_l$ $((A)_r, (A)_i)$ the *left ideal* (*right ideal, ideal*) *of* S *generated by* A.

It is easy to see that

$$\begin{aligned} (A)_l &= S^1A = SA \cup A \\ (A)_r &= AS^1 = A \cup SA \\ (A)_i &= S^1AS^1 = SAS \cup SA \cup AS \cup A. \end{aligned}$$

For $a_1, a_2, \dots, a_n \in S$, we write $(a_1, a_2, \dots, a_n)_l$ instead of $(\{a_1, a_2, \dots, a_n\})_l$ and call it the *left ideal of* S *generated by* $a_1, a_2, \dots, a_n$. Similarly, we can define $(a_1, a_2, \dots, a_n)_r$ and $(a_1, a_2, \dots, a_n)_i$. If A is a left ideal of S and $A = (a)_l$ for some $a \in S$, we then call A the *principal left ideal generated by* a. We can define principal right ideal and principal ideal in the same manner.

Let S be a semigroup. We define the relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ on S as follow:

$$\begin{aligned} a\mathcal{L}b &\Leftrightarrow (a)_l = (b)_l \\ a\mathcal{R}b &\Leftrightarrow (a)_r = (b)_r \\ \mathcal{H} &= \mathcal{L}\cap\mathcal{R} \\ \mathcal{D} &= \mathcal{L}\circ\mathcal{R} \\ a\mathcal{J}b &\Leftrightarrow (a)_i = (b)_i. \end{aligned}$$

Then we have, for all $a, b \in S$

$$\begin{aligned} a\mathcal{L}b &\Leftrightarrow Sa\cup\{a\} = Sb\cup\{b\} \\ &\Leftrightarrow S^1a = S^1b \\ &\Leftrightarrow a = xb \text{ and } b = ya \text{ for some } x, y \in S^1 \\ a\mathcal{R}b &\Leftrightarrow aS\cup\{a\} = bS\cup\{b\} \\ &\Leftrightarrow aS^1 = bS^1 \\ &\Leftrightarrow a = bx \text{ and } b = ay \text{ for some } x, y \in S^1 \\ a\mathcal{H}b &\Leftrightarrow a\mathcal{L}b \text{ and } a\mathcal{R}b \\ a\mathcal{D}b &\Leftrightarrow (a, c)\in\mathcal{L} \text{ and } (c, b)\in\mathcal{R} \text{ for some } c\in S \\ a\mathcal{J}b &\Leftrightarrow SaS\cup Sa\cup aS\cup\{a\} = SbS\cup Sb\cup bS\cup\{b\} \\ &\Leftrightarrow S^1aS^1 = S^1bS^1 \\ &\Leftrightarrow a = xby \text{ and } b = zau \text{ for some } x, y, z, u \in S^1. \end{aligned}$$

Remark 5.1. Let S be a semigroup. Then the following statements hold.

1. $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ are equivalent relations.
2. $\mathcal{H}\subseteq\mathcal{L}\subseteq\mathcal{D}\subseteq\mathcal{J}$ and $\mathcal{H}\subseteq\mathcal{R}\subseteq\mathcal{D}\subseteq\mathcal{J}$.

We call the relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ the *Green's relations on S*. For each $a \in S$, we denote $\mathcal{L}$-class, $\mathcal{R}$-class, $\mathcal{H}$-class, $\mathcal{D}$-class and $\mathcal{J}$-class containing a by L_a, R_a, H_a, D_a and J_a, respectively.

Green's relations on $Hyp(n)$ have been studied by S.L. Wismath [8], and Green's relations on $Hyp_G(2)$ were study by W. Puninagool and S. Leeratanavalee [5]. In this section, we describe some classes of the monoid of generalized hypersubstitutions of type $\tau = (n)$ with $n > 1$.

