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FACTORING AN ODD ABELIAN GROUP BY LACUNARY CYCLIC SUBSETS

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Abstract

It is a known result that if a finite abelian group of odd order is a direct product of lacunary cyclic subsets, then at least one of the factors must be a subgroup. The paper gives an elementary proof that does not rely on characters.

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1. Introduction

In this paper we will use multiplicative notations in connection with abelian groups. Let G be a finite abelian group. The identity element of G will be denoted by e. The order of an element a of G is designated by $|a|$. The number of the elements of a subset A of G is denoted by $|A|$.

Let $A_1, \dots, A_n$ be subsets of G. If the product $A_1 \cdots A_n$ is direct and is equal to G, then we say that the equation $G = A_1 \cdots A_n$ is a factorization of G. A subset A of G in the form

$$A = \{e, a, a^2, \dots, a^{r-1}\}$$

is called a cyclic subset of G. In order to avoid trivial cases we assume that $r \geq 2$ and that $|a| \geq r$. Clearly A is a subgroup of G if and only if $a^r = e$. It is a famous result of G. Hajós [2] that if a finite abelian group is factored as a direct product of its cyclic subsets, then at least one of the factors must be a subgroup.

A subset A of G in the form

$$A = \{e, a, a^2, \dots, a^{r-1}\} \cup g\{e, a, a^2, \dots, a^{s-1}\} \tag{1}$$

is called a lacunary cyclic subset. Here we assume that $|a| \geq r$ since otherwise there would be repetition on the list $e, a, a^2, \dots, a^{r-1}$. From similar reason, we assume that $|a| \geq s$. Further we assume that the subsets

$$\{e, a, a^2, \dots, a^{r-1}\} \tag{2}$$

and

$$g\{e, a, a^2, \dots, a^{s-1}\} \tag{3}$$

are disjoint. Therefore A has $r + s$ elements. We call the cyclic subset

$$C = \{e, a, a^2, \dots, a^{r+s-1}\} \tag{4}$$

a cyclic subset associated with the lacunary cyclic subset A relative to the representation (1). Besides the representation (1) the lacunary cyclic subset A may have another representation as a lacunary cyclic subset, say

$$A = \{e, x, x^2, \dots, x^{\alpha-1}\} \cup y\{e, x, x^2, \dots, x^{\beta-1}\}.$$

We leave the problem of possibility of various representations unresolved. This is why the definition of the cyclic subset associated with a given lacunary cyclic subset contains a reference to the representation.

K. Corrádi and S. Szabó [1] proved that if a finite abelian group of odd order is factored into lacunary cyclic subsets, then at least one of the factors must be a subgroup. The proof heavily relies on the character techniques developed by L. Rédei [3]. Here we give a character free elementary proof.

In 2008 professor A.D. Sands delivered a lecture at the University of Pécs on factoring finite abelian groups. Absorbing his ideas leads us to an elementary character free proof in the case of lacunary cyclic subsets. A part of the lecture has later appeared in printed form [4].

2. Replacement

If from the factorization $G = AB$ it follows that $G = CB$ is also a factorization, then we will say that the factor A in the factorization $G = AB$ can be replaced by C.

Lemma 1. *Let G be a finite abelian group of odd order and let A be a lacunary cyclic subset of G in form* (1)*. If $G = AB$ is a factorization of G, then $G = CB$ is also a factorization of G.*

Proof. If $s = 0$, then $A = C$ and there is nothing to prove. So we may assume that $s \geq 1$.

If $s > r$, then multiply the factorization $G = AB$ by g^{-1}. We get the factorization $G = g^{-1}G = (g^{-1}A)B$. Note that

$$g^{-1}A = g^{-1}\{e, a, a^2, \dots, a^{r-1}\} \cup \{e, a, a^2, \dots, a^{s-1}\}$$

is again a lacunary cyclic subset. Therefore the roles of r and s can be reversed. Thus we may assume that $s \leq r$.

If $r = s$, then $|A| = 2r$. From the factorization $G = AB$ it follows that $|G| = |A||B|$ which implies that $|G|$ is even. This is not the case. Thus we may assume that $1 \leq s < r$.

