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GREEN'S RELATIONS AND THEIR GENERALIZATIONS ON SEMIGROUPS

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Dedicated to the 65th birthday of Klaus Denecke

Abstract

Green's relations and their generalizations on semigroups are useful in studying regular semigroups and their generalizations. In this paper, we first give a brief survey of this topic. We then give some examples to illustrate some special properties of generalized Green's relations which are related to completely regular semigroups and abundant semigroups.

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1. Introduction

In the theory of semigroups, regular semigroups are described as the core semigroups because groups are regular semigroups with a unique idempotent. Let S be a semigroup and $a \in S$. Denote the set of all idempotents of S by $E(S)$. Recall that an element $a \in S$ is *regular* if there exists an element $x \in S$ such that $axa = a$ and $xax = x$. In this case, x is said to be an *inverse* of a and we denote the set of all inverses of a by $V(a)$. A semigroup S is said to be *regular* if every element of S is regular. A regular semigroup S is called *completely regular* if S is a union of some of its subgroups [24]. It is well known that Green's relations are very useful in studying regular semigroups. In particular, Green's relations can be used to describe the structure of regular semigroups (for example, see Theorem 2.1 in this paper). Because of this reason, the most effective method to study the generalized regular semigroups is to modify the Green's relations defined on a given semigroup. In this paper, we shall concentrate on the generalizations of Green's relations on a given semigroup and consider their applications.

The concept of rpp semigroups was first inspired by rings. Recall that a ring R is said to be *rpp* if every principal left ideal of R, regarded as a right R-module, is projective. Similarly, we can define *lpp*-rings. The concepts of *rpp*-rings and *lpp*-rings were first introduced around 1960. Naturally, a ring R is said to be a *pp*-ring if R is both *lpp* and *rpp*. It is well known that the class of *pp*-rings contains the classes of regular rings; hereditary rings; Baer rings; p-q Baer rings and semi-hereditary rings as its proper subclasses. In the literature, *pp*-rings have also been extensively studied by many ring theorists. It is noteworthy that the definition of *pp*-rings can be extended to semigroups, in particular, Fountain [5] has introduced the concept of *rpp*-monoids in 1977. He called a monoid *abundant* [6] if it is both an *lpp*- and *rpp*- monoid. Similar to *pp*-rings, the class of abundant semigroups contains the class of regular semigroups as its proper subclass. The *rpp*-semigroup in which every idempotent is central is called a C-*rpp* semigroup. It was first proved by Fountain [5] that the C-*rpp* monoid can be expressed as a strong semilattice of left cancellative monoids. On the other hand, if S is a completely regular semigroup whose idempotents are central, then it was proved by Clifford that such a semigroup can be expressed as a strong semilattice of groups. Thus, by comparing the result of Fountain [5] with the above well known theorem of Clifford, we can immediately see that a *C-rpp*-semigroup is indeed a generalization of Clifford semigroup. Since a C-rpp semigroups, by definition, need not to be regular, we can regard C-rpp semigroups as a generalization of regular semigroups within the class of rpp-semigroups, from a different point of view which is not just weakening the regularity of the semigroups. Perhaps we need to mention here that the theorem of Fountain on C-rpp monoids has also been recently restricted to perfect abundant semigroups in [11]. It has been proved by Guo-Shum in [11] that a perfect abundant semigroup can be expressed as a strong semilattice of cancellative planks. The structure of left C-rpp semigroups was also considered in [32].

Fountain [5] and [6] first observed that the Green's*-relations can be applied to study rpp-semigroups and abundant semigroups, in particular, super-abundant semigroups. A series of papers have indicated that the Green's*-relations are particularly appropriate in studying abundant semigroups which play exactly the same role as the usual Green's relations in regular semigroups. For super-abundant semigroups, Kong-Shum and Ren-Shum have recently obtained some new results in [18] and [26], respectively.

Within the class of rpp-semigroups, one would naturally ask what kind of semigroups would be similar to the completely regular semigroups (orthogroups) in the class of regular semigroups ? And also, what kind of semigroups would be similar to the super-abundant semigroups (ortho-c-monoids) in the class of abundant semigroups ? For these questions, X.J. Guo, Y.Q. Guo, K.P. Shum, L. Du *et al.* have given some answers (see [9, 10, 12, 13] and [15]), from strong rpp semigroups to super-ample-semigroups, by using a new set of Green-like relations to replace the (l)-Green's relations which is called the $(*, \sim)$-Green's relations. This set of $(*, \sim)$-Green's relations is in fact a generalization of the set of Green's*-relations. An introduction of these generalizations of Green's relations has been given in [14].

In this paper, in Section 2 and Section 3, we state some results related to regular semigroups and rpp-semigroups (abundant semigroups). In particular, we indicate the Green's relations and generalized Green's relations which are useful in studying such kind of semigroups. In Section 4, we define a new set of generalized Green's relations, namely, the Green's $\sharp$-relations, and obtain some elementary results. In Section 5, we explore the relationships between the Green's relations and some generalized Green's relations. In the last Section of this paper, we mention some major differences between regular semigroups and abundant semigroups, in particular, some crucial examples are given.

