Discussiones Mathematicae
General Algebra and Applications 30 (2010) 45–69

FLOCKS IN UNIVERSAL AND BOOLEAN ALGEBRAS

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Abstract

We propose the notion of flocks, which formerly were introduced only in based algebras, for any universal algebra. This generalization keeps the main properties we know from vector spaces, e.g. a closure system that extends the subalgebra one. It comes from the idempotent elementary functions, we call "interpolators", that in case of vector spaces merely are linear functions with normalized coefficients.

The main example, we consider outside vector spaces, concerns Boolean algebras, where flocks form "local" algebras with a sparseness similar to the one of vector spaces. We also outline the problem of generalizing the Segre transformations of based algebras, which used certain flocks, in order to approach a general transformation notion.

Keywords: combinators, elementary functions, closure system, interpolators, semi-affine lattice.

2000 Mathematics Subject Classification: 08B20, 51D99.

0. Preliminaries

0.0. Introduction

We want to generalize flocks (for the word "affine subspaces" see 3.7) from vector spaces to all universal algebras. So far, this was partly done only in based ones, where it was also found that flocks form invariant structures (see **3.1** (C, D) of [16]).

Such an invariance is not a conventional one, under automorphism or similar abstract functions, because it was also found that such functions fail to define transformations [15] in free algebras, as isomorphisms failed in vector spaces. This finding led to the notion of invariance for based algebras [16], but not yet for general (non free) ones.

Due to this lack of a general invariance notion, we cannot state it for our more general flocks. Therefore, this paper *merely proposes* this generalization as an abstract notion like the ones of conventional Universal Algebra.

Yet, two results below support the conjecture that a future general notion of transformation would prove the invariance of flock structures, even after our generalization. The weakest one in 3.4 (A) states that they are automorphism invariant. The other in 3.1 (D) and 3.3 (A) states that they contain the restricted (yet strongly invariant) ones of based algebras.

Anyway, the very problem of finding a general invariance notion motivates our proposal. In fact, **3.6** (B) will hint that from our tentative flocks one might approach such an invariance.

Flocks come from the notion of elementary interpolators, which are the functions that in any algebra provide any indexing of its elements with an element that "interpolates" them. We define them as the idempotent elementary functions. By them we show that flocks form closure systems and we characterize their possible triviality. A natural characterization of flock systems for Boolean algebras follows from such general results.

Such "Intrinsic Algebra" new notions require fit notational tools, in particular the combinatory ones [2]. We recall them below.

0.1. Set-theoretical notation

To denote functional applications we alternate subscripting and right parenthesizing. In spite of this choice of conventional notation, the foundation chosen here is the *pure* set–theoretical one, not the conventional algebraic one. (See 0.6 in [10] for some flaws of the latter.) Hence, we conform to [6], but for the following few differences.

We consider functional composition as a restriction of relational composition, here denoted by $\cdot$, namely $f \cdot g$ is the *composition of g and f* (not "of f and g") and

$$(f \cdot g)(x) = f(g(x)) \ . \tag{0}$$

Accordingly, we perform the restriction of a function f to some set S merely by functional composition: $f \cdot \boldsymbol{i}_S$, where $\boldsymbol{i}_S$ denotes the identity function on S, see also (7). $\mathcal{P}(X)$ denotes the set of subsets of X.

As usual, we write $f:A\to B$ to say that f is a function with arguments in the whole set A and values in B, $f:A\rightarrowtail B$ or $f:A\twoheadrightarrow B$ to say that it also is one to one or onto B and $f:A\rightarrowtail\!\!\!\!\twoheadrightarrow B$ to say it is a bijection onto B. When X is any set, we denote the set-theoretical power ${}^{X}\!A = \{f \mid f: X\to A\}$ of [6] as the arithmetic one A^X. (The context will weaken this ambiguity.) Likewise, in lattices 0 and 1 might not denote the set-theoretical numbers $\emptyset$ and $\{\emptyset\}$.