The factorization $G = AB$ means that the sets

$$eB,\ aB,\ a^2B, \dots, a^{r-1}B,\ geB,\ gaB,\ ga^2B, \dots, ga^{s-1}B \tag{5}$$

form a partition of G. Multiplying the factorization $G = AB$ by a we get the factorization $G = aG = (aA)B$ of G. This means that the sets

$$aB,\ a^2B,\ a^3B, \ldots, a^rB,\ gaB,\ ga^2B,\ ga^3B, \ldots, ga^sB \tag{6}$$

form a partition of G. Comparing the two partitions we get

$$eB \cup gB = a^rB \cup ga^sB. \tag{7}$$

If $gB \cap ga^sB \neq \emptyset$, then $B \cap a^sB \neq \emptyset$ which contradicts the partition (5). Thus $gB \cap ga^sB = \emptyset$ and from (7) it follows that $gB = a^rB$. Plugging this into (5) we get that the sets

$$eB,\ aB,\ a^2B, \ldots, a^{r-1}B,\ a^rB,\ a^{r+1}B,\ a^{r+2}B, \ldots, ga^{r+s-1}B$$

form a partition of G. Thus $G = CB$ is a factorization of G.

This completes the proof. ∎

3. Product of non-periodic subsets

We say that a subset A of G is periodic if there is an element $h \in G \setminus \{e\}$ such that $Ah = A$. The element h is called a period of A.

To a nonempty subset A of a finite abelian group G we assign the subset L defined by

$$L = \bigcap_{a \in A} Aa^{-1}.$$

It turns out that L is a subgroup of G and further that the elements of $L \setminus \{e\}$ are all the periods of A. We will call L the subgroup of periods of A. The next result is Lemma 2.8 of [5].

Lemma 2. *Let A be a nonempty subset of a finite abelian group G. Let L be the subset assigned to A.*

(i) *If $g \in L$, then $gA = A$.*

(ii) *If $gA = A$ for some $g \in G$, then $g \in L$.*

(iii) *L is a subgroup of G.*

(iv) *There is a subset D of A such that the product DL is direct and is equal to A.*

Under certain conditions the product of non-periodic subsets is again a non-periodic subset. The result below is Theorem 3.1 of [5].

Lemma 3. *Let G be a finite abelian group and let H be a subgroup of G. Let A, B subsets of G such that $e \in A$, $e \in B$, $A \subset H$. Assume that the product AB is direct, A, B are not periodic and the elements of B are pair-wise incongruent modulo H. Then the set AB is not periodic.*

4. Periodic lacunary cyclic subsets

A periodic cyclic subset must be a subgroup. The next result is part of the folklore. Most likely it goes back to G. Hajós.

Lemma 4. *Let G be a finite abelian group and let $A = \{e, a, a^2, \ldots, a^{r-1}\}$ be a cyclic subset of G. If A is periodic, then $a^r = e$.*

Under suitable conditions if a lacunary cyclic subset is periodic, then it must be a subgroup.

Lemma 5. *Let G be a finite abelian group of odd order and let A be a lacunary cyclic subset of G in form (1) for which $1 \leq s < r$.*

(i) *A is a subgroup of G if and only if $g = a^r$ and $a^{r+s} = e$.*

(ii) *A is periodic if and only if A is a subgroup of G.*

Proof. (i) Suppose that $g = a^r$ and $a^{r+s} = e$. From $g = a^r$, it follows that $A = C$. Then $a^{r+s} = e$ implies that C is a subgroup of G.

Next we assume that A is a subgroup of G and show that $g = a^r$ and $a^{r+s} = e$ hold. We claim that $g \in \langle a \rangle$. To prove the claim note that as $s \geq 1$, we have $g \in A$ and hence $g^2 \in A$. Since the sets (2) and (3) are disjoint, it follows that $g \neq e$. As $|G|$ is odd, the order of g cannot be 2 and so $g^2 \neq e$.

If $g^2 \in \{e, a, a^2, \ldots, a^{r-1}\}$, then $g^2 \in \langle a \rangle$ and then $g \in \langle a \rangle$, as we claimed.

If $g^2 \in g\{e, a, a^2, \ldots, a^{s-1}\}$, then $g \in \langle a \rangle$, as required.