For notations and terminologies not given in this paper, the reader is referred to the texts of Clifford-Preston [1, 2] and Howie [16]. For information of regular semigroups and abundant semigroups, the reader is referred to Shum-Guo [29] and Shum-Ren [30].

2. Green's relations and regular semigroups

We first denote the set of all transformations of the semigroup S by $\mathcal{T}(S)$. For any $f \in \mathcal{T}(S)$, we denote the image of f by $\mathcal{I}mf$ and the kernel of f by $Kerf$, respectively. Thus, by definition,we have

$$\mathcal{I}mf = \{f(x)|x \in S\}, \quad Kerf = \{(x,y) \in S \times S|f(x) = f(y)\},$$

respectively.

Now, for any $a \in S$, let $a_r(a_l) \in \mathcal{T}(S^1)$ be the inner right [left] translation on S^1 determined by a:

$$a_r : x \longmapsto xa \quad (a_l : x \longmapsto ax).$$

Then, the usual Green's relations can be defined by :

$$a\mathcal{L}b \Longleftrightarrow Im\, a_r = Im\, b_r,$$

$$a\mathcal{R}b \Longleftrightarrow Im\, a_l = Im\, b_l,$$

$$\mathcal{H}{=}\mathcal{L} \wedge \mathcal{R}{=}\mathcal{L} \cap \mathcal{R},$$

$$\mathcal{D}{=}\mathcal{L} \vee \mathcal{R}{=}\mathcal{L} \circ \mathcal{R},$$

$$a\mathcal{J}b \Longleftrightarrow S^1aS^1 = S^1bS^1.$$

Then, the regular semigroups can be characterized by the usual Green's relations as given in the following theorem:

Theorem 2.1 [16]**.** *Let S be a semigroup. Then the following statements are equivalent:*

(1) *S is regular;*

(2) *every $\mathcal{L}$-class of S contains at least one idempotent;*

(3) *every $\mathcal{R}$-class of S contains at least one idempotent;*

(4) *every $\mathcal{D}$-class of S contains at least one idempotent.*

In the theory of semigroups, it is well known that the idempotents of a semigroup play a crucial role and even more important than the idempotents in a ring. In particular, we notice that each $\mathcal{H}$-class of a semigroup S contains at most one idempotent of S. In fact, we have the following result:

Theorem 2.2 [16]**.** *Let S be a semigroup. Then S is completely regular if and only if each $\mathcal{H}$-class of S contains exactly one idempotent.*

Recall that a regular semigroup S is called a *left inverse* [*right inverse, quasi-inverse*] semigroup if $E(S)$ is a *right regular* [*left regular, regular*] band (that is, the identity $efe = fe$ [$efe = ef, efege = efge$] holds for every e, f, $g \in E(S)$). A regular semigroup S is called *inverse* if S is both left inverse and right inverse. An inverse semigroup S is called a *Clifford semigroup* if $E(S)$ lies in the center of S. The following result gives a characterization for the above semigroups by using the usual Green's relations.

Theorem 2.3. *Let S be a semigroup. Then S is a left inverse [right inverse, Clifford] semigroup if and only if each $\mathcal{L}$-class [$\mathcal{R}$-class, $\mathcal{D}$-class] of S contains a unique idempotent.*

By combining the properties of congruences, we can easily obtain the following proposition:

Proposition 2.4. *Let S be a regular semigroup. If $\mathcal{L}$ [$\mathcal{R}$] is a congruence on S, then S is completely regular. But the converse may be not true.*

From the above results, we can see that the Green's relations are indeed quite useful in studying the structure and in classifying the regular semigroups.

3. GREEN'S *-RELATIONS AND ABUNDANT SEMIGROUPS

In studying abundant semigroups, Fountain adopted the Green's *-relations on semigroups first initiated by F. Pastijn in [22].

Before giving the definition of Green's *-relations, we first introduce the notion of *saturated relations.* Let δ be an equivalent relation on a given set X. If $Y \subseteq X$, then δ is said to be a saturated relation on Y whenever Y is a union of some δ-equivalent classes (see [24]).

Now, we define

$$a\mathcal{L}^*b \Longleftrightarrow Ker\, a_l = Ker\, b_l$$

$$a\mathcal{R}^*b \Longleftrightarrow Ker\, a_r = Ker\, b_r$$

$$\mathcal{H}^* = \mathcal{L}^* \wedge \mathcal{R}^* = \mathcal{L}^* \cap \mathcal{R}^*,$$

$$\mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^* \neq \mathcal{L}^* \circ \mathcal{R}^* (\text{in general}),$$

$$a\mathcal{J}^*b \Longleftrightarrow J^*(a) = J^*(b),$$

where $J^*(a)$ is the smallest ideal generated by a which is saturated by $\mathcal{L}^*$ and $\mathcal{R}^*$. Obviously, $\mathcal{L}^*$ [$\mathcal{R}^*$] is a right [left] congruence on S.

It is also clear that $\mathcal{L} \leq \mathcal{L}^*$ [$\mathcal{R} \leq \mathcal{R}^*$] and for $a, b \in RegS$, the set of all regular elements of S, $a\mathcal{L}^*b$ [$a\mathcal{R}^*b$] if and only if $a\mathcal{L}b$ [$a\mathcal{R}b$]. In fact, the class of rpp [abundant] semigroups is actually a larger class of semigroups than the class of abundant [regular] semigroups, for example, a left cancellative [cancellative] monoid is clearly rpp [abundant] but it is not necessarily abundant [regular].

