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# CLAUSAL RELATIONS AND $C$-CLONES*

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**Abstract**

We introduce a special set of relations called clausal relations. We study a Galois connection $\mathrm{Pol} - C\mathrm{Inv}$ between the set of all finitary operations on a finite set $D$ and the set of clausal relations, which is a restricted version of the Galois connection $\mathrm{Pol} - \mathrm{Inv}$. We define $C$-clones as the Galois closed sets of operations with respect to $\mathrm{Pol} - C\mathrm{Inv}$ and describe the lattice of all $C$-clones for the Boolean case $D = \{0, 1\}$. Finally we prove certain results about $C$-clones over a larger set.

**Keywords:** clone, Galois connection, clausal relation, $C$-clone.

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## Introduction

In this paper we introduce a special set $C\mathrm{R}_D$ of relations on a finite set $D$, called *clausal relations* (see Definition 1.4). The definition of clausal relations is based on the notion of a *clausal constraint* as a disjunction of inequalities of the form $x \geq d$ and $x \leq d$, where $d \in D = \{0, 1, \ldots, n-1\}$ and $x$ belongs to a set $X$ of variables. The latter were studied by N. Creignou, M. Hermann, A. Krokhin and G. Salzer (see [1]).

A *clone* on a set $D$ is a set of finitary operations on $D$ that is closed under composition and contains all projections. It is well known (see [3]) that the Galois closed classes of operations on a finite set $D$ of the Galois connection $\mathrm{Pol} - \mathrm{Inv}$ are exactly all clones on $D$. In other words every clone $F$ on $D$ can be described as $F = \mathrm{Pol}\, Q$ for some set $Q$ of relations.

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In [1] N. Creignou, M. Hermann and collaborators classified the complexity of clausal constraints. In this paper we will not deal with such complexity problems. We are rather interested in describing clones which are determined by sets of clausal relations, i.e. describing *$C$-clones* (see Definition 1.14). The restriction to clausal relations implies a restriction of the Galois connection $\mathrm{Pol}-\mathrm{Inv}$ to a Galois connection $\mathrm{Pol}-C\mathrm{Inv}$ where $C\mathrm{Inv}\,F = \mathrm{Inv}\,F \cap C\mathrm{R}_D$ for $F \subseteq \mathrm{O}_D$ (see Definition 1.13). This leads to a much smaller set of clones, a fact motivating us to investigate how many $C$-clones exist on $D$ and to describe them. In particular, this is a contribution to the structure of the lattice of all clones.

The aim of this paper is to give a complete description of Boolean $C$-clones, i.e. when $D = \{0, 1\}$, and to prove that contrary to the Boolean case, we have infinitely many $C$-clones for $3 \leq |D| < \infty$.

The paper is organized as follows: In Section 1 we provide definitions related to relations, clausal relations, $C$-clones and the Galois connection $\mathrm{Pol}-C\mathrm{Inv}$. Furthermore, we present some properties of clausal relations. In Section 2 we describe all Boolean $C$-clones, obtaining an only 5-element sublattice of the lattice described by E. Post (see [2]). Finally, in Section 3 we investigate how many $C$-clones exist for an arbitrary finite set $D$. We show that for $|D| \geq 3$ there exist infinitely many $C$-clones by constructing an infinite descending chain of such clones.

Throughout the paper, $\mathbb{N} = \{0, 1, 2, \ldots\}$ denotes the set of natural numbers, and $\mathbb{N}_+ = \{1, 2, \ldots\}$ denotes the set of positive natural numbers. Furthermore, the domain for our clones is the set $D = \{0, 1, \ldots, n-1\}$ for a fixed natural number $n \geq 2$.

## 1. Clausal relations

In this section we provide definitions and some properties of clausal relations.

**Definition 1.1.** Let $k, m \in \mathbb{N}_+$. An *$m$-ary relation* $\varrho$ on $D$ is a subset of the $m$-fold Cartesian product $D^m$. It is often convenient to represent $\varrho = \{r_1, \ldots, r_k\}$ as a matrix $(r_{ij})_{\substack{1\leq i\leq m\\ 1\leq j\leq k}} \in D^{m\times k}$, whose columns are the tuples in the relation, i.e. $r_j = \big(r_{1j}, r_{2j}, \ldots, r_{mj}\big)$ for all $j \in \{1, \ldots, k\}$.

We define $\mathrm{R}_D^{(m)} := \{\varrho \mid \varrho \subseteq D^m\}$ as the set of all $m$-ary relations defined on $D$ and

$$\mathrm{R}_D := \bigcup_{m=1}^{\infty} \mathrm{R}_D^{(m)}$$

as the set of all finitary relations on $D$.

**Definition 1.2.** Let $m \in \mathbb{N}_+$, $\underline{\mathbf{m}} := \{1, \dots, m\}$, $\varepsilon$ be a partition of $\underline{\mathbf{m}}$ and $\sim_\varepsilon$ be the corresponding equivalence relation on $\underline{\mathbf{m}}$. We define $d_\varepsilon \in \mathrm{R}_D^{(m)}$ to be the relation

$$d_\varepsilon := \{(x_1, \dots x_m) \in D^m \mid i \sim_\varepsilon j \Rightarrow x_i = x_j\}$$

and call it a *trivial* or *diagonal relation*. The set of all diagonal relations together with the empty relation $\emptyset$ is denoted by $\mathrm{diag}(D)$.

**Definition 1.3.** Let $p, q \in \mathbb{N}_+$. For given parameters $\mathbf{a} = (a_1, \dots, a_p) \in D^p$ and $\mathbf{b} = (b_1, \dots, b_q) \in D^q$, the *clausal relation* $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$ *of type* $(p, q)$, is the set of all tuples $(x_1, \dots, x_p, y_1, \dots, y_q) \in D^{p+q}$ satisfying

$$(x_1 \geq a_1) \vee \dots \vee (x_p \geq a_p) \vee (y_1 \leq b_1) \vee \dots \vee (y_q \leq b_q). \tag{1}$$

We observe that if $a_i = 0$ for some $i \in \{1, \dots, p\}$ or $b_j = n - 1$ for some $j \in \{1, \dots, q\}$, then the relation $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$ is total, i.e. $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}} = D^{p+q}$ because (1) is always satisfied.

**Definition 1.4.** Let $p, q \in \mathbb{N}_+$. We use

$$\mathcal{R}_q^p := \{\mathrm{R}_{\mathbf{b}}^{\mathbf{a}} \mid \mathbf{a} \in D^p, \mathbf{b} \in D^q\}$$

to denote the set of all clausal relations of arity * $p + q$ and

$$C\mathrm{R}_D := \bigcup_{(p,q) \in \mathbb{N}_+^2} \mathcal{R}_q^p$$

for the set of all *finitary clausal relations* on $D$.

*If we speak of a clausal relation of arity $p + q$, we implicitely mean also that the clausal relation is of type $(p, q)$.

We will write $\mathrm{R}^{a}_{\mathbf{b}}$ for $\mathrm{R}^{(a)}_{\mathbf{b}}$ in the case $p = 1$ and likewise $\mathrm{R}^{\mathbf{a}}_{b}$ for $\mathrm{R}^{\mathbf{a}}_{(b)}$ in the case $q = 1$. We give two examples of clausal relations.

**Examples 1.5.**

**a)** Let $D = \{0, 1\}$, then

$$\mathrm{R}^{0}_{1} = \{(x_1, y_1) \in D^2 \mid x_1 \geq 0 \vee y_1 \leq 1\} \quad = \quad \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} = D^2$$

**b)** Let $D = \{0, 1, 2\}$, then

$$\begin{aligned} \mathrm{R}^{(2,2)}_{0} = & \ \big\{(x_1, x_2, y_1) \in D^3 \mid x_1 \geq 2 \vee x_2 \geq 2 \vee y_1 \leq 0\big\} \\ = & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 & 2 & 0 & 1 & 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix} \end{aligned}$$

Clausal relations are closed with respect to union but not with respect to intersection as the next lemmata show.

**Lemma 1.6.** *Let $p, q \in \mathbb{N}_+$. If $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$ and $\mathrm{R}^{\mathbf{a}'}_{\mathbf{b}'}$ are clausal relations of arity $p + q$, where $\mathbf{a} = (a_1, \ldots, a_p)$, $\mathbf{a}' = (a'_1, \ldots, a'_p) \in D^p$, $\mathbf{b} = (b_1, \ldots, b_q)$ and $\mathbf{b}' = (b'_1, \ldots, b'_q) \in D^q$. Then it holds*

$$\mathrm{R}^{\mathbf{a}}_{\mathbf{b}} \cup \mathrm{R}^{\mathbf{a}'}_{\mathbf{b}'} = \mathrm{R}^{\mathbf{c}}_{\mathbf{d}}$$

*where* $\mathbf{c} = (\min\{a_1, a'_1\}, \ldots, \min\{a_p, a'_p\})$ *and* $\mathbf{d} = (\max\{b_1, b'_1\}, \ldots, \max\{b_q, b'_q\})$.