Theorem 5.2. *Any $\sigma_{x_i} \in P_G(n)$ is $\mathcal{L}$-related only to itself, but is $\mathcal{R}$-related, $\mathcal{D}$-related and $\mathcal{J}$-related to all elements of $P_G(n)$, and not related to any other generalized hypersubstitutions. Moerover, the set $P_G(n)$ forms a complete $\mathcal{R}$-, $\mathcal{D}$- and $\mathcal{J}$-class.*

Proof. By Lemma 2.1(i), for any $\sigma_{x_i} \in P_G(n)$, $\sigma \circ_G \sigma_{x_i} = \sigma_{x_i}$ for all $\sigma \in Hyp_G(n)$. This shows that any $\sigma_{x_i} \in P_G(n)$ can be $\mathcal{L}$-related only to itself. Since $\sigma_{x_i} \circ_G \sigma_{x_j} = \sigma_{x_j}$ for all $\sigma_{x_i}, \sigma_{x_j} \in P_G(n)$, so any two elements in $P_G(n)$ are $\mathcal{R}$-related. From $\mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}$, we see that any two elements in $P_G(n)$ are $\mathcal{D}$- and $\mathcal{J}$-related. Moreover by Lemma 2.1, $\sigma_s \circ_G \sigma_{x_i} \circ_G \sigma_t \in P_G(n)$ for all $\sigma_s, \sigma_t \in Hyp_G(n)$, and $\sigma_{x_i} \in P_G(n)$. This implies if $\sigma \notin P_G(n)$, then σ cannot be $\mathcal{J}$-related to every element in $P_G(n)$. So $P_G(n)$ is the $\mathcal{J}$-class of its elements. Since any two elements in $P_G(n)$ are $\mathcal{R}$- and $\mathcal{D}$-related, $\mathcal{R} \subseteq \mathcal{J}, \mathcal{D} \subseteq \mathcal{J}$ and $P_G(n)$ is the $\mathcal{J}$-class of its elements, and thus $P_G(n)$ forms a complete $\mathcal{R}$-, $\mathcal{D}$-class. ■

Theorem 5.3. *Any $\sigma_t \in G(n)$ is $\mathcal{R}$-related only to itself, but is $\mathcal{L}$-related, $\mathcal{D}$-related and $\mathcal{J}$-related to all elements of $G(n)$, and not related to any other generalized hypersubstitutions. Moreover, the set $G(n)$ forms a complete $\mathcal{L}$-, $\mathcal{D}$- and $\mathcal{J}$-class.*

Proof. Let $\sigma_t \in G(n)$. Assume that $\sigma_s \in Hyp_G(n)$ where $\sigma_s \mathcal{R} \sigma_t$. By Theorem 5.2, $s \notin X$. Then there exists $\sigma_p \in Hyp_G(n)$ such that $\sigma_s = \sigma_t \circ_G \sigma_p$. Since $s \notin X$ and $\sigma_s = \sigma_t \circ_G \sigma_p$, thus by Lemma 2.1(ii), $p \notin X$. Since $\sigma_t \in G(n)$ and $p \notin X$, thus by Lemma 3.1, $\sigma_t \circ_G \sigma_p = \sigma_t$. So $\sigma_s = \sigma_t$. Thus σ_t is $\mathcal{R}$-related only to itself.

Let $\sigma_s, \sigma_t \in G(n)$. By Lemma 3.1, $\sigma_s \circ_G \sigma_t = \sigma_s$ and $\sigma_t \circ_G \sigma_s = \sigma_t$. Thus $\sigma_s \mathcal{L} \sigma_t$. So any two elements in $G(n)$ are $\mathcal{L}$-related. Since $\mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$, thus any two elements in $G(n)$ are $\mathcal{D}$- and $\mathcal{J}$-related. Assume that $\sigma_t \in G(n)$ and $\sigma_s \in Hyp_G(n)$ where $\sigma_s \mathcal{J} \sigma_t$. By Theorem 5.2, $s \notin X$. Then there exist $\sigma_p, \sigma_q \in Hyp_G(n)$ such that $\sigma_p \circ_G \sigma_t \circ_G \sigma_q = \sigma_s$. Since $s \notin X$ and $\sigma_p \circ_G \sigma_t \circ_G \sigma_q = \sigma_s$, thus by Lemma 2.1, $p, q \notin X$. Since $\sigma_t \in G(n)$ and $q \notin X$, thus by Lemma 3.1, $\sigma_t \circ_G \sigma_q = \sigma_t$. Since $x_1, \ldots, x_n \notin var(t)$, thus by Lemma 4.6 (i), $x_1, \ldots, x_n$ are not variables occurring in the term $(\sigma_p \circ_G \sigma_t)(f) = (\sigma_p \circ_G \sigma_t \circ_G \sigma_q)(f)$. Thus $x_1, \ldots, x_n \notin var(s)$ and so $\sigma_s \in G(n)$. So $G(n)$ is the $\mathcal{J}$-class of its elements. Since any two elements in $G(n)$ are $\mathcal{L}$- and $\mathcal{D}$- related, $\mathcal{L} \subseteq \mathcal{J}, \mathcal{D} \subseteq \mathcal{J}$ and $G(n)$ is the $\mathcal{J}$-class of its elements, and thus $G(n)$ forms a complete $\mathcal{L}$-, $\mathcal{D}$-class. ■

