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# A REDUCTION THEOREM FOR RING VARIETIES WHOSE SUBVARIETY LATTICE IS DISTRIBUTIVE

Mikhail V. Volkov*

*Faculty of Mathematics and Mechanics*
*Ural State University*
*Lenina 51, 620083 Ekaterinburg, Russia*

**e-mail:** Mikhail.Volkov@usu.ru

**Abstract**

We prove a theorem (for arbitrary ring varieties and, in a stronger form, for varieties of associative rings) which basically reduces the problem of a description of varieties with distributive subvariety lattice to the case of algebras over a finite prime field.

**Keywords:** variety of rings, subvariety lattice, distributive lattice, torsion-bounded variety, Mal'tsev product.

**2000 Mathematics Subject Classification:** 06D05, 08B15.

## Introduction

The aim of studying varietal lattices is to achieve a better understanding of the structure of the lattices and to use the information gained for a classification of varieties. As usual, in order to guide research towards this general aim, it is reasonable to set some more concrete (but sufficiently hard) questions such that the theory could grow and mature answering them.

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Apparently, for the case of ring varieties, it is the problem of describing varieties with distributive subvariety lattice that efficiently plays such a role for many years. There is extensive literature on the subject so that even the mere list of relevant publications is far too long to be placed here[†]. Roughly speaking, one may characterize the current stage of investigations as a period of searching for a border separating varieties with distributive and non-distributive subvariety lattice.

In this note we prove a new reduction theorem for ring varieties whose subvariety lattice is distributive. By a reduction theorem we understand a result of the following sort: "for a variety $\mathfrak{X}$, there exists a family of subvarieties $\mathcal{Y}$ such that the subvariety lattice of $\mathfrak{X}$ is distributive whenever the subvariety lattice of $\mathfrak{Y}$ is distributive, for each subvariety $\mathfrak{Y} \in \mathcal{Y}$". From the aforementioned standpoint of looking for a border between varieties with distributive and non-distributive subvariety lattice, such a reduction means in a sense that the border goes through the family $\mathcal{Y}$; of course, it is the more interesting the more restricted $\mathcal{Y}$ is. Several results of this kind may be found in the author's article [9]; we reproduce here one of them as it has served as a departure point for the present note.

For any positive integer $n$ and any (not necessarily associative!) ring variety $\mathfrak{X}$, let $\mathfrak{X}_n$ denote the subvariety of $\mathfrak{X}$ defined within $\mathfrak{X}$ by the identity $nx = 0$. (As usual, the expression $nx$ is just a compact presentation for the sum $\underbrace{x + x + \cdots + x}_{n \text{ times}}$.) By $L(\mathfrak{X})$ we denote the subvariety lattice of $\mathfrak{X}$. A variety $\mathfrak{X}$ is said to be *torsion-bounded*[‡] if for every relatively free ring $R \in \mathfrak{X}$ (that is, a ring that happens to be free in a subvariety of $\mathfrak{X}$), the torsion subgroup $T(R)$ of the additive group of $R$ is bounded, i. e. $nT(R) = 0$ for some positive integer $n$. With this notion and the above notation, we have the following

**Theorem 1** [9, Theorem 1]**.** *Let $\mathfrak{X}$ be a torsion-bounded variety of (not necessarily associative) rings. Then the lattice $L(\mathfrak{X})$ is distributive provided that the lattice $L(\mathfrak{X}_n)$ is distributive for every positive integer $n$.*

In other words, Theorem 1 amounts to saying that if the distributive law fails for some triple of subvarieties of a variety $\mathfrak{X}$, then the subvarieties may

[†]Unfortunately, there exists no appropriate compendium which one could refer to; the paper [3], the only available survey of the area, clearly is out of date.