***Proof.*** Let $\mathbf{z} = (x_1, \ldots, x_p, y_1, \ldots, y_q) \in \mathrm{R}^{\mathbf{a}}_{\mathbf{b}} \cup \mathrm{R}^{\mathbf{a}'}_{\mathbf{b}'}$,

$$\begin{aligned}
&\iff x_1 \geq a_1 \vee \ldots \vee x_p \geq a_p \vee y_1 \leq b_1 \vee \ldots \vee y_q \leq b_q \vee \\
&\qquad x_1 \geq a'_1 \vee \ldots \vee x_p \geq a'_p \vee y_1 \leq b'_1 \vee \ldots \vee y_q \leq b'_q \\
&\iff \bigvee_{1\leq i\leq p} (x_i \geq a_i \vee x_i \geq a'_i) \vee \bigvee_{1\leq j\leq q} (y_j \leq b_j \vee y_j \leq b'_j) \\
&\iff \bigvee_{1\leq i\leq p} (x_i \geq min\{a_i, a'_i\}) \vee \bigvee_{1\leq j\leq q} (y_j \leq max\{b_j, b'_j\}).
\end{aligned}$$

This is equivalent to $\mathbf{z} \in \mathrm{R}^{\mathbf{c}}_{\mathbf{d}}$. ■

Similarly, following lemma can be proved.

**Lemma 1.7.** *Let $p, q \in \mathbb{N}_+$. If $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$ and $\mathrm{R}^{\mathbf{a}'}_{\mathbf{b}'}$ are clausal relations of arity $p+q$, where $\mathbf{a} = (a_1, \ldots, a_p)$, $\mathbf{a}' = (a'_1, \ldots, a'_p) \in D^p$, $\mathbf{b} = (b_1, \ldots, b_q)$ and $\mathbf{b}' = (b'_1, \ldots, b'_q) \in D^q$. Then it holds*

$$\mathrm{R}^{\mathbf{a}}_{\mathbf{b}} \cap \mathrm{R}^{\mathbf{a}'}_{\mathbf{b}'} \supseteq \mathrm{R}^{\mathbf{c}}_{\mathbf{d}}$$

*where $\mathbf{c} = (\max\{a_1, a'_1\}, \ldots, \max\{a_p, a'_p\})$ and $\mathbf{d} = (\min\{b_1, b'_1\}, \ldots, \min\{b_q, b'_q\})$.*

In general equality does not hold in 1.7, as the following example shows.

**Example 1.8.** Let $D = \{0, 1\}$, consider $\mathbf{a} = (0, 1), \mathbf{a}' = (1, 0), b_1 = 0$ and $b'_1 = 1$. Then, $\mathrm{R}^{\mathbf{a}}_{b_1} = D^3 = \mathrm{R}^{\mathbf{a}'}_{b'_1}$. Let $\mathbf{z} = (x_1, x_2, y_1) = (0, 0, 1)$. $\mathbf{z} \in \mathrm{R}^{\mathbf{a}}_{b_1} \cap \mathrm{R}^{\mathbf{a}'}_{b'_1}$, but $\mathbf{z} \notin \mathrm{R}^{(1,1)}_{0}$.

The set $C\mathrm{R}_D$ can be separated in trivial and non-trivial clausal relations as follows.

**Lemma 1.9.** *The set $C\mathrm{R}_D$ can be partitioned as*

$$C\mathrm{R}_D = \{D^{(p+q)} \mid p, q \in \mathbb{N}_+\} \dot\cup C\mathrm{R}^*_D,$$

*where*

$$\{D^{(p+q)} \mid p, q \in \mathbb{N}_+\} = C\mathrm{R}_D \cap \mathrm{diag}(D)$$

*are the trivial clausal relations and*

$$C\mathrm{R}_D^* = \{\mathrm{R}_{\mathbf{b}}^{\mathbf{a}} \mid \mathbf{a} \in (D \setminus \{0\})^p, \mathbf{b} \in (D \setminus \{n-1\})^q; p, q \in \mathbb{N}_+\}$$

*are the non-trivial clausal relations.*

***Proof.*** Let $p, q \in \mathbb{N}_+$, $\mathbf{a} = (a_1, \ldots, a_p) \in D^p$, $\mathbf{b} = (b_1, \ldots, b_q) \in D^q$. We have observed above that if one of the $a_1, \ldots, a_p$ equals 0, or one of the $b_1, \ldots, b_q$ equals $n-1$, then $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$ is a total relation, i.e. $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}} = D^{p+q}$.

We have to show $C\mathrm{R}_D^* \cap \mathrm{diag}(D) = \emptyset$. Let us assume the existence of a relation $\varrho \in C\mathrm{R}_D^* \cap \mathrm{diag}(D)$. Then there exist $\mathbf{a} \in (D \setminus \{0\})^p$ and $\mathbf{b} \in (D \setminus \{n-1\})^q$ such that

$$\varrho = \mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$$

and there exists a partition $\varepsilon$ of $\underline{\mathbf{m}} = \{1, \ldots, m\}$ (where $m := p + q$) such that $\varrho = d_\varepsilon$. Let $\sim_\varepsilon$ be the corresponding equivalence relation. We show

$$\sim_\varepsilon = \{(x, x) \mid x \in \underline{\mathbf{m}}\} := \Delta_{\underline{\mathbf{m}}},$$

thus

$$\varrho = d_\epsilon = D^m.$$

This is a contradiction to $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}} = \varrho = D^m$, because $(0, \ldots, 0, n-1, \ldots, n-1) \notin \mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$.

Let $i, j \in \{1, \ldots, p\}$ with $i \neq j$. Then

$$(\underbrace{0, \ldots, 0, \overset{i}{0}, 0 \ldots, 0, \overset{j}{n-1}, 0, \ldots, 0}_{p}, \underbrace{0, \ldots, 0}_{q}) \in \mathrm{R}_{\mathbf{b}}^{\mathbf{a}} = d_\varepsilon.$$

Thus $i \not\sim_\varepsilon j$. Let $i, j \in \{1, \ldots, q\}$ with $i \neq j$. Then

$$(\underbrace{n-1, \ldots, n-1}_{p}, \underbrace{0, \ldots, 0, \overset{i+p}{0}, 0, \ldots, 0, \overset{j+p}{n-1}, 0, \ldots, 0}_{q}) \in \mathrm{R}_{\mathbf{b}}^{\mathbf{a}} = d_\varepsilon.$$

Thus $i+p \not\sim_\varepsilon j+p$. Let $i \in \{1,\dots,p\}$ and $j \in \{1,\dots,q\}$. Then

$$(\underbrace{n-1,\dots,n-1,\overset{i}{n-1},n-1,\dots,n-1}_{p},\underbrace{0,\dots,0,\overset{j+p}{0},0,\dots,0}_{q}) \in \mathrm{R}^{\mathbf{a}}_{\mathbf{b}} = d_\varepsilon.$$

Thus $i \not\sim_\varepsilon j+p$.

Hence, if $i,j \in \{1,\dots,p+q\}$ with $i \sim_\varepsilon j$ then $i=j$. This shows $\sim_\varepsilon = \Delta_{\underline{\mathbf{m}}}$. ∎

An example of a non-trivial clausal relation can be found in 1.5 b).

**Definition 1.10.** Let $k \in \mathbb{N}_+$. A *$k$-ary operation* on $D$ is a function $f : D^k \longrightarrow D$. We denote $\mathrm{O}_D^{(k)} := \{f \mid f : D^k \longrightarrow D\}$ as the set of all k-ary operations on $D$ and $\mathrm{O}_D := \bigcup_{k=1}^{\infty} \mathrm{O}_D^{(k)}$ as the set of all finitary operations on $D$. For each $j \in \{1,\dots,k\}$ we denote $e_j^k(d_1,\dots,d_k) := d_j$ as the *$j$-th projection of arity* $k$ and $J_D := \{e_j^k \mid k \in \mathbb{N}_+, 1 \le j \le k\}$ as the set of all projections on $D$.

An example of a $k$-ary operation on D is the *$k$-ary constant operation* (briefly, *constant*) $c_a^k : D^k \longrightarrow D$ given by $c_a^k(x_1,\dots,x_k) := a$ for all $x_1,\dots,x_k \in D$, where $a$ is an arbitrary element of $D$.