It is noteworthy that the Green's *-relations on a semigroup S are the usual Green's relations on some oversemigroup of the given semigroup S. Thus, in this connection, every $\mathcal{H}^*$-class of S also contains at least one idempotent [6]. Obviously, the Green's *-relations are analogous to the usual Green's relations on the generalized regular semigroup S and hence, they are the most appropriate relations to study the structure of generalized regular semigroups.

We observe that the Green's *-relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on a semigroups S can also be explicitly defined as follows [5]:

For any two elements $a, b \in S$, $a\mathcal{L}^*b$ $[a\mathcal{R}^*b]$ if and only if

$$(\forall x, y \in S^1) \quad ax = ay \ [xa = ya] \Longleftrightarrow bx = by \ [xb = yb].$$

Recall that a semigroup S is rpp [lpp, abundant] if and only if each $\mathcal{L}^*$-class [$\mathcal{R}^*$-class, $\mathcal{L}^*$-class and $\mathcal{R}^*$-class] of S contains idempotents, and a semigroup S is said to be *superabundant* [6] if each $\mathcal{H}^*$-class of S contains one unique idempotent. Obviously, superabundant semigroups play a similar role in the class of abundant semigroups as the case of completely regular semigroups in the class of regular semigroups.

We first let $L^*_a(S)$ and $R^*_a(S)$ be the respectively $\mathcal{L}^*$-class and $\mathcal{R}^*$-class containing $a \in S$. Also, we let a^+ and a^* be the idempotents in $R^*_a(S)$ and $L^*_a(S)$, respectively. For the set E of idempotents of S, we let $B = \langle E \rangle$ be the semiband generated by E, which is a regular semigroup whose elements are products of finitely many idempotents (see Howie [16]). Now, for any $e \in E$, let $\langle e \rangle$ be the subsemigroup generated by all idempotent in eBe. Then, it is clear that $\langle e \rangle$ is generated by all idempotents f with $f \leq e$.

The following important condition on abundant semigroups, namely, the *idempotent connected condition*, in brevity, the *IC condition*, was first introduced by El-Qallali and Fountain [3] in 1981. They called an abundant semigroup S an *idempotent-connected semigroup*(*in brevity IC semigroup*) if for every $a \in S$, with its corresponding idempotents $a^+ \in R^*_a(S)$ and $a^* \in L^*_a(S)$, respectively, there exists a bijection mapping $\alpha : \langle a^+ \rangle \to \langle a^* \rangle$ satisfying the condition $xa = a(x\alpha)$, where $x \in \langle a^+ \rangle$.

Recall that an abundant semigroup S is *adequate* [*quasi-adequate*] if the set of idempotents of S forms a semilattice [band] with respect to the multiplication of S. In particular, an adequate [*quasi-adequate*] semigroup satisfying the IC condition is called a type A semigroup [type W semigroup] (see El-Qallali-Fountain [3, 4], Guo-Guo-Shum [10] and Shum-Guo [29]). One can easily see that a type A semigroup [type W semigroup] is precisely an analogue of an inverse semigroup [orthodox semigroup] in the class of abundant semigroups. Observe that the regular [inverse, orthodox] semigroups are IC abundant [type A, type W] semigroup. This is because for any $a \in S$ and any $a' \in V(a)$, if we define a mapping $\alpha : \langle aa' \rangle \to \langle a'a \rangle$ such that $x\alpha = a'xa$ for $x \in \langle aa' \rangle$, then we can easily observe that such mapping α is the required mapping that makes the regular semigroup S an IC semigroup.

As inspired by Yamada [33] and El-Qallali-Fountain [3], Ren-Shum [27] have recently introduced the concepts of $\mathcal{L}^*$-*inverse semigroups* and *quasi**-*inverse semigroups*. An IC abundant semigroup is said to be an

$\mathcal{L}^*$*-inverse semigroup* if its idempotents form a left regular band. In fact, Ren-Shum [27] have constructed an $\mathcal{L}^*$-inverse semigroup by using a special left wreath product of semigroups, namely the so called left cohert product of semigroups. They have proved that a semigroup S is an $\mathcal{L}^*$-inverse semigroup if and only if S is a left cohort product of a left regular band B and a type A semigroup Γ. We provide below an example of an $\mathcal{L}^*$-inverse semigroup and show that such a semigroup contains a type A semigroup and a left inverse semigroup as its major components. On the other hand, this example also illustrates that the Green's *-relations are particularly useful in constructing such special kind of abundant semigroups with some specific properties.