[‡]In [9] such varieties were called *joined*; this strange term was borrowed from the (rather unlucky) English translation of the paper [7] where the notion first appeared.

always be chosen to satisfy the identity $nx = 0$ for some positive integer $n$. In the present paper we refine Theorem 1 by showing that, under the same premise, either the three subvarieties may be chosen to satisfy the identity $px = 0$ for some *prime* number $p$ or one of the subvarieties in the "bad" triple equals $\mathfrak{X}_p$ where $p$ is again a prime number. In order to formulate the result in the same manner as Theorem 1, recall that an element $a$ of a lattice $\langle L, \vee, \wedge \rangle$ is said to be *distributive* if, for any $x, y \in L$, the equality

$$a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$$

holds true.

**Theorem 2.** *Let $\mathfrak{X}$ be a torsion-bounded variety of (not necessarily associative) rings. Then the lattice $L(\mathfrak{X})$ is distributive provided that, for every prime number $p$, the lattice $L(\mathfrak{X}_p)$ is distributive and the variety $\mathfrak{X}_p$ is a distributive element of the lattice $L(\mathfrak{X})$.*

In [9] it is shown that for associative rings Theorem 1 holds without the restriction that $\mathfrak{X}$ is torsion-bounded. Here we prove that the same is true for the associative case of Theorem 1. However the condition that the subvariety $\mathfrak{X}_p$ is a distributive element of the lattice $L(\mathfrak{X})$ cannot be omitted even when $\mathfrak{X}$ consists of associative, commutative and nilpotent rings as we demonstrate by studying the subvariety lattice of the variety $\mathfrak{N}_{p^2}$ ($p$ is an arbitrary prime number) defined within the class of all associative-commutative rings by the identities

$$p^2x = 0,\ pxy = 0,\ x_1x_2 \cdots x_{p+1} = 0.$$

We notice that any ring variety satisfying the identity $px = 0$ may be treated as a variety of algebras over the $p$-element field. Thus, for associative rings, Theorem 2 in a sense reduces the problem of a description of varieties with distributive subvariety lattice to the case of algebras over a field where lots of information are already available, cf., for example, [6].

The note is structured as follows. Section 1 contains some preliminaries. Theorem 2 is proved in Section 2. In Section 3 we analyze the subvariety lattice of the variety $\mathfrak{N}_{p^2}$.

## 1. Preliminaries

No acquaintance with the structure theory of rings is presupposed since all proofs in the paper are based on either lattice-theoretical arguments or combinatorial manipulations with ring identities. Here we fix the terminology and recall a couple of auxiliary results that will be used in the sequel.

An element $a$ of a lattice $\langle L, \vee, \wedge\rangle$ is called *neutral* if, for any $x, y \in L$, the sublattice generated by $x$, $y$ and $a$ is distributive. We need the following property of neutral elements:

**Lemma 1** [4, Theorem III.2.4]**.** *If $a$ is a neutral element of a lattice $L$, then $L$ is a subdirect product of the principal ideal $(a] = \{x \in L \mid x \leq a\}$ and the principal filter $[a) = \{x \in L \mid x \geq a\}$.*

Clearly, every neutral element of a lattice is distributive. The converse is not true in general but in the presence of the modular law the two concepts coincide. We register this fact in the next lemma:

**Lemma 2** [4, Theorem III.2.6]**.** *Every distributive element of a modular lattice is neutral.*

Recall that the lattice of all ring varieties is modular because it is anti-isomorphic to the lattice of all fully invariant ideals of the absolutely free ring $F\langle x_1, x_2, \dots\rangle$ on countably many generators. (Recall that an ideal is said to be *fully invariant* if it contains all of its endomorphic images.) Therefore, Lemma 2 applies to subvariety lattices of ring varieties, and so does the following well-known property of modular lattices:

**Lemma 3** [4, Theorem IV.1.2]**.** *If $\langle L, \vee, \wedge\rangle$ is a modular lattice, then for all $x, y \in L$, the intervals $[x, x \vee y]$ and $[x \wedge y, y]$ are isomorphic.*

Next we recall an argument that belongs to the PI-ring folklore. Since we have failed to find any source in which the argument would be presented in a form convenient for the usage in this note, we provide an appropriate formulation and, for the sake of completeness, supply it with a short proof.