Lemma 5.4. *Let* $\sigma_s, \sigma_t \in Hyp_G(n) \setminus (P_G(n) \cup G(n))$. *Then* $\sigma_s \mathcal{R} \sigma_t$ *if and only if* $s = C_{\alpha[t]}$ *for some bijection* α *on* J.

Proof. Assume that $s = C_{\alpha[t]}$ for some bijection α on J. So σ_α and $\sigma_{\alpha^{-1}}$ are inverse generalized hypersubstitutions. So by Lemma 4.7(ii), $\sigma_t \circ_G \sigma_\alpha = \sigma_{C_{\alpha[t]}} = \sigma_s$ and $\sigma_s \circ_G \sigma_{\alpha^{-1}} = \sigma_t$. Thus $\sigma_s \mathcal{R} \sigma_t$. Conversely, assume that $\sigma_s \mathcal{R} \sigma_t$. Then there exist $p, q \in W_{(n)}(X) \setminus X$ such that $\sigma_s \circ_G \sigma_p = \sigma_t$ and $\sigma_t \circ_G \sigma_q = \sigma_s$. Let $p = f(p_1, \ldots, p_n)$ and $q = f(q_1, \ldots, q_n)$. So we have two equations

$$S^n(s, \hat{\sigma}_s[p_1], \ldots, \hat{\sigma}_s[p_n]) = t \tag{1}$$

$$S^n(t, \hat{\sigma}_t[q_1], \ldots, \hat{\sigma}_t[q_n]) = s. \tag{2}$$

Now, if neither of these equations satisfies the condition (Q) of Lemma 4.3, we would have the length of the term t is longer than the length of the term s and also the length of s is longer than the length of t, which is clearly impossible. Thus, at least one of two equations must fit the condition (Q). But if one equation fits the condition (Q), Lemma 4.3 tells us that s and t have the same length, and therefore, the second equation also fits the condition (Q). By Lemma 4.3, if $x_i \in var(t) \cap X_n$, then $q_i \in X$. If such $q_i \notin X_n$, then from (2) we get $q_i \in var(s)$. So $S^n(s, \hat{\sigma}_s[p_1], \ldots, \hat{\sigma}_s[p_n]) \neq t$ which contradicts to (1). Thus such $q_i \in X_n$. Let $\alpha(i) = j$ if $x_i \in var(t) \cap X_n$ and $q_i = x_j$. This defines a partial function on J. It is clear that α is injective. Extending this map to a bijection on J, which we shall also call α. So $s = C_{\alpha[t]}$. ■

Lemma 5.5. *Let* $t \in W_{(n)}(X)$ *and* π *be a permutation on* J. *Then* $\pi^{-1}[\pi[t]] = t$.