**Definition 1.11.** We say that a $k$-ary operation $f \in \mathrm{O}_D^{(k)}$ *preserves* an $m$-ary relation $\varrho \in \mathrm{R}_D^{(m)}$, denoted by $f \rhd \varrho$, if whenever

$$r_1 = (a_{11},\dots,a_{m1}) \in \varrho, \dots, r_k = (a_{1k},\dots,a_{mk}) \in \varrho$$

it follows that also $f$ applied to these tuples belongs to $\varrho$, i.e.

$$f[r_1,\dots,r_k] := (f(a_{11},\dots,a_{1k}),\dots,f(a_{m1},\dots,a_{mk})) \in \varrho.$$

**Definition 1.12.** Let $F \subseteq \mathrm{O}_D$ be a set of operations on $D$. Then we define $\mathrm{Inv}_D F$ as the set of all relations that are invariant for all $f \in F$:

$$\mathrm{Inv}_D F := \{\varrho \in \mathrm{R}_D \mid \forall f \in F : f \rhd \varrho\}.$$

Similarly, for a set $Q \subseteq \mathrm{R}_D$ of relations, $\mathrm{Pol}_D\, Q$ is the set of all operations that preserve every relation $\varrho \in Q$:

$$\mathrm{Pol}_D\, Q := \{f \in F \mid \forall \varrho \in Q : f \rhd \varrho\}.$$

Furthermore, for $k \in \mathbb{N}_+$ we abbreviate

$$\mathrm{Pol}_D^{(k)}\, Q := \mathrm{O}_D^{(k)} \cap \mathrm{Pol}_D\, Q.$$

If $D$ is known from the context we write Pol instead of $\mathrm{Pol}_D$, and Inv instead of $\mathrm{Inv}_D$. The operators Pol and Inv define the Galois connection $\mathrm{Pol} - \mathrm{Inv}$, which is induced by the relation $\rhd$.

It is well known that for $F \subseteq \mathrm{O}_D$ the Galois closed set $\mathrm{Pol}\,\mathrm{Inv}\, F$ of operations on $D$ is precisely the clone $\langle F\rangle_{O_D}$ generated by $F$, i.e. $\mathrm{Pol}\,\mathrm{Inv}\, F = \langle F\rangle_{O_D}$. Let $L_D$ be the set of clones on $D$, we denote $\mathcal{L}_D := (L_D, \subseteq)$ as the lattice of all clones on $D$, with greatest element $\mathrm{O}_D$ and least element $J_D$.

Note that by virtue of the Galois connection $\mathrm{Pol} - \mathrm{Inv}$ the study of the Galois closed sets of operations is equivalent to the study of the Galois closed sets of relations, $\mathrm{Inv}\,\mathrm{Pol}\, Q = \left[Q\right]_{\mathrm{R}_D}$ for $Q \subseteq \mathrm{R}_D$.

Next we present a restriction of the Galois connection $\mathrm{Pol} - \mathrm{Inv}$ where the relations are confined to be clausal relations. This restriction gives us a much smaller number of Galois closed sets of operations, so called $C$-clones.

**Definition 1.13.** For $F \subseteq \mathrm{O}_D$ we define $C\mathrm{Inv}\, F := \mathrm{Inv}\, F \cap C\mathrm{R}_D$.
The operators

$$C\mathrm{Inv} : \mathcal{P}(\mathrm{O}_D) \longrightarrow \mathcal{P}(C\mathrm{R}_D) : \qquad F \mapsto C\mathrm{Inv}\, F$$

and

$$\mathrm{Pol} : \mathcal{P}(C\mathrm{R}_D) \longrightarrow \mathcal{P}(\mathrm{O}_D) : \qquad Q \mapsto \mathrm{Pol}\, Q$$

define a *Galois connection* $\mathrm{Pol} - C\mathrm{Inv}$ between operations and clausal relations.

We will call the Galois closed sets of operations of this Galois connection $C$-clones, more formally:

**Definition 1.14.** A set $F \subseteq \mathrm{O}_D$ of operations is called a *$C$-clone* if $F = \mathrm{Pol}\, Q$ for some set $Q \subseteq C\mathrm{R}_D$ of clausal relations, and a set $Q \subseteq C\mathrm{R}_D$ is called *relational $C$-clone* if $Q = C\mathrm{Inv}\, F$ for a set $F$ of operations.

Every Galois connection naturally gives rise to a pair of closure operators. For one of them we introduce a special notation.

**Definition 1.15.** For any $F \subseteq \mathrm{O}_D$ we define $\langle F\rangle_{\mathcal{C}} := \mathrm{Pol}\, C\mathrm{Inv}\, F$.

We finish this section with a lemma clarifying the relationship of this closure operator and the clone generation, i.e. the corresponding closure operator of the Galois connection $\mathrm{Pol} - \mathrm{Inv}$.

**Lemma 1.16.** *For any $F \subseteq \mathrm{O}_D$ it holds:*

(a) $\langle F\rangle_{O_D} \subseteq \langle F\rangle_{\mathcal{C}}$.

(b) $\langle\langle F\rangle_{\mathcal{C}}\rangle_{O_D} = \langle F\rangle_{\mathcal{C}}$, *in particular every $C$-clone is a clone.*

***Proof.*** The first statement follows from $C\mathrm{Inv}\, F \subseteq \mathrm{Inv}\, F$ for any $F \subseteq \mathrm{O}_D$, hence $\langle F\rangle_{O_D} = \mathrm{Pol}\,\mathrm{Inv}\, F \subseteq \mathrm{Pol}\, C\mathrm{Inv}\, F = \langle F\rangle_{\mathcal{C}}$. For the second statement observe that $\mathrm{Pol}\,\mathrm{Inv}\,\langle F\rangle_{\mathcal{C}} = \langle F\rangle_{\mathcal{C}}$. ■

Let $CL_D = \{\mathrm{Pol}\, Q \mid Q \subseteq C\mathrm{R}_D\}$ be the set of all $C$-clones on $D$, we denote $\mathcal{CL}_D := (CL_D, \subseteq)$ as the lattice of all $C$-clones on $D$, with greatest element $\mathrm{O}_D$ and least element $\mathrm{Pol}\, C\mathrm{R}_D$. The main goal is to describe the lattice of all $C$-clones on $D$, the first step towards this goal is to describe the lattice of all $C$-clones for $D = \{0, 1\}$.

## 2. Boolean $C$-clones

In this section we will describe all Boolean $C$-clones, i.e. when $D$ is the set $\{0, 1\}$ that we also denote by $\mathbf{2}$ for short.

From Lemma 1.9, the set $C\mathrm{R}_{\mathbf{2}}$ can be written as in the following corollary.

**Corollary 2.1.**

$$C\mathrm{R}_{\mathbf{2}} = \{\mathbf{2}^{p+q} \mid p, q \in \mathbb{N}_+\} \dot\cup C\mathrm{R}^*_{\mathbf{2}},$$

where

$$C\mathrm{R}^*_{\mathbf{2}} = \{\mathrm{R}^{\mathbf{1},p}_{\mathbf{0},q} \mid p, q \in \mathbb{N}_+\}$$

and

$$\mathrm{R}^{\mathbf{1},p}_{\mathbf{0},q} := R^{\overbrace{(1,\ldots,1)}^{p \text{ times}}}_{\underbrace{(0,\ldots,0)}_{q \text{ times}}}.$$

Observe that

$$\begin{aligned}\mathrm{R}^{\mathbf{1},p}_{\mathbf{0},q} &= \{(\mathbf{x},\mathbf{y}) \in \mathbf{2}^p \times \mathbf{2}^q \mid \exists i \in \{1,\ldots,p\} : x_i = 1 \vee \exists j \in \{1,\ldots,q\} : y_j = 0\} \\ &= \mathbf{2}^{p+q} \setminus \{(0,\ldots,0,1,\ldots,1)\}.\end{aligned}$$

The following lemma shows that every Boolean $C$-clone can be determined by sets of non-trivial Boolean clausal relations.

**Lemma 2.2.** *For $Q \subseteq C\mathrm{R}_{\mathbf{2}}$, it holds* $\mathrm{Pol}(Q) = \mathrm{Pol}(Q \cap C\mathrm{R}^*_{\mathbf{2}})$.

We shall describe $\{\mathrm{Pol}\, Q \mid Q \subseteq C\mathrm{R}_{\mathbf{2}}\}$. Since $\mathrm{Pol} - C\mathrm{Inv}$ is a Galois connection, this set is dually isomorphic to $\{C\mathrm{Inv}\, F \mid F \subseteq \mathrm{O}_{\mathbf{2}}\}$, and furthermore, we have

$$C\mathrm{Inv}\, F = \bigcap_{f \in F} C\mathrm{Inv}\, f$$

for $F \subseteq \mathrm{O}_{\mathbf{2}}$. Consequently, it suffices to regard the closed relational sets $C\mathrm{Inv}\, f$ for $f \in \mathrm{O}_{\mathbf{2}}$. Since there is a one to one correspondence between $C\mathrm{Inv}\, f$ and $\langle f \rangle_{\mathcal{C}}$ via the operators Pol and $C$Inv, we will first consider one-generated $C$-clones. By Lemma 2.2 and Definition 1.15,

$$\langle f \rangle_{\mathcal{C}} = \mathrm{Pol}\, C\mathrm{Inv}\, f = \mathrm{Pol}(C\mathrm{Inv}\, f \cap C\mathrm{R}^*_{\mathbf{2}})$$

i.e. $\langle f \rangle_{\mathcal{C}}$ is the set of all the functions that preserve all the non-trivial invariant clausal relations of $f$.