We refer to non-zero elements of the absolutely free ring $F\langle x_1, x_2, \dots\rangle$ as to *polynomials in variables* $x_1, x_2, \dots$. For each $i = 1, 2, \dots$, let $\delta_i$ denote the *deletion endomorphism* of the ring $F\langle x_1, x_2, \dots\rangle$ that annihilates the

generator $x_i$, that is, the endomorphism defined by the rule

$$\delta_i(x_j) = \begin{cases} x_j & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$

A polynomial $h \in F\langle x_1, x_2, \dots \rangle$ is called *uniform* if for every $i = 1, 2, \dots$ either $\delta_i(h) = h$ or $\delta_i(h) = 0$ (in the latter case we say that $h$ *depends on the variable* $x_i$). Clearly, every polynomial $h \in F\langle x_1, x_2, \dots \rangle$ is a sum of uniform summands each being the image of $h$ under some composition of endomorphisms $\delta_i$. In particular, this means that every fully invariant ideal $T$ of $F\langle x_1, x_2, \dots \rangle$ is generated (even as an additive subgroup of $F\langle x_1, x_2, \dots \rangle$) by its uniform polynomials. Therefore, the above notions make sense also for non-zero elements of the relatively free ring $F\langle x_1, x_2, \dots \rangle / T$: for instance, if such an element $h$ happens to be represented by a uniform polynomial $f \in F\langle x_1, x_2, \dots \rangle$, then any other uniform polynomial $g \in F\langle x_1, x_2, \dots \rangle$ representing $h$ depends on the same set $X$ of variables as $f$ does. In this case we say that the element $h$ is uniform and depends on $X$.

By a *monomial* we mean an element of the free groupoid freely generated by the set $\{x_1, x_2, \dots\}$. Clearly, $F\langle x_1, x_2, \dots \rangle$ is the free abelian group freely generated by the set of all monomials. Hence, for each polynomial $h \in F\langle x_1, x_2, \dots \rangle$, there exists a unique minimal representation of $h$ as a linear combination of monomials with integer coefficients. We refer to monomials that occur in this representation as to the *monomials of* $h$.

Now let $f(x, y_1, \dots, y_r)$ be a uniform polynomial depending on the variables $x, y_1, \dots, y_r \in \{x_1, x_2, \dots\}$. We denote by $f^{(k)}(x, y_1, \dots, y_r)$ the sum of all monomials of $f(x, y_1, \dots, y_r)$ in which the distinguished variable $x$ occurs exactly $k$ times. Then for some positive integer $n$ we have

$$f(x, y_1, \dots, y_r) = \sum_{k=1}^{n} f^{(k)}(x, y_1, \dots, y_r).$$

**Lemma 4.** *Suppose that a ring $R$ satisfies the identity*

$$f(x, y_1, \dots, y_r) = 0 \tag{1}$$

*where $f(x, y_1, \ldots, y_r)$ is a uniform polynomial such that the variable $x$ occurs at most $n$ times in each of its monomials. Then, for each positive integer $s$ and for each $k = 1, \ldots, n$, the ring $R$ also satisfies the identity*

$$\Delta(s, s^2, \ldots, s^n) f^{(k)}(x, y_1, \ldots, y_r) = 0$$

*where $\Delta(s, s^2, \ldots, s^n)$ is the determinant of the Vandermonde matrix*

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ s & s^2 & \ldots & s^n \\ \vdots & \vdots & \ddots & \vdots \\ s^{n-1} & s^{2(n-1)} & \ldots & s^{n(n-1)} \end{pmatrix}. \tag{2}$$

***Proof.*** For $i = 1, \ldots, n$, substitute $s^{i-1}x$ for $x$ in the identity (1). We then obtain that $R$ satisfies each of the identities

$$\sum_{k=1}^{n} s^{k(i-1)} f^{(k)}(x, y_1, \ldots, y_r) = 0, \quad i = 1, \ldots, n. \tag{3}$$