Proof. We will proceed by induction on the complexity of the term t. If $t \in X$ then $\pi^{-1}[\pi[t]] = \pi^{-1}[t] = t$. Assume that $t = f(t_1, \ldots, t_n)$ and $\pi^{-1}[\pi[t_i]] = t_i$ for all $1 \leq i \leq n$. So $\pi^{-1}[\pi[t]] = \pi^{-1}[\pi[f(t_1, \ldots, t_n)]] = \pi^{-1}[f(\pi[t_{\pi(1)}], \ldots, \pi[t_{\pi(n)}])] = f(\pi^{-1}[\pi[t_{\pi(\pi^{-1}(1))}]], \ldots, \pi^{-1}[\pi[t_{\pi(\pi^{-1}(n))}]]) = f(\pi^{-1}[\pi[t_1]], \ldots, \pi^{-1}[\pi[t_n]]) = f(t_1, \ldots, t_n) = t$. ■

Lemma 5.6. *Let $\sigma_t \in Hyp_G(n) \setminus P_G(n)$. Then, for any permutation π on J, σ_t is $\mathcal{L}$-related to the generalized hypersubstitution $\sigma_{\pi[t]}$.*

Proof. We know from Lemma 4.7(i) that $\sigma_\pi \circ_G \sigma_t = \sigma_{\pi[t]}$. From Lemma 4.7(i) and Lemma 5.4, $\sigma_{\pi^{-1}} \circ_G \sigma_{\pi[t]} = \sigma_{\pi^{-1}[\pi[t]]} = \sigma_t$. So $\sigma_t \mathcal{L} \sigma_{\pi[t]}$. ■

Lemma 5.7. *Two idempotents σ_s and σ_t in $Hyp_G(n) \setminus P_G(n)$ are $\mathcal{L}$-related if and only if $var(s) \cap X_n = var(t) \cap X_n$.*

Proof. Assume that $\sigma_s \mathcal{L} \sigma_t$. Then there exist $u, v \in W_{(n)}(X)$ such that $\sigma_u \circ_G \sigma_t = \sigma_s$ and $\sigma_v \circ_G \sigma_s = \sigma_t$. By Lemma 4.6(i), $var(s) \cap X_n \subseteq var(t) \cap X_n$ and $var(t) \cap X_n \subseteq var(s) \cap X_n$. So $var(s) \cap X_n = var(t) \cap X_n$. Conversely, we use the fact that for any two idempotents e and f in any semigroup, $e\mathcal{L}f$ if and only if $ef = e$ and $fe = f$. Since $var(s) \cap X_n = var(t) \cap X_n$, by Theorem 3.2 we can prove that $\sigma_t \circ_G \sigma_s = \sigma_t$ and $\sigma_s \circ_G \sigma_t = \sigma_s$. ■

Theorem 5.8. *Let σ_t be an idempotent in $Hyp_G(n) \setminus (P_G(n) \cup G(n))$. Then $L_{\sigma_t} = \{\sigma_{\pi[w]} | \pi$ is a permutation of $J, w \notin X, var(w) \cap X_n = var(t) \cap X_n$ and σ_w is an idempotent$\}$.*

Proof. Let $\sigma_{\pi[w]} \in Hyp_G(n)$ where π is a permutation of $J, w \notin X, var(w) \cap X_n = var(t) \cap X_n$ and σ_w is an idempotent. By Lemma 5.7, $\sigma_w \mathcal{L} \sigma_t$. By Lemma 5.6, $\sigma_w \mathcal{L} \sigma_{\pi[w]}$. So $\sigma_{\pi[w]} \mathcal{L} \sigma_t$. Let $t = f(u_1, \ldots, u_n)$ and $s = f(v_1, \ldots, v_n)$ with $\sigma_s \mathcal{L} \sigma_t$. Then there exists $f(b_1, \ldots, b_n) \in W_{(n)}(X)$ such that $\sigma_{f(b_1,\ldots,b_n)} \circ_G \sigma_{f(v_1,\ldots,v_n)} = \sigma_{f(u_1,\ldots,u_n)}$. We write $\sigma = \sigma_{f(b_1,\ldots,b_n)}$. From $\sigma_{f(b_1,\ldots,b_n)} \circ_G \sigma_{f(v_1,\ldots,v_n)} = \sigma_{f(u_1,\ldots,u_n)}$, we get $S^n(f(b_1, \ldots, b_n), \hat{\sigma}[v_1], \ldots, \hat{\sigma}[v_n]) = f(u_1, \ldots, u_n)$. If $x_i \in var(t) \cap X_n$, then $u_i = x_i$ since σ_t is an idempotent. So $b_i = x_j$ for some $x_j \in X_n$. This implies $\hat{\sigma}[v_j] = x_i$ and then $v_j = x_i$. Let β be a partial function on J defined by $\beta(i) = j$ if $x_i \in var(t) \cap X_n$ and $v_j = x_i$. If $\beta(i) = \beta(k) = j$, then $v_j = x_i = x_k$. So $i = k$ and β is injective. So β can be extended to a permutation α on J.