For the rest of this section we are going to characterize, one-generated $C$-clones for some special functions $f \in \mathrm{O}_{\mathbf{2}}$, (namely $f \in \{\neg, h, \vee, \wedge, g\}$). We also use the notation from Figure 1 without further explanation.

**Lemma 2.3.** *Let $\neg : \mathbf{2} \longrightarrow \mathbf{2}$ be the negation operation, i.e. $\neg(0) = 1$ and $\neg(1) = 0$. Then it holds*

$$\langle \neg \rangle_{\mathcal{C}} = \mathrm{O}_{\mathbf{2}}.$$

***Proof.*** By Definition 1.15 and Lemma 2.2, we have

$$(2)\qquad \langle\neg\rangle_{\mathcal{C}} \;=\; \{f \in \mathrm{O}_{\mathbf{2}} \mid \forall p, q \in \mathbb{N}_+ : \quad \neg \rhd \mathrm{R}_{\mathbf{0},q}^{\mathbf{1},p} \implies f \rhd \mathrm{R}_{\mathbf{0},q}^{\mathbf{1},p}\}.$$

Because of (2) it is enough to show $\neg \not\rhd \mathrm{R}_{\mathbf{0},q}^{\mathbf{1},p}$ for all $p, q \in \mathbb{N}_+$. Indeed, the tuple $\mathbf{r} = (1, \ldots, 1, 0, \ldots, 0) \in \mathrm{R}_{\mathbf{0},q}^{\mathbf{1},p} = \mathbf{2}^{p+q} \setminus \{(0, \ldots, 0, 1, \ldots, 1)\}$ for all $p, q \in \mathbb{N}_+$ but

$$\neg[\mathbf{r}] = (\neg(1), \ldots, \neg(1), \neg(0), \ldots, \neg(0)) = (0, \ldots, 0, 1, \ldots, 1) \notin \mathrm{R}_{\mathbf{0},q}^{\mathbf{1},p}.$$

■

Note that for $p = q = 1$ it holds

$$\mathrm{R}_{\mathbf{0},q}^{\mathbf{1},p} = \mathrm{R}_0^1 = \{(0,0), (1,0), (1,1)\} = \geqq,$$

and hence

$$\mathrm{Pol}\left(\mathrm{R}_0^1\right) = \mathrm{Pol}(\geqq) = \mathrm{Pol}(\leqq) = M,$$

where $M$ is the clone of all monotone Boolean functions.

**Lemma 2.4.** *Let $h \in \mathrm{O}_{\mathbf{2}}^{(3)}$ be the ternary majority function on* $\mathbf{2}$ *(median), i.e. $h(x,y,z) := (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ for $x, y, z \in \mathbf{2}$. Then it holds*

$$\langle h\rangle_{\mathcal{C}} = \mathrm{Pol}\left(\mathrm{R}_0^1\right).$$

***Proof.*** We show that $h \not\rhd \mathrm{R}_{\mathbf{0},q}^{\mathbf{1},p}$, unless $p = 1, q = 1$. We consider several cases:

- $p \geq 2, q \geq 1$ : The schema

$$\begin{array}{lcll}
h(1 \;\; 0 \;\; 0) & = & 0 & \\
h(0 \;\; 1 \;\; 0) & = & 0 & \\
h(0 \;\; 0 \;\; 0) & = & 0 & \\
& \vdots & & \\
h(0 \;\; 0 \;\; 0) & = & 0 & \text{(row } p\text{)} \\
h(1 \;\; 1 \;\; 0) & = & 1 & \\
h(1 \;\; 1 \;\; 1) & = & 1 & \\
& \vdots & & \\
h(1 \;\; 1 \;\; 1) & = & 1 & \text{(row } p+q\text{)}
\end{array}$$

shows that $h \not\triangleright \mathrm{R}_{\mathbf{0},q}^{\mathbf{1},p}$, because the tuples (columns of the arguments of $h$) all belong to $\mathrm{R}_{\mathbf{0},q}^{\mathbf{1},p}$, but after applying $h$ to the tuples, one obtains a tuple (column) that does not belong to $\mathrm{R}_{\mathbf{0},q}^{\mathbf{1},p}$.

- $p \geq 1, q \geq 2:$ Likewise, the schema

$$\begin{array}{rcll} h(1 \quad 0 \quad 0) & = & 0 & \\ h(0 \quad 0 \quad 0) & = & 0 & \\ & \vdots & & \\ h(0 \quad 0 \quad 0) & = & 0 & (\text{row } p) \\ h(1 \quad 0 \quad 1) & = & 1 & \\ h(1 \quad 1 \quad 0) & = & 1 & \\ h(1 \quad 1 \quad 1) & = & 1 & \\ & \vdots & & \\ h(1 \quad 1 \quad 1) & = & 1 & (\text{row } p+q) \end{array}$$

shows that $h \not\triangleright \mathrm{R}_{\mathbf{0},q}^{\mathbf{1},p}$.

If $p = q = 1$, then $h \triangleright \mathrm{R}_0^1$, because $h$ is a monotone operation and $M = \mathrm{Pol}(\mathrm{R}_0^1)$. ∎

**Lemma 2.5.** *It holds*

$$\langle \wedge \rangle_{\mathcal{C}} = \mathrm{Pol}\{\mathrm{R}_{\mathbf{0},q}^{\mathbf{1},p} \mid p = 1, q \in \mathbb{N}_+\} \quad \textit{and} \quad \langle \vee \rangle_{\mathcal{C}} = \mathrm{Pol}\{\mathrm{R}_{\mathbf{0},q}^{\mathbf{1},p} \mid q = 1, p \in \mathbb{N}_+\}.$$

***Proof.*** At first we have a look at $\wedge$. We show that $\wedge \not\triangleright \mathrm{R}_{\mathbf{0},q}^{\mathbf{1},p}$, unless $p = 1$ and $q \in \mathbb{N}_+$. We consider two cases:

- $p \geq 2:$ The schema

$$\begin{array}{ccccll} 1 & \wedge & 0 & = & 0 & \\ 0 & \wedge & 1 & = & 0 & \\ 0 & \wedge & 0 & = & 0 & \\ & & \vdots & & & \\ 0 & \wedge & 0 & = & 0 & (\text{row } p) \\ 1 & \wedge & 1 & = & 1 & \\ & & \vdots & & & \\ 1 & \wedge & 1 & = & 1 & (\text{row } p+q) \end{array}$$

shows that $\wedge \not\triangleright \mathrm{R}^{\mathbf{1},p}_{\mathbf{0},q}$.

- $p = 1, q \in \mathbb{N}_+$: We show
$$\wedge \triangleright \mathrm{R}^{\mathbf{1},p}_{\mathbf{0},q}.$$
We assume the existence of tuples
$$(x_1, y_1, \ldots, y_q), (x_2, z_1, \ldots, z_q) \in \mathrm{R}^{\mathbf{1},p}_{\mathbf{0},q}$$
such that
$$x_1 \wedge x_2 = 0$$
and for all $j \in \{1, \ldots, q\}$
$$y_j \wedge z_j = 1.$$
Because of $x_1 \wedge x_2 = 0$ w.l.o.g. $x_1 = 0$. Then our assumption
$$(x_1, y_1, \ldots, y_q) \in \mathrm{R}^{\mathbf{1},p}_{\mathbf{0},q}$$
implies that there is one $j \in \{1, \ldots, q\}$ such that $y_j = 0$. Thus, $y_j \wedge z_j = 0$, a contradiction.

Similarly, the result for $\vee$ can be proved. ■

In 2.1 we saw that clausal relations are either total or total without one tuple, and none of the diagonals except for total relations are clausal relations. Furthermore, $C\mathrm{Inv}\,\mathrm{O}_{\mathbf{2}} = \mathrm{Inv}\,\mathrm{O}_{\mathbf{2}} \cap C\mathrm{R}_{\mathbf{2}} = \mathrm{diag}(\mathbf{2}) \cap C\mathrm{R}_{\mathbf{2}}$, hence we obtain the following lemma.

**Lemma 2.6.** *It holds*
$$C\mathrm{Inv}\,\mathrm{O}_{\mathbf{2}} = \{\mathbf{2}^{(p+q)} \mid p, q \in \mathbb{N}_+\}.$$

Let $c_0, c_1$ be the unary constant operations on $\mathbf{2}$. For any $p, q \in \mathbb{N}_+$ we have
$$c_0 \triangleright \mathrm{R}^{\mathbf{1},p}_{\mathbf{0},q} \text{ and } c_1 \triangleright \mathrm{R}^{\mathbf{1},p}_{\mathbf{0},q}$$
because $c_0(y_1) = 0$ and $c_1(x_1) = 1$ for any $(x_1, \ldots, x_p, y_1, \ldots, y_q) \in \mathrm{R}^{\mathbf{1},p}_{\mathbf{0},q}$.

In the rest of the section we will freely use $\vee$ to denote supremum of two clones in Post's Lattice (see Figure 1). Nevertheless, we hope not to confuse the reader and be clear.