Now consider (3) as a system of simultaneous linear equations whose matrix is (2). Performing a series of elementary transformations of the system so that the effect of the series is the same as that of multiplying the system on the left by the integer matrix adjoined to (2), we deduce from (3) the system

$$\Delta(s, s^2, \ldots, s^n) f^{(k)}(x, y_1, \ldots, y_r) = 0, \quad k = 1, \ldots, n,$$

as required. ■

Recall that a ring variety $\mathfrak{V}$ is said to be *pure* if the additive group of the $\mathfrak{V}$-free ring of countably infinite rank is torsion-free. Applying Lemma 4 to this ring immediately yields

**Corollary 5.** *If a pure variety $\mathfrak{V}$ satisfies the identity* (1)*, then $\mathfrak{V}$ also satisfies all the identities $f^{(k)}(x, y_1, \ldots, y_r) = 0$ where $k = 1, \ldots, n$.*

The *degree* of a monomial is merely its length as an element of the free groupoid over $\{x_1, x_2, \dots\}$. For a polynomial $h \in F\langle x_1, x_2, \dots\rangle$ we define $\max\deg(h)$ and $\min\deg(h)$ to be respectively the maximum and the minimum degree of the monomials of $h$ and call these numbers the *upper degree* and the *lower degree* of $h$. If $\max\deg(h) = \min\deg(h) = d$ we call $h$ *homogeneous of degree* $d$ and denote its degree by $\deg(h)$. Every polynomial $g$ can be written as a sum of homogeneous polynomials; if one chooses such a representation with the smallest number of summands, then the (uniquely determined) summands are called the *homogeneous components* of $g$. If a fully invariant ideal of $F\langle x_1, x_2, \dots\rangle$ happens to contain the homogeneous components of all its polynomials the corresponding ring variety is called *homogeneous*. Even though not all ring varieties are homogeneous, many important varieties are. For instance, from Corollary 5 it readily follows that every pure variety is homogeneous.

Recall that an identity $f = 0$ is said to be *multilinear* if each variable occurs in each monomial of the polynomial $f$ at most once.

**Lemma 6** [12, Corollary of Theorem 1.5]**.** *Every variety defined by multilinear identities is homogeneous.*

It should be clear that the notions of upper and lower degrees make sense for non-zero elements of the $\mathfrak{V}$-free ring of countably many generators in an arbitrary homogeneous variety $\mathfrak{V}$. For example, if such an element $h$ happens to be represented by a polynomial $f \in F\langle x_1, x_2, \dots\rangle$ with lower degree $d$, then $\min\deg(g) = d$ for every polynomial $g \in F\langle x_1, x_2, \dots\rangle$ representing $h$ so that we may (and will) call the number $d$ the *lower degree* of $h$ and denote it by $\min\deg(h)$.

## 2. Proof of Theorem 2

Given a positive integer $n$, we denote by $\mathfrak{B}_n$ the ring variety defined by the identity $nx = 0$. Our proof of Theorem 2 relies on a certain nice property of the mapping arising from multiplying ring varieties by $\mathfrak{B}_n$ on the left. Recall that the *Mal'tsev product* a variety $\mathfrak{V}$ with a variety $\mathfrak{W}$ is the class $\mathfrak{V}\mathfrak{W}$ of all rings $R$ possessing an ideal $I \in \mathfrak{V}$ such that the quotient ring $R/I$ belongs to $\mathfrak{W}$. It is shown in [5, Theorem 7] that $\mathfrak{V}\mathfrak{W}$ is again a variety which obviously contains both $\mathfrak{V}$ and $\mathfrak{W}$.

**Lemma 7** [7, Lemma 7]**.** *Let $\mathfrak{Y}$ be an arbitrary ring variety, $n$ a positive integer. The mapping defined by $\mathfrak{V} \mapsto \mathfrak{B}_n\mathfrak{V}$ where $\mathfrak{V}$ runs over $L(\mathfrak{Y})$ is an isomorphism of the lattice $L(\mathfrak{Y})$ onto the interval $[\mathfrak{B}_n, \mathfrak{B}_n\mathfrak{Y}]$.*

Lemma 7 implies the following result which basically ensures the induction step in the proof of Theorem 2.