Let $w = f(p_1, \ldots, p_n)$ where $p_i = x_i$ if $x_i \in var(t) \cap X_n$ and $p_i = \alpha[v_{\alpha(i)}]$ if $x_i \notin var(t) \cap X_n$. We will show that $var(w) \cap X_n = var(t) \cap X_n, \sigma_w$ is an idempotent and $s = f(v_1, \ldots, v_n) = \pi[w]$ where $w = \alpha^{-1}$. We show first that $var(w) \cap X_n = var(t) \cap X_n$. Since $\sigma_s \mathcal{L} \sigma_t$, thus by Lemma 5.7, $var(s) \cap X_n = var(t) \cap X_n$. Let $x_j \in var(w) \cap X_n$. Then $x_j \in var(p_i)$ for some $i \in J$ and $x_j \in X_n$. If $p_i = x_i$ where $x_i \in var(t) \cap X_n$, then $x_j = x_i \in var(t)$. If $p_i = \alpha[v_{\alpha(i)}]$, then $x_j \in var(p_i) = var(\alpha[v_{\alpha(i)}]) = var(v_{\alpha(i)}) \subseteq var(s)$. But $var(s) \cap X_n = var(t) \cap X_n$, so $x_j \in var(t)$. Let $x_j \in var(t) \cap X_n$. Then $p_j = x_j$ and so $x_j \in var(s) \cap X_n$. Next we show that σ_w is an idempotent. Let $x_i \in var(w) \cap X_n$. Then $x_i \in var(t) \cap X_n$. So $p_i = x_i$. Thus σ_w is an idempotent. Finally we show that $s = f(v_1, \ldots, v_n) = \pi[w]$ where $\pi = \alpha^{-1}$. To do this we will show that for all $1 \leq k \leq n$, $v_k = \pi[p_{\pi(k)}]$. Let $1 \leq k \leq n$. If there exists $i \in J$ such that $\beta(i) = k$, then $\alpha(i) = k$ and $\pi(k) = \alpha^{-1}(k) = i$. So $p_i = x_i = v_k$. Thus $\pi[p_{\pi(k)}] = \pi[p_i] = \pi[x_i] = x_i = v_k$. If no such index i exists, then $\pi[p_{\pi(k)}] = \pi[\alpha[v_{\alpha(\pi(k))}]] = \pi[\alpha[v_{\alpha(\alpha^{-1}(k))}]] = \pi[\alpha[v_k]] = \alpha^{-1}[\alpha[v_k]] = v_k$. ∎

Corollary 5.9. *Let σ_t be an idempotent in $Hyp_G(n) \backslash (P_G(n) \cup G(n))$. Then $D_{\sigma_t} = \{\sigma_w | w = C_{\alpha[\pi[s]]}$ for some α bijection on J, π a permutation on $J, s \notin X$, and σ_s an idempotent with $var(s) \cap X_n = var(t) \cap X_n\}$.*

Theorem 5.10. *Let σ_t be an idempotent in $Hyp_G(n) \backslash (P_G(n) \cup G(n))$. Then its $\mathcal{J}$-class is equal to its $\mathcal{D}$-class.*