**Lemma 2.7.** *The least $C$-clone is*

$$\langle\emptyset\rangle_{\mathcal{C}} = \mathrm{Pol}(C\mathrm{R}_{\mathbf{2}}) = \{f \in \mathrm{O}_{\mathbf{2}} \mid \forall p, q \in \mathbb{N}_{+} : f \rhd \mathrm{R}_{\mathbf{0},q}^{\mathbf{1},p}\}.$$

*It holds*

$$\langle\emptyset\rangle_{\mathcal{C}} \supseteq \langle c_0\rangle_{\mathrm{O}_{\mathbf{2}}} \vee \langle c_1\rangle_{\mathrm{O}_{\mathbf{2}}} \overset{Fig.1}{=} \langle c_0, c_1\rangle_{\mathrm{O}_{\mathbf{2}}},$$

*and*

$$\langle\emptyset\rangle_{\mathcal{C}} \subsetneq \langle\wedge\rangle_{\mathcal{C}} \subsetneq M,$$

$$\langle\emptyset\rangle_{\mathcal{C}} \subsetneq \langle\vee\rangle_{\mathcal{C}} \subsetneq M.$$

*Furthermore, $\langle\wedge\rangle_{\mathcal{C}}$ and $\langle\vee\rangle_{\mathcal{C}}$ are incomparable $C$-clones.*

***Proof.*** From the previous observation we obtain $c_0, c_1 \in \langle\emptyset\rangle_{\mathcal{C}}$. Because $\langle\emptyset\rangle_{\mathcal{C}}$ is a Boolean clone, we have

$$\langle c_0\rangle_{\mathrm{O}_{\mathbf{2}}}, \langle c_1\rangle_{\mathrm{O}_{\mathbf{2}}} \subseteq \langle\langle\emptyset\rangle_{\mathcal{C}}\rangle_{\mathrm{O}_{\mathbf{2}}} \overset{1.16}{=} \langle\emptyset\rangle_{\mathcal{C}}.$$

Thus, $\langle\emptyset\rangle_{\mathcal{C}}$ is an upper bound for $\langle c_0\rangle_{\mathrm{O}_{\mathbf{2}}}$ and $\langle c_1\rangle_{\mathrm{O}_{\mathbf{2}}}$. Hence,

$$\langle c_0\rangle_{\mathrm{O}_{\mathbf{2}}} \vee \langle c_1\rangle_{\mathrm{O}_{\mathbf{2}}} \subseteq \langle\emptyset\rangle_{\mathcal{C}}.$$

Because $\langle\emptyset\rangle_{\mathcal{C}} = \mathrm{Pol}(C\mathrm{R}_{\mathbf{2}})$, we have that neither $\wedge \in \langle\emptyset\rangle_{\mathcal{C}}$ nor $\vee \in \langle\emptyset\rangle_{\mathcal{C}}$, because $\wedge \not\rhd \mathrm{R}_{\mathbf{0},q}^{\mathbf{1},p}$ for $p \geq 2$ and $\vee \not\rhd \mathrm{R}_{\mathbf{0},q}^{\mathbf{1},p}$ for $q \geq 2$. Thus,

$$\wedge \in \langle\wedge\rangle_{\mathcal{C}} \setminus \langle\emptyset\rangle_{\mathcal{C}},$$

$$\vee \in \langle\vee\rangle_{\mathcal{C}} \setminus \langle\emptyset\rangle_{\mathcal{C}}.$$

Lemma 2.4 implies $M = \mathrm{Pol}(\mathrm{R}_0^1)$. Thence, (cf. 2.5)

$$\langle \wedge \rangle_{\mathcal{C}}\,, \langle \vee \rangle_{\mathcal{C}} \subseteq M \overset{2.4}{=} \langle h \rangle_{\mathcal{C}}\,.$$

This inclusion is proper since

$$h \in (M \setminus \langle \wedge \rangle_{\mathcal{C}}) \cap (M \setminus \langle \vee \rangle_{\mathcal{C}}).$$

This holds because $h$ is a monotone operation and $h \not\triangleright \mathrm{R}_{\mathbf{0},q}^{\mathbf{1},p}$ for $p = 1, q > 1$ and for $q = 1, p > 1$.

Because of $\wedge \not\triangleright \mathrm{R}_{\mathbf{0},q}^{\mathbf{1},p}$ for $p \geq 2$ and $q = 1$, we have $\wedge \in \langle \wedge \rangle_{\mathcal{C}} \setminus \langle \vee \rangle_{\mathcal{C}}$, and because of $\vee \not\triangleright \mathrm{R}_{\mathbf{0},q}^{\mathbf{1},p}$ for $p = 1$ and $q \geq 2$ we have $\vee \in \langle \vee \rangle_{\mathcal{C}} \setminus \langle \wedge \rangle_{\mathcal{C}}$. Consequently, the two $C$-clones are incomparable. ■

**Lemma 2.8.** *For any subset $F \subseteq \mathrm{O}_{\mathbf{2}}$ it holds*

$$\langle c_0, c_1 \rangle_{\mathrm{O}_{\mathbf{2}}} \vee \langle F \rangle_{\mathrm{O}_{\mathbf{2}}} \subseteq \langle F \rangle_{\mathcal{C}}\,.$$

***Proof.*** From 2.7 we infer

$$\langle c_0, c_1 \rangle_{\mathrm{O}_{\mathbf{2}}} \subseteq \langle \emptyset \rangle_{\mathcal{C}} \subseteq \langle F \rangle_{\mathcal{C}}\,.$$

Because of $\langle F \rangle_{\mathcal{C}} \in \boldsymbol{\mathcal{L}}_{\mathbf{2}}$ and $F \subseteq \langle F \rangle_{\mathcal{C}}$ we have

$$\langle F \rangle_{\mathrm{O}_{\mathbf{2}}} \subseteq \langle \langle F \rangle_{\mathcal{C}} \rangle_{\mathrm{O}_{\mathbf{2}}} \overset{1.16}{=} \langle F \rangle_{\mathcal{C}}\,.$$

Consequently, we obtain $\langle c_0, c_1 \rangle_{\mathrm{O}_{\mathbf{2}}} \vee \langle F \rangle_{\mathrm{O}_{\mathbf{2}}} \subseteq \langle F \rangle_{\mathcal{C}}$. ■

**Lemma 2.9.** *It holds*

$$\langle c_0, c_1, \wedge \rangle_{\mathrm{O}_{\mathbf{2}}} = \langle \wedge \rangle_{\mathcal{C}} \qquad \textit{and} \qquad \langle c_0, c_1, \vee \rangle_{\mathrm{O}_{\mathbf{2}}} = \langle \vee \rangle_{\mathcal{C}}\,.$$

***Proof.*** From 2.8 and Figure 1 we obtain $\langle c_0, c_1, \wedge \rangle_{\mathrm{O}_{\mathbf{2}}} \overset{Fig.1}{=} \langle c_0, c_1 \rangle_{\mathrm{O}_{\mathbf{2}}} \vee \langle \wedge \rangle_{\mathrm{O}_{\mathbf{2}}} \subseteq \langle \wedge \rangle_{\mathcal{C}}$. Let us assume

$$\langle c_0, c_1, \wedge \rangle_{\mathrm{O}_{\mathbf{2}}} \subsetneqq \langle \wedge \rangle_{\mathcal{C}}\,.$$

Then, $\langle\wedge\rangle_{\mathcal{C}}$ has to be a clone in $\mathcal{L}_{\mathbf{2}}$ being above $\langle c_0, c_1, \wedge\rangle_{\mathrm{O}_{\mathbf{2}}}$. Because the upper cover of $\langle c_0, c_1, \wedge\rangle_{\mathrm{O}_{\mathbf{2}}}$ in $\mathcal{L}_{\mathbf{2}}$ is $M$ (see Post's Lattice, Figure 1), it follows

$$M \subseteq \langle\wedge\rangle_{\mathcal{C}}\,,$$

which is a contradiction to $M \supset \langle\wedge\rangle_{\mathcal{C}}$ (cf. Lemma 2.7). Similarly, the claim for $\langle\vee\rangle_{\mathcal{C}}$ can be proved. ■