**Lemma 8.** *Let $\mathfrak{Y}$ and $\mathfrak{Z}$ be ring varieties such that $\mathfrak{Z} \subseteq \mathfrak{B}_n\mathfrak{Y}$ for some $n$. If the lattice $L(\mathfrak{Y})$ is distributive, then so is the interval $[\mathfrak{Z}_n, \mathfrak{Z}]$.*

***Proof.*** By Lemma 7 the interval $[\mathfrak{B}_n, \mathfrak{B}_n\mathfrak{Y}]$ is distributive. Since both $\mathfrak{Z} \subseteq \mathfrak{B}_n\mathfrak{Y}$ and $\mathfrak{B}_n \subseteq \mathfrak{B}_n\mathfrak{Y}$, the join $\mathfrak{Z} \vee \mathfrak{B}_n$ of the varieties $\mathfrak{Z}$ and $\mathfrak{B}_n$ is also contained in the product $\mathfrak{B}_n\mathfrak{Y}$. Therefore the interval $[\mathfrak{B}_n, \mathfrak{Z} \vee \mathfrak{B}_n]$ is distributive too. By Lemma 3 the intervals $[\mathfrak{B}_n, \mathfrak{Z} \vee \mathfrak{B}_n]$ and $[\mathfrak{B}_n \cap \mathfrak{Z}, \mathfrak{Z}]$ are isomorphic. The latter interval is then distributive, but due to the evident equality $\mathfrak{Z}_n = \mathfrak{B}_n \cap \mathfrak{Z}$, this is precisely the interval we are interested in. ■

We shall also need another simple result from [7]:

**Lemma 9** [7, Corollary 3]**.** *For any positive integers $k$ and $\ell$,*

$$\mathfrak{B}_k\mathfrak{B}_\ell = \mathfrak{B}_{k\ell}.$$

Now we can prove Theorem 2. In view of Theorem 1, it suffices to verify that the lattice $L(\mathfrak{X}_n)$ is distributive for every positive integer $n$. We induct on $n$. The case $n = 1$ is obvious because $\mathfrak{X}_1$ is the trivial variety, and thus, $L(\mathfrak{X}_1)$ is the one-element lattice. Suppose that $n > 1$ and let $n = pm$ where $p$ is a prime number. Since $m < n$, the lattice $L(\mathfrak{X}_m)$ is distributive by the induction assumption. Calculating the Mal'tsev product

$$\begin{aligned}
\mathfrak{B}_p\mathfrak{X}_m =& \mathfrak{B}_p(\mathfrak{X} \cap \mathfrak{B}_m) \\
=& \mathfrak{B}_p\mathfrak{X} \cap \mathfrak{B}_p\mathfrak{B}_m && \text{by Lemma 7} \\
=& \mathfrak{B}_p\mathfrak{X} \cap \mathfrak{B}_{pm} && \text{by Lemma 9} \\
=& \mathfrak{B}_p\mathfrak{X} \cap \mathfrak{B}_n,
\end{aligned}$$

we see that it contains the variety $\mathfrak{X} \cap \mathfrak{B}_n = \mathfrak{X}_n$. Now we are in a position to apply Lemma 8 (with $\mathfrak{Y} = \mathfrak{X}_m$ and $\mathfrak{Z} = \mathfrak{X}_n$) which yields the conclusion that

the interval $[\mathfrak{X}_p, \mathfrak{X}_n]$ is distributive. On the other hand, the lattice $L(\mathfrak{X}_p)$ is given to be distributive. Since $\mathfrak{X}_p$ is a distributive element of the lattice $L(\mathfrak{X})$, it is also a distributive element in the lattice $L(\mathfrak{X}_n)$. Now by Lemmas 2 and 1 the lattice $L(\mathfrak{X}_n)$ embeds into the direct product $L(\mathfrak{X}_p) \times [\mathfrak{X}_p, \mathfrak{X}_n]$, and therefore, it is distributive as well. □

As already mentioned in the Introduction, for associative rings, the restriction of $\mathfrak{X}$ being torsion-bounded may be dropped for it automatically holds true under the conditions of Theorem 2.