Example 3.1. Let $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. Put

$$T = \{2^n a,\ 2^n b \quad | \quad n \geqslant 0\}.$$

Then, it is easy to see that T is a semigroup under the usual matrix multiplication. Now, we form a semigroup $S = \{a, b, c, d, e, f, g, h, u, v, w, x, y, z, a_n, b_m\}$ by the following Cayley table:

$*$	a	b	c	d	e	f	g	h	u	v	w	x	y	z	a_n	b_j
a	a	a	d	d	f	f	h	h	v	v	x	x	z	z	a_n	a_j
b	b	b	d	d	f	f	h	h	v	v	x	x	z	z	b_n	b_j
c	c	c	c	c	e	e	g	g	u	u	w	w	y	y	c	c
d	d	d	d	d	f	f	h	h	v	v	x	x	z	z	d	d
e	e	e	e	e	c	c	w	w	y	y	g	g	u	u	e	e
f	f	f	f	f	d	d	x	x	z	z	h	h	v	v	f	f
g	g	g	g	g	y	y	c	c	w	w	u	u	e	e	g	g
h	h	h	h	h	z	z	d	d	x	x	v	v	f	f	h	h
u	u	u	u	u	w	w	y	y	c	c	e	e	g	g	u	u
v	v	v	v	v	x	x	z	z	d	d	f	f	h	h	v	v
w	w	w	w	w	u	u	e	e	g	g	y	y	c	c	w	w
x	x	x	x	x	v	v	f	f	h	h	z	z	d	d	x	x
y	y	y	y	y	g	g	u	u	e	e	c	c	w	w	y	y
z	z	z	z	z	h	h	v	v	f	f	d	d	x	x	z	z
a_m	a_m	a_m	d	d	f	f	h	h	v	v	x	x	z	z	a_{m+n}	a_{m+j}
b_i	b_i	b_i	d	d	f	f	h	h	v	v	x	x	z	z	b_{i+n}	b_{i+j}

where $a_n = 2^n a$, $a_m = 2^m a$, $b_i = 2^i b$ and $b_j = 2^j b$ for any $n, m, i, j \geqslant 1$.

In fact, the multiplication on S is defined by extending the multiplication on the matrix semigroup T so that S becomes an infinite semigroup, where $E(S) = \{a, b, c, d\}$ is the set of all idempotents of S. We can check that the $\mathcal{L}^*$-classes of S are $\{a, b, a_n, b_i \mid n, i \geqslant 1\}, \{c, d, e, f, g, h, u, v, w, x, y, z\}$ and the $\mathcal{R}^*$-classes of S are $\{a, a_n \mid n \geqslant 1\}, \{b, b_i \mid i \geqslant 1\}$, $\{c, e, g, u, w, y\}$ and $\{d, f, h, v, x, z\}$, respectively. Thus each $\mathcal{R}^*$-class and each $\mathcal{L}^*$-class of S contains an idempotent and hence by definition, S is an abundant semigroup.

Furthermore, we can easily observe that $E(S)$ forms a left regular band and

$$S_1 = \{a, b, c, d, e, f, g, h, u, v, w, x, y, z\}$$

is a left inverse subsemigroup of S. Since $\langle a \rangle = \{a, d\}, \langle b \rangle = \{b, d\}, \langle c \rangle = c$ and $\langle d \rangle = d$ in S, where $\langle a \rangle$ denotes the subsemigroup of S generated by all $x \in E(S)$ with $x \leqslant a$, it can be also easily checked that S is an IC abundant semigroup and by definition, S is an $\mathcal{L}^*$-inverse semigroup. Because every element of $S \setminus S_1$ is non-regular, S is not a left inverse semigroup. Also, since every idempotent of S does not commute with each other, S is not a type A semigroup.

Another interesting subclass of type W semigroups is the class of $\mathcal{Q}^*$-*inverse semigroups*. A $\mathcal{Q}^*$-inverse semigroup is an IC abundant semigroup whose idempotents form a regular band. It is easy to see that the $\mathcal{L}^*$-inverse semigroup [quasi*-inverse semigroups] is a special kind of abundant semigroups sitting between the type A semigroups [$\mathcal{L}^*$-inverse semigroups] and the quasi*-inverse semigroups [type W semigroups]. Thus, we can regard the $\mathcal{L}^*$-inverse [quasi*-inverse] semigroups as an analogue of the left inverse [quasi-inverse] semigroups within the class of abundant semigroups. In a recent paper, Ren-Shum has proved that a semigroup S is a $\mathcal{Q}^*$-inverse semigroup if and only if S is a spined product of an $\mathcal{L}^*$-inverse semigroup and an $\mathcal{R}^*$-inverse semigroup [28]. Thus, $\mathcal{L}^*$-inverse semigroup is indeed a special quasi*-inverse semigroup.

By the discussion in Section 2, one can easily see that the Clifford semigroups, inverse semigroups, left inverse semigroups, quasi-inverse semigroups, orthodox semigroups and so on form a semigroup hierarchy within the class of regular semigroups. Their corresponding analogues in the class of abundant semigroups are therefore the C-abundant semigroups, type A semigroups, $\mathcal{L}^*$-inverse semigroups, quasi*-inverse semigroups and type W semigroups, respectively. Thus this set of special abundant semigroups in the class of abundant semigroups also forms a corresponding regular semigroup hierarchy in the class of abundant semigroups. Some generalizations of the above semigroups were described in [30] and [32].