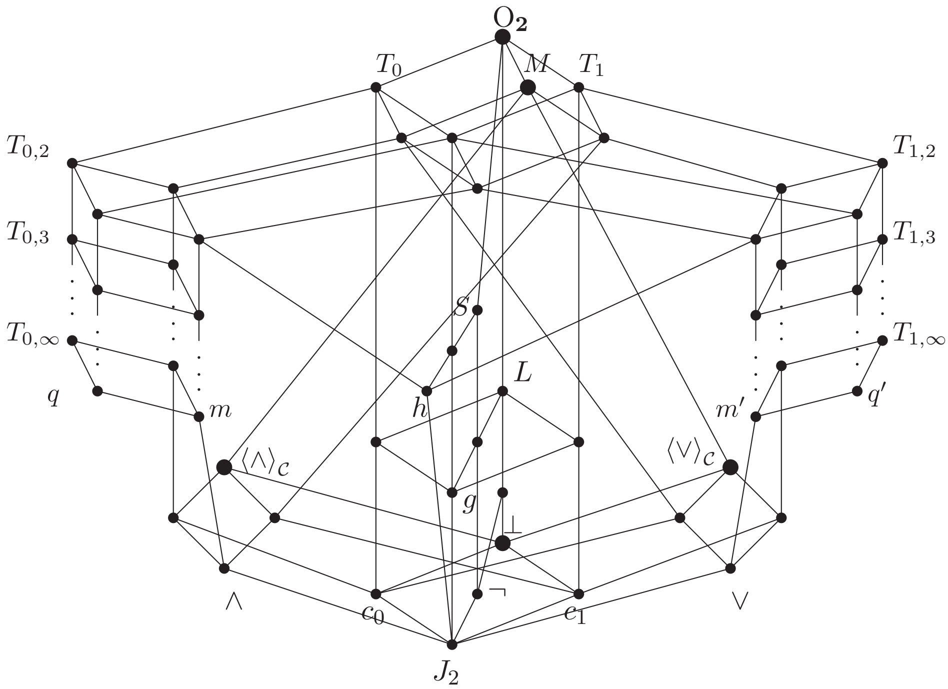


Figure 1. Post's Lattice

**Lemma 2.10.** *For any two subsets $F, G \subseteq \mathrm{O}_{\mathbf{2}}$ the following implication holds*

$$F \subseteq G \subseteq \langle F\rangle_{\mathcal{C}} \implies \langle G\rangle_{\mathcal{C}} = \langle F\rangle_{\mathcal{C}}\,.$$

***Proof.***

$$F \subseteq G \implies \langle F\rangle_{\mathcal{C}} \subseteq \langle G\rangle_{\mathcal{C}}$$

$$G \subseteq \langle F\rangle_{\mathcal{C}} \implies \langle G\rangle_{\mathcal{C}} \subseteq \langle\langle F\rangle_{\mathcal{C}}\rangle_{\mathcal{C}} = \langle F\rangle_{\mathcal{C}}\,.$$

■

**Lemma 2.11.** *Let* $c_0, c_1$ *be the unary constant operations on* $\mathbf{2}$. *Then it holds*

$$\langle \emptyset \rangle_{\mathcal{C}} = \langle c_0, c_1 \rangle_{\mathrm{O}_{\mathbf{2}}}.$$

***Proof.*** $\langle \emptyset \rangle_{\mathcal{C}} \subseteq \langle \vee \rangle_{\mathcal{C}} \cap \langle \wedge \rangle_{\mathcal{C}} \overset{2.9}{=} \langle c_0, c_1, \wedge \rangle_{\mathrm{O}_{\mathbf{2}}} \cap \langle c_0, c_1, \vee \rangle_{\mathrm{O}_{\mathbf{2}}} \overset{Fig.1}{=} \langle c_0, c_1 \rangle_{\mathrm{O}_{\mathbf{2}}} \overset{2.7}{\subseteq} \langle \emptyset \rangle_{\mathcal{C}}$. ∎

**Lemma 2.12.** *Let* $g$ *be the ternary minority operation, i.e.*

$$g(x,x,y) = g(x,y,x) = g(y,x,x) = y.$$

*Then it holds:*

$$\langle g \rangle_{\mathcal{C}} = \langle L \rangle_{\mathcal{C}} = \mathrm{O}_{\mathbf{2}}.$$

***Proof.*** Because of $\neg \in L$, it follows that

$$\mathrm{O}_{\mathbf{2}} \overset{2.3}{=} \langle \neg \rangle_{\mathcal{C}} \subseteq \langle L \rangle_{\mathcal{C}} \subseteq \mathrm{O}_{\mathbf{2}},$$

hence $\langle L \rangle_{\mathcal{C}} = \mathrm{O}_{\mathbf{2}}$. Applying Lemma 2.8 to $\{g\}$ leads to

$$L \overset{Fig.1}{=} \langle c_0, c_1 \rangle_{\mathrm{O}_{\mathbf{2}}} \vee \langle g \rangle_{\mathrm{O}_{\mathbf{2}}} \subseteq \langle g \rangle_{\mathcal{C}}.$$

Together with 2.10 we infer

$$\langle g \rangle_{\mathcal{C}} = \langle L \rangle_{\mathcal{C}}.$$

∎

**Remark 2.13.** Let $F \leq \mathrm{O}_{\mathbf{2}}$ denote a clone in Post's Lattice. Then we have

$$\mathcal{CL}_{\mathbf{2}} = \{\mathrm{Pol}(C\mathrm{Inv}(F)) \mid F \subseteq \mathrm{O}_{\mathbf{2}}\} = \{\mathrm{Pol}(C\mathrm{Inv}(F)) \mid F \leq \mathrm{O}_{\mathbf{2}}\}.$$

***Proof.*** It is obvious that

$$\{\mathrm{Pol}(C\mathrm{Inv}(F)) \mid F \leq \mathrm{O}_{\mathbf{2}}\} \subseteq \{\mathrm{Pol}(C\mathrm{Inv}(F)) \mid F \subseteq \mathrm{O}_{\mathbf{2}}\}.$$

To show the other inclusion we regard $F \subseteq \mathrm{O}_{\mathbf{2}}$. Let $G := \operatorname{Pol}\operatorname{Inv}(F) \leq \mathrm{O}_{\mathbf{2}}$.

$$\begin{aligned}\operatorname{Pol} C\operatorname{Inv}(G) &= \operatorname{Pol}((\operatorname{Inv}(G)) \cap C\mathrm{R}_D) = \operatorname{Pol}((\operatorname{Inv}(\operatorname{Pol}\operatorname{Inv}(F))) \cap C\mathrm{R}_D)\\ &= \operatorname{Pol}(\operatorname{Inv}(F) \cap C\mathrm{R}_D) = \operatorname{Pol} C\operatorname{Inv}(F).\end{aligned}$$

■

In the following we prove that there are no more Boolean $C$-clones than the ones already described in the previous Lemmata 2.4, 2.9, 2.11, 2.12.

**Theorem 2.14.** *The lattice of all Boolean $C$-clones is*

$$\mathcal{CL}_{\mathbf{2}} = \{\perp, \langle\wedge\rangle_{\mathcal{C}}, \langle\vee\rangle_{\mathcal{C}}, \langle h\rangle_{\mathcal{C}}, \mathrm{O}_{\mathbf{2}}\},$$

*where*

$$\perp \;:=\; \langle c_0, c_1\rangle_{\mathrm{O}_{\mathbf{2}}} \quad \langle\wedge\rangle_{\mathcal{C}} = \langle c_0, c_1, \wedge\rangle_{\mathrm{O}_{\mathbf{2}}} \quad \langle\vee\rangle_{\mathcal{C}} = \langle c_0, c_1, \vee\rangle_{\mathrm{O}_{\mathbf{2}}} \quad \langle h\rangle_{\mathcal{C}} = M.$$

***Proof.*** The next six equalities are consequences of Lemma 2.10 and the previous lemmata.

$$\left\{\langle C\rangle_{\mathcal{C}} \mid C \in \left[J_{\mathbf{2}}, \langle c_0, c_1\rangle_{\mathrm{O}_{\mathbf{2}}}\right]_{\mathcal{L}_{\mathbf{2}}}\right\} = \left\{\langle c_0, c_1\rangle_{\mathrm{O}_{\mathbf{2}}}\right\}$$

$$\left\{\langle C\rangle_{\mathcal{C}} \mid C \in \left[\langle\wedge\rangle_{\mathrm{O}_{\mathbf{2}}}, \langle\wedge, c_0, c_1\rangle_{\mathrm{O}_{\mathbf{2}}}\right]_{\mathcal{L}_{\mathbf{2}}}\right\} = \left\{\langle\wedge, c_0, c_1\rangle_{\mathrm{O}_{\mathbf{2}}}\right\}$$

$$\left\{\langle C\rangle_{\mathcal{C}} \mid C \in \left[\langle\vee\rangle_{\mathrm{O}_{\mathbf{2}}}, \langle\vee, c_0, c_1\rangle_{\mathrm{O}_{\mathbf{2}}}\right]_{\mathcal{L}_{\mathbf{2}}}\right\} = \left\{\langle\vee, c_0, c_1\rangle_{\mathrm{O}_{\mathbf{2}}}\right\}$$

$$\left\{\langle C\rangle_{\mathcal{C}} \mid C \in \left[\langle\neg\rangle_{\mathrm{O}_{\mathbf{2}}}, \mathrm{O}_{\mathbf{2}}\right]_{\mathcal{L}_{\mathbf{2}}}\right\} = \{\mathrm{O}_{\mathbf{2}}\}$$

$$\left\{\langle C\rangle_{\mathcal{C}} \mid C \in \left[\langle g\rangle_{\mathrm{O}_{\mathbf{2}}}, \mathrm{O}_{\mathbf{2}}\right]_{\mathcal{L}_{\mathbf{2}}}\right\} = \{\mathrm{O}_{\mathbf{2}}\}$$

$$\left\{\langle C\rangle_{\mathcal{C}} \mid C \in \left[\langle h\rangle_{\mathrm{O}_{\mathbf{2}}}, M\right]_{\mathcal{L}_{\mathbf{2}}}\right\} = \{M\}.$$