**Corollary 10.** *Let $\mathfrak{X}$ be a variety of associative rings. Then the lattice $L(\mathfrak{X})$ is distributive provided that, for every prime number $p$, the lattice $L(\mathfrak{X}_p)$ is distributive and the variety $\mathfrak{X}_p$ is a distributive element of the lattice $L(\mathfrak{X})$.*

***Proof.*** In the proof of [9, Proposition 1] it is shown that, for every prime number $p$, the subvariety lattice of the variety $\mathfrak{C}_p$ of all associative and commutative rings of characteristic $p$ is non-distributive. Therefore, if the lattice $L(\mathfrak{X}_p)$ is distributive for each prime $p$, then $\mathfrak{X}$ contains none of the varieties $\mathfrak{C}_p$ and by the main result of [8], we conclude that $\mathfrak{X}$ satisfies an identity of the form

$$x^k = x^{k+\ell}$$

for some positive integers $k$ and $\ell$. By Corollary 5, a torsion-free ring satisfying this identity also satisfies $x^k = 0$. By a well-known result of the theory of associative rings (usually referred to as Nagata–Higman's theorem, see, e.g., [12, Section 6.1]), every torsion-free ring satisfying $x^k = 0$ is nilpotent. Thus, torsion-free rings in $\mathfrak{X}_p$ are nilpotent whence $\mathfrak{X}_p$ is a torsion-bounded variety by [7, Corollary 1]. Now Theorem 2 applies. ■

We notice that, in a similar manner, one can get rid of the condition of being torsion-bounded when restricting Theorem 2 to other important classes of rings. For example, using the Lie ring analogues of the cited results from [9] and [8] found in [1] and respectively [10] as well as Zelmanov's theorem [11] on nilpotency of torsion-free Lie rings satisfying the $k^{th}$ Engel condition, we may literally transfer Corollary 10 to varieties of locally almost solvable Lie rings, that is, Lie rings in which every finitely generated subring possesses a solvable ideal of a finite index.

## 3. An example

In this section the word "ring" is always assumed to mean "associative-commutative ring" and every identity system is assumed to contain the associative and the commutative laws. If $\Sigma$ is an identity system, then $\operatorname{var}\Sigma$ denotes the variety defined by $\Sigma$. With this notation, we recall that

$$\mathfrak{N}_{p^2} = \operatorname{var}\{p^2x = 0,\ pxy = 0,\ x_1x_2\cdots x_{p+1} = 0\}$$

where $p$ is a fixed prime number. Consider the following subvariety of the variety $\mathfrak{N}_{p^2}$:

$$\mathfrak{D}_p = \operatorname{var}\{px = 0,\ x_1x_2\cdots x_{p+1} = 0\}.$$

Observe that $\mathfrak{D}_p = \mathfrak{N}_{p^2} \cap \mathfrak{B}_p$ and for each prime number $q \neq p$, the variety $\mathfrak{N}_{p^2} \cap \mathfrak{B}_q$ is trivial. It is known (see [2]) that subvarieties of $\mathfrak{D}_p$ form a chain. Thus, the lattice $L(\mathfrak{N}_{p^2} \cap \mathfrak{B}_q)$ is distributive for all prime $q$. We aim to show that nevertheless the lattice $L(\mathfrak{N}_{p^2})$ is not distributive. Thus, in Theorem 2 the extra condition that the variety $\mathfrak{X}_p$ is a distributive element in the lattice $L(\mathfrak{X})$ is essential. Let

$$\mathfrak{A}_{p^2} = \operatorname{var}\{p^2x = 0,\ xy = 0\}.$$

As the first step towards our aim, we prove.