Proof. Let $t = f(u_1, \ldots, u_n)$ and let c be the number of distinct variables in X_n which occur in t. Let $s = f(v_1, \ldots, v_n)$ with $\sigma_s \mathcal{J} \sigma_t$. Then there exist $f(a_1, \ldots, a_n), f(b_1, \ldots, b_n), f(p_1, \ldots, p_n), f(r_1, \ldots, r_n) \in W_{(n)}(X)$ such that

$$\sigma_{f(a_1,\ldots,a_n)} \circ_G \sigma_{f(v_1,\ldots,v_n)} \circ_G \sigma_{f(b_1,\ldots,b_n)} = \sigma_{f(u_1,\ldots,u_n)} \tag{1}$$

$$\sigma_{f(p_1,\ldots,p_n)} \circ_G \sigma_{f(u_1,\ldots,u_n)} \circ_G \sigma_{f(r_1,\ldots,r_n)} = \sigma_{f(v_1,\ldots,v_n)}. \tag{2}$$

Let $f(q_1, \ldots, q_n)$ be the term for $\sigma_{f(v_1,\ldots,v_n)} \circ_G \sigma_{f(b_1,\ldots,b_n)}$. We write $\sigma = \sigma_{f(a_1,\ldots,a_n)}$. From (1), we get $S^n(f(a_1, \ldots, a_n), \hat{\sigma}[q_1], \ldots, \hat{\sigma}[q_n]) = f(u_1, \ldots, u_n)$.

If $x_k \in var(t) \cap X_n$, then $u_k = x_k$ since σ_t is an idempotent. So $a_k = x_j$ for some $x_j \in X_n$. This implies $\hat{\sigma}[q_j] = x_k$ and then $q_j = x_k$. Let α be a function from $J(t)$ to J defined by $\alpha(k) = j$ if $x_k \in var(t) \cap X_n$ and $a_k = x_j$ where $J(t) = \{k \in J | x_k \in var(t)\}$. So α can be extended to a permutation on J.

We write $\sigma_1 = \sigma_{f(v_1,\ldots,v_n)}$. Since $f(q_1, \ldots, q_n)$ is the term for $\sigma_{f(v_1,\ldots,v_n)} \circ_G \sigma_{f(b_1,\ldots,b_n)}$, thus $S^n(f(v_1, \ldots, v_n), \hat{\sigma}_1[b_1], \ldots, \hat{\sigma}_1[b_n]) = f(q_1, \ldots, q_n)$. For each $k \in J(t), q_{\alpha(k)} = x_k$. So $v_{\alpha_k} = x_l$ for some $x_l \in X_n$. So $\hat{\sigma}_1[b_l] = x_k$ and then $b_l = x_k$. Let $\beta : \alpha(J(t)) \to J$ defined by $\beta(\alpha(k)) = l$ where $k \in J(t)$ and $v_{\alpha(k)} = x_l$. So β can be extended to a permutation on J. Since α and β are injective, thus at least c distinct variables in X_n occur as v_i in entries of $s = f(v_1 \ldots, v_n)$. We claim that the only variables in X_n which occur in s are those c variables. Let $f(c_1, \ldots, c_n)$ be the term for $\sigma_{f(u_1,\ldots,u_n)} \circ_G \sigma_{f(r_1,\ldots,r_n)}$.

We write $\sigma_2 = \sigma_{f(p_1,\ldots,p_n)}$. From (2), we get $S^n(f(p_1, \ldots, p_n), \hat{\sigma}_2[c_1], \ldots, \hat{\sigma}_2[c_n]) = f(v_1, \ldots, v_n)$. Since at least c distinct variables in X_n occur as v_i in entries $s = f(v_1 \ldots, v_n)$, thus at least c distinct variables in X_n occur as p_i in entries $s = f(p_1 \ldots, p_n)$ and then at least c distinct variables in X_n occur as c_i in entries $f(c_1 \ldots, c_n)$.