The next four equalities will be shown below.

$$\left\{\langle C\rangle_{\mathcal{C}} \mid C \in \left[\langle\wedge\rangle_{\mathrm{O}_\mathbf{2}}, M\right]_{\mathcal{L}_\mathbf{2}} \setminus \left[\langle\wedge\rangle_{\mathrm{O}_\mathbf{2}}, \langle\wedge, c_0, c_1\rangle_{\mathrm{O}_\mathbf{2}}\right]_{\mathcal{L}_\mathbf{2}}\right\} = \{M\}$$

$$\left\{\langle C\rangle_{\mathcal{C}} \mid C \in \left[\langle\wedge\rangle_{\mathrm{O}_\mathbf{2}}, \mathrm{O}_\mathbf{2}\right]_{\mathcal{L}_\mathbf{2}} \setminus \left[\langle\wedge\rangle_{\mathrm{O}_\mathbf{2}}, \langle\wedge, c_0, c_1\rangle_{\mathrm{O}_\mathbf{2}}\right]_{\mathcal{L}_\mathbf{2}}, C \not\subseteq M\right\} = \{\mathrm{O}_\mathbf{2}\}$$

$$\left\{\langle C\rangle_{\mathcal{C}} \mid C \in \left[\langle\vee\rangle_{\mathrm{O}_\mathbf{2}}, M\right]_{\mathcal{L}_\mathbf{2}} \setminus \left[\langle\vee\rangle_{\mathrm{O}_\mathbf{2}}, \langle\vee, c_0, c_1\rangle_{\mathrm{O}_\mathbf{2}}\right]_{\mathcal{L}_\mathbf{2}}\right\} = \{M\}$$

$$\left\{\langle C\rangle_{\mathcal{C}} \mid C \in \left[\langle\vee\rangle_{\mathrm{O}_\mathbf{2}}, \mathrm{O}_\mathbf{2}\right]_{\mathcal{L}_\mathbf{2}} \setminus \left[\langle\vee\rangle_{\mathrm{O}_\mathbf{2}}, \langle\vee, c_0, c_1\rangle_{\mathrm{O}_\mathbf{2}}\right]_{\mathcal{L}_\mathbf{2}}, C \not\subseteq M\right\} = \{\mathrm{O}_\mathbf{2}\}.$$

Regarding a clone $C \leq \mathrm{O}_\mathbf{2}$ with $\langle\wedge\rangle_{\mathrm{O}_\mathbf{2}} \subseteq C$, but $C \notin \left[\langle\wedge\rangle_{\mathrm{O}_\mathbf{2}}, \langle\wedge, c_0, c_1\rangle_{\mathrm{O}_\mathbf{2}}\right]_{\mathcal{L}_\mathbf{2}}$ yields (using Lemma 2.8)

$$C_1 := \langle c_0, c_1\rangle_{\mathrm{O}_\mathbf{2}} \vee C \subseteq \langle C\rangle_{\mathcal{C}}.$$

If $C \subseteq M$ then we have that $M = C_1 \subseteq \langle C\rangle_{\mathcal{C}}$, and because of monotonicity of $\langle\cdot\rangle_{\mathcal{C}}$ yields $\langle C\rangle_{\mathcal{C}} \subseteq \langle M\rangle_{\mathcal{C}} = M$. Thus,

$$\langle C\rangle_{\mathcal{C}} = M.$$

Otherwise, (i. e. $C \not\subseteq M$) leads to

$$\mathrm{O}_\mathbf{2} = C_1 \subseteq \langle C\rangle_{\mathcal{C}} \subseteq \mathrm{O}_\mathbf{2}.$$

The proof for $\vee$ instead of $\wedge$ is similar. Knowing all those 10 equalities and applying Remark 2.13, one obtains

$$\mathcal{C}\mathcal{L}_\mathbf{2} = \{\perp, \langle\wedge\rangle_{\mathcal{C}}, \langle\vee\rangle_{\mathcal{C}}, M, \mathrm{O}_\mathbf{2}\},$$

the clones of which are described in the previous lemmata. ■

The previous theorem does not only describes the set of all the Boolean $C$-clones but also the operations that these contain. For example, $\langle\vee\rangle_{\mathcal{C}}$ contains the operations $c_0, c_1, \vee$, all the projections and compositions of these fuctions.

As we mentioned, when we describe $C$-clones at the same time we are describing relational $C$-clones. The lattices of $C$-clones and of relational $C$-clones are dually isomorphic as is shown in the next figure.

$$
\begin{tikzcd}
 & \mathrm{O}_{\mathbf{2}} \arrow[d, dash] & \\
 & M \arrow[dl, dash] \arrow[dr, dash] & \\
\langle\wedge\rangle_{\mathcal{C}} \arrow[dr, dash] & & \langle\vee\rangle_{\mathcal{C}} \arrow[dl, dash] \\
 & \bot &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & C\mathrm{Inv}\,\bot \arrow[dl, dash] \arrow[dr, dash] & \\
C\mathrm{Inv}\,\langle\wedge\rangle_{\mathcal{C}} \arrow[dr, dash] & & C\mathrm{Inv}\,\langle\vee\rangle_{\mathcal{C}} \arrow[dl, dash] \\
 & C\mathrm{Inv}\,M \arrow[d, dash] & \\
 & C\mathrm{Inv}\,\mathrm{O}_{\mathbf{2}} &
\end{tikzcd}
$$

For the remainder of this section, we restrict ourselves to the following: Given a $C$-clone $\mathrm{Pol}(Q)$ with $Q \subseteq C\mathrm{R}^*_{\mathbf{2}}$, find a minimal subset $Q_1 \subseteq Q$, such that $\mathrm{Pol}(Q) = \mathrm{Pol}(Q_1)$. The motivation for the restriction is to establish that all Boolean $C$-clones can be described by a finite number of clausal relations.

**Lemma 2.15.** *The following equalities hold*

$$
\begin{aligned}
\mathrm{O}_{\mathbf{2}} &= \mathrm{Pol}\,(\emptyset)\,, \\
M &= \mathrm{Pol}\left(\mathrm{R}^1_0\right), \\
\langle\wedge\rangle_{\mathcal{C}} &= \mathrm{Pol}\{\mathrm{R}^{\mathbf{1},p}_{\mathbf{0},q} \mid p = 1, q \in \mathbb{N}_+\} = \mathrm{Pol}\left(\mathrm{R}^1_{(0,0)}\right), \\
\langle\vee\rangle_{\mathcal{C}} &= \mathrm{Pol}\{\mathrm{R}^{\mathbf{1},p}_{\mathbf{0},q} \mid q = 1, p \in \mathbb{N}_+\} = \mathrm{Pol}\left(\mathrm{R}^{(1,1)}_0\right), \\
\langle c_0, c_1\rangle_{\mathrm{O}_{\mathbf{2}}} &= \mathrm{Pol}\{\mathrm{R}^{\mathbf{1},p}_{\mathbf{0},q} \mid p, q \in \mathbb{N}_+\} = \mathrm{Pol}\left(\mathrm{R}^{(1,1)}_{(0,0)}\right).
\end{aligned}
$$

***Proof.*** The characterization of $\mathrm{O}_{\mathbf{2}}$ is trivial. The second statement follows from Lemma 2.4. The arguments for the rest of the equalities are very similar, so w.l.o.g. we will only deal with the characterization of $\langle\wedge\rangle_{\mathcal{C}}$.

"$\subseteq$": We have

$$\{\mathrm{R}^1_{(0,0)}\} \subseteq \{\mathrm{R}^{\mathbf{1},p}_{\mathbf{0},q} \mid p = 1, q \in \mathbb{N}_+\}$$

hence

$$W := \mathrm{Pol}\{\mathrm{R}^1_{(0,0)}\} \supseteq \mathrm{Pol}\{\mathrm{R}^{\mathbf{1},p}_{\mathbf{0},q} \mid p = 1, q \in \mathbb{N}_+\} = \langle \wedge \rangle_{\mathcal{C}}\,.$$

"$\supseteq$": From the proof of Lemma 2.4 we know $M = \langle h \rangle_{\mathcal{C}} \nsubseteq W$, applying the above-established facts that $\langle c_0, c_1, \wedge \rangle_{\mathbf{O_2}} \overset{2.9}{=} \langle \wedge \rangle_{\mathcal{C}} \subseteq W$ and $C$-clones are clones, one can read off of Post's Lattice that $W = \langle c_0, c_1, \wedge \rangle_{\mathbf{O_2}} = \langle \wedge \rangle_{\mathcal{C}}$. ∎

## 3. $C$-clones

In the previous section we showed that there are five different Boolean $C$-clones. Next we show that for $|D| \geq 3$ there are infinitely many $C$-clones by exhibiting an infinite descending chain of such clones.