**Lemma 11.** $\mathfrak{N}_{p^2} = \mathfrak{A}_{p^2} \vee \mathfrak{D}_p$.

***Proof.*** Obviously, $\mathfrak{N}_{p^2} \supseteq \mathfrak{A}_{p^2} \vee \mathfrak{D}_p$. In order to prove the converse inclusion, is suffices to show that the $\mathfrak{N}_{p^2}$-free ring $F$ with countably many generators belongs to the join $\mathfrak{A}_{p^2} \vee \mathfrak{D}_p$. Consider the ideal $pF = \{pf \mid f \in F\}$ of $F$; in view of the identities $pxy = 0$ and $p^2x = 0$ every element $pf \in pF$ can be represented as

$$pf = p \cdot \sum \alpha_i x_i$$

for some non-negative integers $\alpha_i < p$. Now observe that the variety $\mathfrak{N}_{p^2}$ is defined by multilinear identities. Therefore, it is homogeneous by Lemma 6, and so we can speak of degree (or lower degree) of the elements of $F$. Thus, $\deg(pf) = 1$ for every non-zero element $pf \in pF$. On the other hand, every non-zero element of the ideal $F^2$ generated by all products $gh$ with $g, h \in F$ has lower degree at least 2.

Hence $pF \cap F^2 = 0$ and the ring $F$ is a subdirect product of the quotient rings $F/pF$ and $F/F^2$. It is easy to see that $F/F^2 \in \mathfrak{A}_{p^2}$ while $F/pF \in \mathfrak{D}_p$ whence $F$ belongs to $\mathfrak{A}_{p^2} \vee \mathfrak{D}_p$ as required. ■

We fix an integer $\alpha$ such that $1 \leq \alpha < p$ and consider the variety

$$\mathfrak{F}^{\alpha}_{p^2} = \mathrm{var}\{p^2x = 0,\ x_1x_2\cdots x_{p+1} = 0,\ x^p = \alpha px\}.$$

It is easy to see that $\mathfrak{F}^{\alpha}_{p^2} \subseteq \mathfrak{N}_{p^2}$: if one multiplies the identity

$$x^p = \alpha px \tag{4}$$

by $y$, one gets the identity $\alpha pxy = 0$ which, being combined with $p^2x = 0$, implies $pxy = 0$. On the other hand, we have

**Lemma 12.** $\mathfrak{F}^{\alpha}_{p^2} \nsubseteq \mathfrak{D}_p$.

***Proof.*** It suffices to prove that the identity $px = 0$ fails in $\mathfrak{F}^{\alpha}_{p^2}$. Consider two one-generator rings:

$$A_{p^2} = \langle a \mid a^2 = p^2a = 0\rangle\,,$$
$$D_p = \langle d \mid d^{p+1} = pd = 0\rangle\,.$$

(It is clear that the ring $A_{p^2}$ generates the variety $\mathfrak{A}_{p^2}$, and it can be shown that the ring $D_p$ generates the variety $\mathfrak{D}_p$ but we do not need this for our proof.) Let $C$ be the subring of the direct product $A_{p^2} \times D_p$ generated by the pair $c = (a, d)$. Observe that $(pa, 0), (0, d^p) \in C$ since $(pa, 0) = pc$ and $(0, d^p) = c^p$. Now consider the additive subgroup $B$ of $C$ generated by the element $b = (\alpha pa, -d^p) = \alpha pc - c^p$. In view of the relations $a^2 = 0$ and $d^{p+1} = 0$, one has $bc = 0$ whence $B$ annihilates $C$, and thus it is an ideal of $C$. Now let $F$ denote the quotient ring $C/B$. The ring $F$ satisfies the identities $p^2x = 0$ and $x_1x_2\cdots x_{p+1} = 0$ because they hold in both $A$ and $D$; moreover, it can be easily checked that $F$ also satisfies the identity (4). Thus, the ring $F$ belongs to the variety $\mathfrak{F}^{\alpha}_{p^2}$. On the other hand, $F$ does not satisfy the identity $px = 0$ because the element $pc = (pa, 0)$ does not lie in the ideal $B$. Hence the ring $F$ does not belong to the variety $\mathfrak{D}_p$. ■