4. Green's $\sharp$-relations and $\mathcal{H}^\sharp$-abundant semigroups

In this section, we generalize the usual Green's relations to the following Green's $^\sharp$-relations defined by:

$$\mathcal{L}^\sharp = \{(a,b) \in S \times S : (\forall x, y \in S^1)(ax, ay) \in \mathcal{L} \Leftrightarrow (bx, by) \in \mathcal{L}\},$$

$$\mathcal{R}^\sharp = \{(a,b) \in S \times S : (\forall x, y \in S^1)(xa, ya) \in \mathcal{R} \Leftrightarrow (xb, yb) \in \mathcal{R}\},$$

$$\mathcal{H}^\sharp = \mathcal{L}^\sharp \cap \mathcal{R}^\sharp, \ \mathcal{D}^\sharp = \mathcal{L}^\sharp \vee \mathcal{R}^\sharp.$$

We observe that $\mathcal{L}^\sharp$ and $\mathcal{R}^\sharp$ are equivalences on the semigroup S. We denote the $\mathcal{L}^\sharp$-class containing an element a of S by $L^\sharp_a(S)$ or simply by $L^\sharp_a$ if no ambiguity arises.

The following lemma is a new result.

Proposition 4.1. *Let S be a semigroup and $a, b \in S$. Then $\mathcal{L}^\sharp$ is a right congruence and $\mathcal{L} \subseteq \mathcal{L}^\sharp$. In particular, if $a, b \in S$ and a, b are regular, then $a\mathcal{L}b$ if and only if $a\mathcal{L}^\sharp b$.*

Proof. Let $(a,b) \in \mathcal{L}^\sharp$ and $c \in S$. If $((ac)x, (ac)y) \in \mathcal{L}$ for $x, y \in S^1$, that is, $(a(cx), a(cy)) \in \mathcal{L}$, then $(b(cx), b(cy)) \in \mathcal{L}$. Because $(a,b) \in \mathcal{L}^\sharp$, $((bc)x, (bc)y) \in \mathcal{L}$. Similarly, if $((bc)x, (bc)y) \in \mathcal{L}$, then $((ac)x, (ac)y) \in \mathcal{L}$, and hence $(ac, bc) \in \mathcal{L}^\sharp$ so that $\mathcal{L}^\sharp$ is right compatible with the semigroup multiplication. Assume that $(a,b) \in \mathcal{L}$, that is, $S^1a = S^1b$ and $(ax, ay) \in \mathcal{L}$ for some $x, y \in S^1$. Then $S^1ax = S^1ay$. In this case, we derive that $S^1bx = S^1by$ and thereby $(bx, by) \in \mathcal{L}$. Dually, $(bx, by) \in \mathcal{L}$ implies $(ax, ay) \in \mathcal{L}$ for all $x, y \in S$. Consequently, $\mathcal{L} \subseteq \mathcal{L}^\sharp$.

If a and b are both regular elements of S and $a\mathcal{L}^\sharp b$, then there exist idempotents e, f in S such that $e\mathcal{L}a\mathcal{L}^\sharp b\mathcal{L}f$. From $(ee, e1) \in \mathcal{L}$, we have $(fe, f) \in \mathcal{L}$ which implies that $f = xfe$ for some $x \in S^1$. Hence, $fe = f$. Similarly, $ef = e$. Thus $e\mathcal{L}f$ and so $a\mathcal{L}b$. ∎

If S is a regular semigroup, then it is clear that every $\mathcal{H}$-class of S contains at most one idempotent. Similarly, it can be easily seen that there is at most one idempotent contained in each $\mathcal{H}^\sharp$-class. If $e \in \mathcal{H}^\sharp_a \cap E(S)$ for some $a \in S$, then we just write e as x^0, for any $x \in \mathcal{H}^\sharp_a$. By Theorem 2.2, if S is a completely regular semigroup, then every $\mathcal{H}$-class of S contains at least one idempotent of S. Following the terminology of Fountain in [6], we naturally call a semigroup S an *$\mathcal{H}^\sharp$-abundant semigroup* if each $\mathcal{H}^\sharp$-class of S contains at least one idempotent of S.

In an $\mathcal{H}^\sharp$-abundant semigroup S, it is clear that for any $x \in \mathcal{H}^\sharp_a$ with $a \in S$, we have $x = xx^0 = x^0x$. Thus, the $\mathcal{H}^\sharp$-abundant semigroups are the generalizations of completely regular semigroups and completely regular semigroups have been thoughtfully investigated by Petrich-Reilly in [24]. Also, an $\mathcal{H}^\sharp$-abundant semigroup S is called an $\mathcal{H}^\sharp$-cryptogroup if $S/\mathcal{H}^\sharp$ is a band. Cryptogroups have been recently studied by a number of authors in [17]–[21].

We now give a theorem on $\mathcal{H}^\sharp$-abundant semigroup.

Theorem 4.2. *Let S be an $\mathcal{H}^\sharp$-abundant semigroup. Then S is an $\mathcal{H}^\sharp$-cryptic group (that is, $\mathcal{H}^\sharp$ is a congruence on S) if and only if for any $a, b \in S$, $(ab)^0 = (a^0b^0)^0$.*

Proof. (*Necessity*). Let $a, b \in S$. Then $a\mathcal{H}^\sharp a^0$ and $b\mathcal{H}^\sharp b^0$. Since S is an $\mathcal{H}^\sharp$-cryptic group, $\mathcal{H}^\sharp$ is a congruence on S and so $ab\mathcal{H}^\sharp a^0b^0$. Clearly, $ab\mathcal{H}^\sharp(ab)^0$ and hence $(ab)^0 = (a^0b^0)^0$ because every $\mathcal{H}^\sharp$-class contains a unique idempotent of S.