Now $A \subset \langle a \rangle$ and $\langle a \rangle \subset A$ imply $\langle a \rangle = A$. Using $|A| = r + s$, we get $a^{r+s} = e$, as required. Further $a^{r+s} = e$ gives $A = C$. From

$$\{e, a, a^2, \dots, a^{r-1}\} \cup g\{e, a, a^2, \dots, a^{s-1}\}$$
$$= \{e, a, a^2, \dots, a^{r-1}\} \cup \{a^r, a^{r+1}, a^{r+2}, \dots, a^{r+s-1}\}$$

one can see that

$$g\{e, a, a^2, \dots, a^{s-1}\} = \{a^r, a^{r+1}, a^{r+2}, \dots, a^{r+s-1}\}.$$

If $a^r \in g\{a, a^2, \dots, a^{s-1}\}$, then it follows that $a^r = ga^i$, $1 \leq i \leq s-1$, then $a^{r-i} = g$ which contradicts that the sets (2) and (3) are disjoint. Hence $a^r = g$, as required.

(ii) If A is a subgroup of G, then since $A \neq \{e\}$, A is periodic.

Assume that A is periodic and let h be a period of A. We claim that $g \in \langle a \rangle$. In order to prove the claim notice that as $e \in A$, it follows that $h \in A$. Hence either

$$h \in \{e, a, a^2, \dots, a^{r-1}\}$$

or

$$h \in g\{e, a, a^2, \dots, a^{s-1}\}.$$

Let us suppose that $h = ga^i$ and distinguish two cases depending on either

$$gh \in \{e, a, a^2, \dots, a^{r-1}\}$$

or

$$gh \in g\{e, a, a^2, \dots, a^{s-1}\}.$$

If $gh = a^j$, then $g^2 \in \langle a \rangle$ and so $g \in \langle a \rangle$, as we claimed. If $gh = ga^j$, then $g \in \langle a \rangle$, as required.

Let us turn to the $h = a^i$ possibility. If $(ga^j)h \in \{e, a, a^2, \dots, a^{r-1}\}$ for some j, $0 \leq j \leq s-1$, then we get $g \in \langle a \rangle$. If $(ga^j)h \in g\{e, a, a^2, \dots, a^{s-1}\}$ for some j, $0 \leq j \leq s-1$, then h is a period of $g\{e, a, a^2, \dots, a^{s-1}\}$. As $\{e, a, a^2, \dots, a^{s-1}\}$ is periodic, by Lemma 4, it follows that $a^s = e$.

Since $s < r$ and $|a| \geq r$ we get a contradiction. Thus $g \in \langle a \rangle$ and so $A \subset \langle a \rangle$.

Let H be the subgroup of periods of A. Clearly A is a cyclic subgroup and can be written in the form $H = \langle a^t \rangle$. Let $|H| = k$. As $|G|$ is odd, it follows that $k \geq 3$. If $A = \langle a \rangle$, then A is a subgroup of G and we are done. Thus we may assume that there is an a^i such that $a^i \notin A$. There is an integer v for which

$$e, a, a^2, \ldots, a^{v-1} \in A$$

and $a^v \notin A$, $v < t$. Set $D = \{e, a, a^2, \ldots, a^{v-1}\}$ and $E = \{a^v\}$. Note that

$$D,\ Da^t,\ Da^{2t}, \ldots, Da^{(k-1)t}$$

are subsets of A and

$$E,\ Ea^t,\ Ea^{2t}, \ldots, Ea^{(k-1)t}$$

are not subsets of A. It follows that A has at least $k-1$ gaps. But we know that A has at most one gap.

This completes the proof. ∎

5. The result

We are in position now to prove the main result of the paper.

Theorem 1. *Let G be a finite abelian group of odd order. If $G = A_1 \cdots A_n$ is a factorization of G, where each A_i is a lacunary cyclic subset, then at least one of the factors must be a subgroup of G.*

Proof. In the $n = 1$ case $G = A_1$ and so A_1 is a subgroup of G. We assume that $n \geq 2$ and start an induction on n. We consider a factorization $G = A_1 \cdots A_n$ and show that one of the factors is a subgroup of G using the fact that the result holds for each smaller values of n. If one of the factors $A_1, \ldots, A_n$ is periodic, then, by Lemma 5, one of the factors is a subgroup of G and we are done. Thus we may assume that none of the factors $A_1, \ldots, A_n$ is periodic.

In the factorization $G = A_1 \cdots A_n$ replace each factor A_i by the associated cyclic subset C_i relative to A_i to get the factorization $G = C_1 \cdots C_n$. By Hajós's theorem, one of the factors $C_1, \ldots, C_n$ is a subgroup of G.

We may assume that $C_1 = H_1$ is a subgroup of G since this is only a matter of indexing the factors $C_1, \ldots, C_n$.