**Proposition 13.** *The lattice $L(\mathfrak{N}_{p^2})$ is not distributive.*

***Proof.*** From Lemma 11 we have

$$\mathfrak{F}_{p^2}^{\alpha} = \mathfrak{F}_{p^2}^{\alpha} \cap \mathfrak{N}_{p^2} = \mathfrak{F}_{p^2}^{\alpha} \cap (\mathfrak{A}_{p^2} \vee \mathfrak{D}_p).$$

On the other hand, the identities (4) and $xy = 0$ together imply the identity $\alpha px = 0$ which, being combined with $p^2x = 0$, implies $px = 0$. Therefore the intersection $\mathfrak{F}_{p^2}^{\alpha} \cap \mathfrak{A}_{p^2}$ satisfies $px = 0$ and hence is contained in the variety $\mathfrak{D}_p$. We conclude that

$$(\mathfrak{F}_{p^2}^{\alpha} \cap \mathfrak{A}_{p^2}) \vee (\mathfrak{F}_{p^2}^{\alpha} \cap \mathfrak{D}_p) \subseteq \mathfrak{D}_p.$$

Now Lemma 12 shows that

$$\mathfrak{F}_{p^2}^{\alpha} \cap (\mathfrak{A}_{p^2} \vee \mathfrak{D}_p) \neq (\mathfrak{F}_{p^2}^{\alpha} \cap \mathfrak{A}_{p^2}) \vee (\mathfrak{F}_{p^2}^{\alpha} \cap \mathfrak{D}_p).$$

Thus, the distributive law fails. ■

In fact, it is not difficult to give a complete description of the lattice $L(\mathfrak{N}_{p^2})$.

The lattice is shown in Figure 1 below.

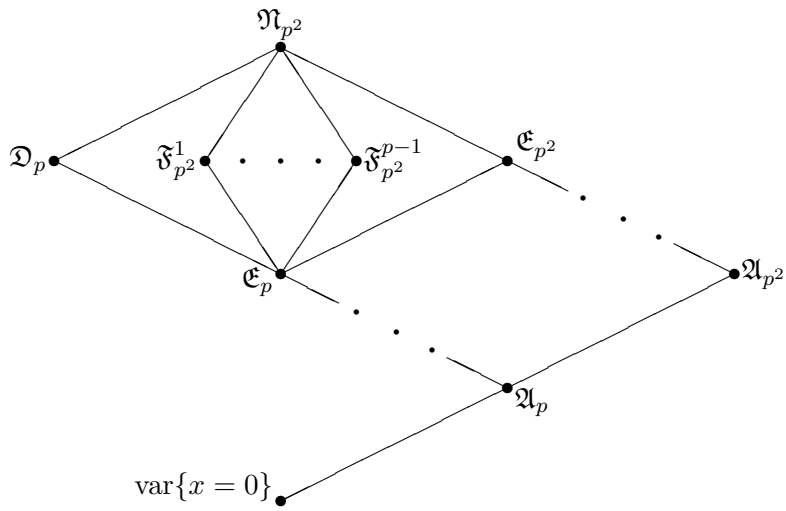


Figure 1. The lattice $L(\mathfrak{N}_{p^2})$.

In addition to the varieties already introduced, Figure 1 refers to the following varieties:

$$\begin{aligned}\mathfrak{E}_{p^2} &= \mathrm{var}\{p^2x = 0,\ pxy = 0,\ x_1x_2\cdots x_{p+1} = 0,\ x^p = 0\},\\ \mathfrak{E}_p &= \mathrm{var}\{px = 0,\ x_1x_2\cdots x_{p+1} = 0,\ x^p = 0\},\\ \mathfrak{A}_p &= \mathrm{var}\{px = 0,\ xy = 0\}.\end{aligned}$$

Each of the two intervals $[\mathfrak{A}_p, \mathfrak{E}_p]$ and $[\mathfrak{A}_{p^2}, \mathfrak{E}_{p^2}]$ is a chain of $p$ varieties that are defined within $\mathfrak{E}_p$ (respectively $\mathfrak{E}_{p^2}$) by the identities $x_1\cdots x_k = 0$ where $k = 2, \ldots, p+1$. Altogether, $L(\mathfrak{N}_{p^2})$ consists of $3p+2$ varieties.

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