(*Sufficiency*). In order to prove that S is a cryptic group, we only need to prove that $\mathcal{H}^\sharp$ is a congruence on S, that is, to prove that $\mathcal{H}^\sharp$ is compatible with semigroup multiplication. Let $(a, b) \in \mathcal{H}^\sharp$ and $c \in S$. Then $(ca)^0 = (c^0a^0)^0 = (c^0b^0)^0 = (cb)^0$ and whence, $\mathcal{H}^\sharp$ is left compatible with semigroup multiplication. Dually, $\mathcal{H}^\sharp$ is also right compatible with semigroup multiplication. Hence, $\mathcal{H}^\sharp$ is a congruence on S. ■

The proofs of the following results are similar to the corresponding results given in [7, 10] and [16]. We hence omit the proofs.

Proposition 4.3. *If e, f are $\mathcal{D}^\sharp$-related idempotents of an $\mathcal{H}^\sharp$-abundant semigroup S, then $e\mathcal{D}f$.*

Proposition 4.4. *If S is an $\mathcal{H}^\sharp$-abundant semigroup, then $\mathcal{D}^\sharp = \mathcal{L}^\sharp \circ \mathcal{R}^\sharp = \mathcal{R}^\sharp \circ \mathcal{L}^\sharp$.*

Proposition 4.5. *Let e, f be idempotents in an $\mathcal{H}^\sharp$-abundant semigroup S and $e\mathcal{J}f$. Then $e\mathcal{D}f$.*

5. The relationship between Green's relations and Green's (l)-relations

In order to amend the defect $\mathcal{L}^* \vee \mathcal{R}^* \neq \mathcal{L}^* \circ \mathcal{R}^*$ (in general) in [23], Pastijn modified the Green's *-relations $\mathcal{D}^*$, $\mathcal{L}^*$, $\mathcal{R}^*$ and $\mathcal{H}^*$ to the relations $\mathcal{D}^{(l)}$, $\mathcal{L}^{(l)}$, $\mathcal{R}^{(l)}$ and $\mathcal{H}^{(l)}$, and he had done some of the initial work on this kind of Green-like relations. We can verify that these relations together with $\mathcal{J}^{(l)}$ form a new set of Green-like relations on the semigroup S. In fact, this set of Green-like relations lies in between the set of the usual Green's relations and the set of the Green's *-relations on S and we call them the Green's (l)-relations. These Green's (l)-relations are now displayed below:

$$\mathcal{L}^{(l)} = \mathcal{L}^*,$$

$$\mathcal{R}^{(l)} = \mathcal{R},$$

$$\mathcal{H}^{(l)}=\mathcal{L}^{(l)} \wedge \mathcal{R}^{(l)} = \mathcal{L}^{(l)} \cap \mathcal{R}^{(l)},$$

$$\mathcal{D}^{(l)} = \mathcal{L}^{(l)} \vee \mathcal{R}^{(l)},$$

$$a\mathcal{J}^{(l)}b \Longleftrightarrow J^{(l)}(a) = J^{(l)}(b),$$

where $J^{(l)}(a)$ is the smallest ideal containing a which is saturated by $\mathcal{L}^{(l)}$. In fact, the relation $\mathcal{D}^{(l)}$ refines the relation $\mathcal{J}^{(l)}$ since $J^{(l)}(a)$ is also saturated by $\mathcal{R}$.

The following theorem given in [15] describes the relationships between the Green's relations with the Green's *-relations and the Green's (l)-relations.

Theorem 5.1 (see [15])**.**

(i) $\mathcal{L} \lneq \mathcal{L}^{(l)}$, $\mathcal{D} \lneq \mathcal{D}^{(l)}$, $\mathcal{J} \lneq \mathcal{J}^{(l)}$;

(ii) $\mathcal{R}^{(l)} \lneq \mathcal{R}^*$, $\mathcal{D}^{(l)} \lneq \mathcal{D}^*$, $\mathcal{J}^{(l)} \lneq \mathcal{J}^*$.

Proof. All the relations "$\leq$" in the above theorem can be easily verified since they are direct consequences of definitions (see [15]). We now give some examples to illustrate that the above relations "$\leq$" are indeed strict.

Example (i): Let S be a left simple and left cancellative semigroup without idempotent (for example, the dual of Baer-Levi semigroup is such a semigroup ([2] 8.1,-Vol.II)). Then, in this semigroup, we have $ab \neq a$ for any elements $a, b \in S$. Now let $M = S^1$. Clearly, M is a nontrivial left cancellative monoid, and so M is $\mathcal{J}^{(l)} = \mathcal{D}^{(l)} = \mathcal{L}^{(l)}$-simple. However, since $J(1) \supsetneq J(s)$ for every $s \in S$, and so we have $\mathcal{J} \lneqq \mathcal{J}^{(l)}$, $\mathcal{D} \lneqq \mathcal{D}^{(l)}$, $\mathcal{L} \lneqq \mathcal{L}^{(l)}$.