Let $D \supseteq \{0, 1, 2\}$ and $m \in \mathbb{N}_+$. Consider the following clausal relation:

$$\mathrm{R}^{(1,\dots,1)}_{(1,\dots,1)} =$$

$$\Big\{(x_1, \dots, x_m, y_1, \dots, y_m) \in D^{2m} \,|\, x_1 \geq 1 \vee \dots \vee x_m \geq 1 \vee y_1 \leq 1 \vee \dots \vee y_m \leq 1\Big\}.$$

We define

$$\varrho_m := \mathrm{R}^{(1,\dots,1)}_{(1,\dots,1)}.$$

Observe that the tuple $(0, \dots, 0, 2, \dots, 2) \notin \varrho_m$.

**Proposition 3.1.** *If $\varrho_m$ is the $2m$-ary relation defined above, then*

$$\mathrm{Pol}(\varrho_{m-1}) \supsetneqq \mathrm{Pol}(\varrho_m)$$

*holds for any $m \in \mathbb{N}_+$.*

***Proof.*** Let $n \in \mathbb{N}_+$ and $f \in \mathrm{Pol}^{(n)}(\varrho_m)$. We have to show $f \in \mathrm{Pol}^{(n)}(\varrho_{m-1})$. Let $r_1, \dots, r_n \in \varrho_{m-1}$, where $r_k =: (x_{1k}, \dots, x_{m-1k}, y_{1k}, \dots, y_{m-1k})$ for $k$, that belongs to $\{1, \dots, n\}$. Then

$$x_{1k} \geq 1 \vee \dots \vee x_{m-1k} \geq 1 \vee y_{1k} \leq 1 \vee \dots \vee y_{m-1k} \leq 1$$

$$\Leftrightarrow x_{1k} \geq 1 \vee \dots \vee x_{m-1k} \geq 1 \vee x_{m-1k} \geq 1 \vee y_{1k} \leq 1 \vee \dots \vee y_{m-1k} \leq 1 \vee y_{m-1k} \leq 1.$$

We define for $k \in \{1, \dots, n\}$ a tuple

$$\begin{aligned} r'_k &= (\overline{x}_{1k}, \dots, \overline{x}_{m-1k}, \overline{x}_{mk}, \overline{y}_{1k}, \dots, \overline{y}_{m-1k}, \overline{y}_{mk}) \\ &:= (x_{1k}, \dots, x_{m-1k}, x_{m-1k}, y_{1k}, \dots, y_{m-1k}, y_{m-1k}). \end{aligned}$$

The new tuples $r'_1, \dots, r'_n$ belong to $\varrho_m$ because of the above expression.

Because $f \rhd \varrho_m$ we have

$$(c_1, \dots, c_{m-1}, c_m, d_1, \dots, d_{m-1}, d_m) := f[r'_1, \dots, r'_n] \in \varrho_m.$$

Then, by construction, we have $c_{m-1} = c_m$, $d_{m-1} = d_m$ and

$$\begin{aligned} c_{m-1} \geq 1 \vee c_m \geq 1 &\iff c_{m-1} \geq 1 \\ d_{m-1} \leq 1 \vee d_m \leq 1 &\iff d_{m-1} \leq 1. \end{aligned}$$

Therefore, $(c_1, \dots, c_{m-1}, d_1, \dots, d_{m-1}) = f[r_1, \dots, r_n] \in \varrho_{m-1}$, i.e. $f \rhd \varrho_{m-1}$. Now we show that the inclusion is proper. Let $f$ be an $2m$-ary operation such that

$$f(x_1, \dots, x_m, \dots, x_{2m}) = \begin{cases} 0 & \text{if there is only one 1 among the first } m \text{ entries} \\ & \text{and 0 in the other entries.} \\ 2 & \text{if there is only one 1 between the } m+1, \dots, 2m \\ & \text{entries and 2 in the other entries.} \\ 1 & \text{otherwise.} \end{cases}$$

That is

$$f(1, 0, \dots, \overset{m}{0}, 0, \dots, \overset{2m}{0}) =, \dots, = f(0, \dots, 0, \overset{m}{1}, 0, \dots, 0) = 0$$

$$f(2, 2, \dots, \overset{m+1}{1}, 2, \dots, 2) =, \dots, = f(2, \dots, 2, \overset{m+1}{2}, \dots, 2, 1) = 2.$$

We show $f \notin \mathrm{Pol}^{(2m)}(\varrho_m)$.

Consider the tuples $r_1, \ldots r_{2m} \in \varrho_m$, such that

$$r_1 = (1, 0, \ldots, \overset{m}{0}, 2, \ldots, \overset{2m}{2}), \ldots, r_{2m} = (0, \ldots, 0, 2, \ldots, 2, \overset{2m}{1}).$$

$f[r_1, \ldots, r_{2m}] \notin \varrho_m$ because

| | | 1 | 2 | … | m | m+1 | m+2 | … | 2m | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $f($ | 1 | 0 | … | 0 | 0 | 0 | … | 0 | $)$ | =0 |
| 2 | $f($ | 0 | 1 | … | 0 | 0 | 0 | … | 0 | $)$ | =0 |
| | | | | $\ddots$ | | | | $\vdots$ | | | $\vdots$ |
| m | $f($ | 0 | 0 | … | 1 | 0 | 0 | … | 0 | $)$ | =0 |
| m+1 | $f($ | 2 | 2 | … | 2 | 1 | 2 | … | 2 | $)$ | =2 |
| m+2 | $f($ | 2 | 2 | … | 2 | 2 | 1 | … | 2 | $)$ | =2 |
| | | | | $\vdots$ | | | | $\ddots$ | | | $\vdots$ |
| 2m | $f($ | 2 | 2 | … | 2 | 2 | 2 | … | 1 | $)$ | =2 |
| | | | | | | | | | | | $\notin \varrho_m$ |

However, $f \in \mathrm{Pol}^{(2m)}(\varrho_{m-1})$ as the following argument shows. Let $r'_1, \ldots, r'_{2m} \in \varrho_{m-1}$, where

$$r'_k = (x_{1k}, \ldots, x_{m-1k}, y_{1k}, \ldots, y_{m-1k})$$

for $k \in \{1, \ldots, 2m\}$. Note that, by definition, every tuple $r'_k$ satisfies the following expression:

$$x_{1k} \geq 1 \vee \ldots \vee x_{m-1k} \geq 1 \vee y_{1k} \leq 1 \vee \ldots \vee y_{m-1k} \leq 1.$$

Let us regard the tuples $r'_k \in \varrho_{m-1}$ as the columns of a matrix $A \in D^{2(m-1)\times(2m)}$. We construct a matrix

$$(b_{ij})_{\substack{1\leq i\leq 2(m-1)\\ 1\leq j\leq 2m}} = B \in \{0,1\}^{2(m-1)\times(2m)}$$

in the following way:

$$b_{ij} := \begin{cases} 1 & \text{if } a_{ij} \geq 1 \text{ and } 1 \leq i \leq m-1, \\ 1 & \text{if } a_{ij} \leq 1 \text{ and } m \leq i \leq 2(m-1), \\ 0 & \text{otherwise.} \end{cases}$$

In every column $r'_k$ there is at least one element $d \in D$ such that $d \geq 1$ or $d \leq 1$, hence any column of $B$ contains at least one entry $b_{ij} = 1$. Since there are only $2(m-1)$ rows but $2m$ columns, there is at least one row of B, say the $l$-th row, containing two entries 1. Consequently,

$$f(r_{l1}, \ldots, r_{l2m}) = 1$$

from the definition of $f$. Then $f[r_1, \ldots, r_{2m}] \in \varrho_{m-1}$, hence $f \rhd \varrho_{m-1}$. ■

## 4. Conclusion

We are interested in exhibiting the clones that can be determined by sets of clausal relations. The first step towards this goal was to give a complete characterization of the Boolean $C$-clones and to prove certain results about $C$-clones over $|D| \geq 3$. Although some results obtained in Section 2 (c.f. Lemmata 2.4, 2.5, 2.3) can be generalized to the case $|D| \geq 3$, the task to describe all $C$-clones for $|D| \geq 3$ seems to be rather difficult due to the existence of an infinite number of $C$-clones.

Therefore, in future investigations we will restrict our studies to the characterization of only the unary parts of the $C$-clones. That is we will try to characterize all sets of unary functions $\mathrm{Pol}^{(1)} Q$ where $Q \subseteq C\mathrm{R}_D$. The motivation for this restriction is that over a finite set $D$ there are only finitely many weak Krasner clones, i.e. clones of the form $F = \mathrm{Pol}^{(1)} Q$ where $Q \subseteq \mathrm{R}_D$.

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