We write $\sigma_3 = \sigma_{f(u_1,\ldots,u_n)}$. Since $f(c_1, \ldots, c_n)$ is the term for $\sigma_{f(u_1,\ldots,u_n)} \circ_G \sigma_{f(r_1,\ldots,r_n)}$, thus $S^n(f(u_1, \ldots, u_n), \hat{\sigma}_3[r_1], \ldots, \hat{\sigma}_3[r_n]) = f(c_1, \ldots, c_n)$. But $f(u_1, \ldots, u_n)$ has only c distinct variables in X_n. Thus all the $r_j's$ used in the composition in (2) are variables in X_n. So the number of distinct variables in X_n which occur in $f(v_1, \ldots, v_n)$ is at most c. Thus the number of distinct variables in X_n which occur in $f(v_1, \ldots, v_n)$ is c and every variable in X_n which occurs in it occurs as a v_i. Let $w_1 = C_{(\beta\circ\alpha)^{-1}}[f(v_1, \ldots, v_n)]$. So $var(w_1) \cap X_n = var(t) \cap X_n$. From Lemma 5.4, we get $\sigma_{w_1} \mathcal{R} \sigma_s$. Let $w_2 = \alpha[w_1]$. From Lemma 5.4, $\sigma_{w_2} \mathcal{L} \sigma_{w_1}$. We will show that σ_{w_2} is an idempotent. Let $w_1 = C_{(\beta\circ\alpha)^{-1}}[f(v_1, \ldots, v_n)] = f(d_1, \ldots, d_n)$. For each $x_k \in var(t) \cap X_n$, $v_{\alpha(k)} = x_{\beta(\alpha(k))}$. So $d_{\alpha(k)} = x_k$. From $w_2 = \alpha[w_1]$, we get $w_2 = \alpha[f(d_1, \ldots, d_n)] = f(\alpha[d_{\alpha(1)}], \ldots, \alpha[d_{\alpha(n)}])$ and $var(w_2) \cap X_n = var(t) \cap X_n$. Let $x_j \in var(w_2) \cap X_n$. Then $x_j \in var(t) \cap X_n$. So $\alpha[d_{\alpha(j)}] = \alpha[x_j] = x_j$. So σ_{w_2} is an idempotent. By Lemma 5.7, $\sigma_{w_2} \mathcal{L} \sigma_t$. So $\sigma_{w_1} \mathcal{L} \sigma_t$. Thus $\sigma_s \mathcal{D} \sigma_t$.

■

Corollary 5.11. *Let σ_s, σ_t be idempotents in $Hyp_G(n) \setminus (P_G(n) \cup G(n))$. Then σ_s and σ_t are $\mathcal{J}$- or $\mathcal{D}$-related if and only if the number of distinct variables in X_n which occur in s and t are equal.*

Proof. One direction follows immediately from Corollary 5.9. Conversely, let $s = f(u_1, \ldots, u_n), t = f(v_1, \ldots, v_n), var(s) \cap X_n = \{x_{k_1}, \ldots, x_{k_c}\}$ and $var(t) \cap X_n = \{x_{l_1}, \ldots, x_{l_c}\}$. Since σ_s, σ_t are idempotents, thus $u_{k_j} = x_{k_j}$ and $v_{l_j} = x_{l_j}$ for all $1 \leq j \leq c$. Let $s' = f(u'_1, \ldots, u'_n), t' = f(v'_1, \ldots, v'_n)$ where $u'_{k_j} = x_{k_j}$ and $v'_{l_j} = x_{l_j}$ for all $1 \leq j \leq c$ and other $u'_j = x_{k_1}, v'_j = x_{l_1}$. By Lemma 5.7, $\sigma_s \mathcal{L} \sigma_{s'}$ and $\sigma_t \mathcal{L} \sigma_{t'}$. Let $\pi(l_j) = k_j$ for all $1 \leq j \leq c$. Then π is injective. So π can be extended to a permutation on J, which we will also call π. So $\pi[s'] = f(p_1, \ldots, p_n)$ where $p_{l_j} = x_{k_j}$ for all $1 \leq j \leq c$ and other $p_j = x_{k_1}$. Let $\alpha(k_j) = l_j$ for all $1 \leq j \leq c$. So α can be extended to a bijection on J, which we will also call α. So $C_{\alpha[\pi[s']]} = t'$. Thus $\sigma_s \mathcal{J} \sigma_{s'} \mathcal{J} \sigma_{C_{\alpha[\pi[s']]} = t'} \mathcal{J} \sigma_t$. ■

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