Example (ii): Let $A(B)$ be a monogenic semigroup (monogenic monoid) including the generator a (generator b and identity e). Write $T = A \cup B \cup \{1\}$, and extend the multiplications of A and B to T by using the following rules:

$$a^m b^n = b^{m+n}, b^n a^m = a^{m+n}, b^0 = e, m = 1, 2, \cdots, n = 0, 1, 2, \cdots .$$

Then we have the following situations on the semigroup T:

$\mathcal{D}^* = \mathcal{J}^*$ is the universal relation;

$\mathcal{L}^{(l)} = \mathcal{L}^*$ has two classes: $A \cup \{1\}, B$;

$\mathcal{R}^*$ has also two classes: $\{1\}, A \cup B$.

Clearly, in the above example, $\mathcal{R} = \mathcal{L}$ which is the equality relation on T. Thereby, $\mathcal{D}^{(l)} = \mathcal{L}^{(l)} \vee \mathcal{R} = \mathcal{L}^{(l)} \lneqq \mathcal{D}^*$ and $J^{(r)}(a) \subsetneq T = J^{(r)}(1)$ in T. Consequently, $\mathcal{J}^{(r)} \lneqq \mathcal{J}^*$ on T and so $\mathcal{J}^{(l)} \lneqq \mathcal{J}^*$ on the reciprocal semigroup T' of T. ■

Theorem 5.2 (see [15]).

(i) $\mathcal{H} \lneqq \mathcal{H}^{(l)}$;

(ii) $\mathcal{H}^{(l)} \lneqq \mathcal{H}^*$.

Proof. (i) We now give an example to show that $\mathcal{H} \lneqq \mathcal{H}^{(l)}$. For this purpose, we let M_1 be a monoid satisfying the following condition:

$$(\exists a, b, x_0 \in M_1) \begin{cases} a = b x_0, \\ b = a x_0, \\ a \text{ and } b \text{ are not } \mathcal{L}-\text{related.} \end{cases}$$

Such a monoid exists, for example, the full left transformation monoid $\mathcal{T}_l(\{1, 2, 3\})$ on the set $\{1, 2, 3\}$ is such a monoid, where

$$a = \begin{pmatrix} 1\,2\,3 \\ 2\,2\,3 \end{pmatrix}, b = \begin{pmatrix} 1\,2\,3 \\ 3\,2\,2 \end{pmatrix} \text{ and } x_0 = \begin{pmatrix} 1\,2\,3 \\ 3\,2\,1 \end{pmatrix}.$$

We now construct a new semigroup $S_1 = (S_1, *)$ from M_1 by letting

$$S_1 = \begin{array}{c} M_1 \\ \times \\ M_1 \end{array} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid \ x, y \in M_1 \right\},$$

where the multiplication is defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} * \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} xv \\ yv \end{pmatrix}, \text{for all } x, y, u, v \in M_1.$$

Consider $\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} b \\ a \end{pmatrix} \in S_1$. Then for every $x \in M$, we have

$$\begin{pmatrix} a \\ b \end{pmatrix} * \begin{pmatrix} x \\ x_0 \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix},$$

and

$$\begin{pmatrix} b \\ a \end{pmatrix} * \begin{pmatrix} x \\ x_0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Hence, $\begin{pmatrix} a \\ b \end{pmatrix} \mathcal{R} \begin{pmatrix} b \\ a \end{pmatrix}$ in S_1. Clearly, $S_1 \neq S_1^1$, and for any $x \in M_1$, $\begin{pmatrix} x \\ l \end{pmatrix}$ is a right identity of S_1, where l is the identity of monoid M_1. Therefore, for all $u, v, x \in M_1$, we have

$$\begin{pmatrix} u \\ v \end{pmatrix} * \begin{pmatrix} x \\ l \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} * 1.$$

In order to verify the $\mathcal{L}^{(l)}$-relation of two elements $s, t \in S_1$, we need to find the kernels of the left translations s_l, t_l on S_1 (not on S_1^1). For any $\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \in S_1$, we have,

$$\begin{pmatrix} a \\ b \end{pmatrix} * \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} * \begin{pmatrix} x' \\ y' \end{pmatrix}$$

if and only if

$$\begin{pmatrix} b \\ a \end{pmatrix} * \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix} * \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

This implies that $\begin{pmatrix} a \\ b \end{pmatrix} \mathcal{L}^{(l)} \begin{pmatrix} b \\ a \end{pmatrix}$, and thereby $\begin{pmatrix} a \\ b \end{pmatrix} \mathcal{H}^{(l)} \begin{pmatrix} b \\ a \end{pmatrix}$. Since a and b are not $\mathcal{L}$-related in M_1, $\begin{pmatrix} a \\ b \end{pmatrix}$ is not $\mathcal{L}$-related to $\begin{pmatrix} b \\ a \end{pmatrix}$ in S_1. This shows that $\begin{pmatrix} a \\ b \end{pmatrix}$ is not $\mathcal{H}$-related to $\begin{pmatrix} b \\ a \end{pmatrix}$ and consequently, $\mathcal{H} \lneqq \mathcal{H}^{(l)}$.