In the factorization $G = A_1 \cdots A_n$ replace the factor A_1 by $C_1 = H_1$ to get the factorization $G = H_1 A_2 \cdots A_n$. Considering the factor group G/H_1 we get the factorization

$$G/H_1 = (A_2H_1)/H_1 \cdots (A_nH_1)/H_1$$

of G/H_1, where

$$(A_iH_1)/H_1 = \{a_iH_1 : \ a_i \in A_i\}.$$

Note that $(a_iH_1)/H_1$ is a lacunary cyclic subset of G/H_1 and so, by the inductive assumption, it follows that one of the factors

$$(A_2H_1)/H_1, \ldots, (A_nH_1)/H_1$$

is a subgroup of G/H_1. We may assume that $(A_2H_1)/H_1$ is a subgroup of G/H_1. There is a subgroup H_2 of G such that $H_1A_2 = H_2$. Therefore $G = H_2A_3 \cdots A_n$ is a factorization of G. Considering the factor group G/H_2 we get the factorization

$$G/H_2 = (A_3H_2)/H_2 \cdots (A_nH_2)/H_2$$

of G/H_2. Continuing in this way finally we have that there are subgroups $H_1, H_2, \ldots, H_n$ of G such that $H_n = G$ and

$$H_1A_2 = H_2, \ H_2A_3 = H_3, \ldots, H_{n-1}A_n = H_n.$$

The factorization $H_1A_2 = H_2$ implies that $A_2 \subset H_2$. The factorization $H_2A_3 = H_3$ shows that the elements of A_3 are incongruent modulo H_2. Thus Lemma 3 is applicable and provides that the product A_2A_3 cannot be periodic.

The factorization $H_1(A_2A_3) = H_3$ implies that $A_2A_3 \subset H_2$. From the factorization $H_3A_4 = H_4$ on can see that the elements of A_4 are incongruent modulo H_3. By Lemma 3, the product $(A_2A_3)A_4$ is not periodic. Continuing in this way finally we get that the product $(A_2 \cdots A_{n-1})A_n$ is not periodic.

Set $B = A_2 \cdots A_n$, $A = A_1$, $C = C_1$ and suppose that A, C are in forms (1), (4), respectively. Now $G = AB$ is a factorization of G. From $C = C_1 = H_1$, by Lemma 5, it follows that $a^{r+s} = e$.

In the way we have seen in the proof of Lemma 1, from the factorization $G = AB$ we can conclude that $a^r B = gB$. If $a^r g^{-1} \neq e$, then B is periodic. This is not the case so $a^r = g$ and consequently, by Lemma 5, $A = C$. Therefore A_1 is equal to H_1.

This completes the proof. ∎

If a finite abelian group cannot be written as a direct product of lacunary cyclic subsets, then Theorem 1 is vacuously true. The next example shows that there are genuine factorizations of finite abelian groups into lacunary cyclic subsets.

Let

$$\{e\} = H_0 \subset H_1 \subset \cdots \subset H_{n-1} \subset H_n = G$$

be subgroups of a finite abelian group G such that the factor groups

$$H_1/H_0,\ H_2/H_1, \ldots, H_n/H_{n-1}$$

are cyclic. Let

$$C_i = \left\{e, c_i, c_i^2, \ldots, c_i^{r(i)+s(i)-1}\right\}$$

be a complete set of representatives in H_i modulo H_{i-1}. Choose an $h_i \in H_{i-1}$. Note that

$$A_i = \left\{e, c_i, c_i^2, \ldots, c^{r(i)-1}, h_i c_i^{r(i)}, \ldots, h_i c_i^{r(i)+s(i)-1}\right\}$$

is also a complete set of representatives in H_i modulo H_{i-1}. It follows that

$$H_n = H_{n-1}A_n,\ H_{n-1} = H_{n-2}A_{n-1}, \ldots, H_1 = H_0 A_1$$

are factorizations and so

$$G = A_1 A_2 \cdots A_n$$

is a factorization of G. Set $g_i = h_i c_i^{r(i)}$. The representation

$$A_i = \left\{e, c_i, c_i^2, \ldots, c_i^{r(i)-1}\right\} \cup g_i \left\{e, c_i, c_i^2, \ldots, c_i^{s(i)-1}\right\}$$

makes clear that A_i is a lacunary cyclic subset of G.

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