(ii) In the above semigroup $S_1 = (S_1, *)$, we have $\begin{pmatrix} a \\ b \end{pmatrix} \mathcal{L}^{(l)} (= \mathcal{L}^*) \begin{pmatrix} b \\ a \end{pmatrix}$ but $\begin{pmatrix} a \\ b \end{pmatrix}$ is not $\mathcal{L}$-related to $\begin{pmatrix} b \\ a \end{pmatrix}$. Hence, in the reciprocal semigroup $S_1' = (S_1, \circ)$ of S_1, the multiplication in S_1' is defined by $s \circ t = t * s$, for any $s, t \in S_1$. We now see that $\begin{pmatrix} a \\ b \end{pmatrix} \mathcal{R}^* \begin{pmatrix} b \\ a \end{pmatrix}$ but $\begin{pmatrix} a \\ b \end{pmatrix}$ is not $\mathcal{R}$-related to $\begin{pmatrix} b \\ a \end{pmatrix}$. This shows that $\mathcal{R}^{(l)} = \mathcal{R} \lneqq \mathcal{R}^*$ in S_1'. Also, we have $\begin{pmatrix} a \\ b \end{pmatrix} \mathcal{H}^{(l)} \begin{pmatrix} b \\ a \end{pmatrix}$ in S_1, and hence $\begin{pmatrix} a \\ b \end{pmatrix} \mathcal{H}^* \begin{pmatrix} b \\ a \end{pmatrix}$ in S_1. Since $\begin{pmatrix} a \\ b \end{pmatrix}$ is not $\mathcal{L}$-related to $\begin{pmatrix} b \\ a \end{pmatrix}$ in S_1, $\begin{pmatrix} a \\ b \end{pmatrix} \mathcal{H}^* \begin{pmatrix} b \\ a \end{pmatrix}$ but $\begin{pmatrix} a \\ b \end{pmatrix}$ is not $\mathcal{R}$-related to $\begin{pmatrix} b \\ a \end{pmatrix}$ in S_1'. This shows that $\begin{pmatrix} a \\ b \end{pmatrix}$ is not $\mathcal{H}^{(l)}$-related to $\begin{pmatrix} b \\ a \end{pmatrix}$, and so $\mathcal{H}^{(l)} \lneqq \mathcal{H}^*$ on S_1'. ■

6. Some differences between regular semigroups and abundant semigroups

It was observed by El-Qallali [3, 4] and others that the class of abundant semigroups not only contains regular semigroups, but also contains cancellative monoids, semilattices of cancellative monoids, bands of cancellative monoids and so on, as its special subclasses. Although many properties of abundant semigroups are inherited from the properties of regular semigroups, but there are some remarkable differences between these two kinds of semigroups, both in structure and properties(see [3]–[7] and [23]). We list some of their major differences below:

(i) The homomorphic image of a regular semigroup is still regular, but the homomorphic image of an abundant semigroup is not necessarily abundant [3]. In other words, if ρ is a congruence on an abundant semigroup S then its quotient semigroup S/ρ is not necessarily abundant.

(ii) It is easy to see that a semilattice of regular semigroups is always regular. However, this is not the case for abundant semigroups.

Example 6.1 [6]**.** Let A be a free monoid generated by elements x, y with identity 1. Let $B = \{e, f\}$ be a left zero band and form $S = A \cup B$. Define the multiplication on S as follows: for elements in A and B, the multiplications in A and B are the same and we always let 1 be the identity element of S. For any $w \in A \setminus \{1\}$and any $b \in B$, we define $wb = e$ if w is a word starting from x. Otherwise, we define $wb = f$. Also, we define $bw = b$. Clearly A and B are both abundant semigroups and $S = A \cup B$ is a semilattice of the abundant semigroups A and B, however, S itself is not an abundant semigroup. This is because $1, e, f$ are the only idempotents of S, but there does not exist any $\mathcal{L}^*$-relation between the element x and these three idempotents.

(iii) In regular semigroups, it is well-known that the Lallement Lemma holds, that is, if ρ is a congruence on a regular semigroup S and $a\rho$ is an idempotent of S/ρ, then there exists an idempotent $e \in S$ such that $e\rho = a\rho$. However, in abundant semigroups, the Lallement Lemma may not hold. The following is an example.

Example 6.2. Let S be a free monoid generated by x, y. Define a relation ρ on S by $(u, v) \in \rho$ if and only if the words u, v have the same first letter. Clearly, such a semigroup is an abundant semigroup(in this case, $\mathcal{L}^* = \mathcal{R}^* = S \times S$) and ρ is a congruence on S. Also, we can observe that there are only two ρ-classes on S, and consequently, S/ρ forms a left zero band of two elements. Clearly, there must exist $x \in S$ such that the ρ-class containing x contains no idempotent, since S contains only one idempotent. Thus, Lallement Lemma fails to be true in abundant semigroups.

In the above examples, we can see that although abundant semigroups are generalized regular semigroups, they still have many important properties which do not hold in regular semigroups. Because of these deviations, it is unjustifiable to simply say that abundant semigroups are analogues of regular semigroups.

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