0.2. Combinatory notation

Some notation is borrowed from Combinatory Logic [2]. From a set-theoretical point of view it is *locally* incomplete, yet the context will always make it unambiguous.

Among the functions in A^X we consider the constant ones. For $a\in A\neq\emptyset$ we denote the one with value a by $\boldsymbol{k}_a$:

$$\boldsymbol{k}_a(x) = a\ ,\ \text{ for all } x\in X\neq\emptyset\ . \tag{1}$$

Also, this always defines a *constant generating function* $\boldsymbol{k}:A\to A^X$. In fact, for $X=\emptyset$ and $A\neq\emptyset$ there only are the trivial cases $\boldsymbol{k}_a=\emptyset$ and for $A=\emptyset$ the case $\boldsymbol{k}=\emptyset$.

We will use constant generating functions for several different A and X. As typefaces run out, we will not distinguish them (the context will mend this set-theoretical abuse). E.g., given sets Y, B and $a\in A$, we write the identities

$$\boldsymbol{k}_a\cdot M \;=\; \boldsymbol{k}_a\ ,\qquad \text{for all } M:X\to Y \text{ and} \tag{2}$$

$$M\cdot\boldsymbol{k}_a \;=\; \boldsymbol{k}_{M(a)}\ ,\ \text{for all } M:A\to B\ , \tag{3}$$

which follow either immediately, when $X=\emptyset$, or from (1), when $X\neq\emptyset$. Yet, in (2) $\boldsymbol{k}:A\to A^Y$ on the left, whereas $\boldsymbol{k}:A\to A^X$ on the right, while in (3) $\boldsymbol{k}:A\to A^X$ on the left, whereas $\boldsymbol{k}:B\to B^X$ on the right.

Given a function $m:I\to A^J$, $\mathbf{C}^{(J)}_m$ denotes the *exchanged function of* m $\mathbf{C}^{(J)}_m:J\to A^I$ such that $[\mathbf{C}^{(J)}_m(j)]_i = m_i(j)$ for all $j\in J$ and $i\in I$. When $I\neq\emptyset$, m determines J. Then, we can simplify this notation as $\boldsymbol{c}_m: J\to A^I$ and, when I and J are fixed, we get an *exchange function* $\boldsymbol{c}:(A^J)^I\to(A^I)^J$. However, we will use several different exchange functions, again without distinguishing their notation.

Moreover, we will also leave the duty of specifying J to the context, when $I=\emptyset$. Then, all such conventions allow us to write

$$\mathbf{C}_{\emptyset}^{(J)} = \boldsymbol{c}_{\emptyset} = \begin{cases} \emptyset & \text{when } J=\emptyset \text{ and} \\ \boldsymbol{k}_{\emptyset}\colon J \to 1 & \text{when } J\neq\emptyset \ , \end{cases} \tag{4}$$

and in general

$$[\boldsymbol{c}_m(j)]_i = m_i(j) \quad , \quad \text{for all } j\in J \text{ and } i\in I \ . \tag{5}$$

This notation serves to generalize Menger's superposition [4, 5], which concerns finitary operations. In fact, here such cardinality restrictions will fail even within algebras with finitary operations.

In addition to typeface savings, such conventions highlight functional features of algebraic interest better than a set-theoretically complete notation. For instance, they exhibit the operational sameness of two different $\boldsymbol{c}$ in the identity $\boldsymbol{c}_{\boldsymbol{c}(m)} = m$ (for all such m), which follows from (5) or (4) and implies both $\boldsymbol{c}\cdot\boldsymbol{c} = \boldsymbol{i}_{(A^J)^I}$ and $\boldsymbol{c}:(A^J)^I \rightarrowtail\!\!\twoheadrightarrow (A^I)^J$.

In spite of this functional notation, our $\boldsymbol{c}$ and $\boldsymbol{k}$ are not (set-theoretical) functions, but define them by the context. When we implicitly redefine such symbols, we exploit the exchange combinator ("elementary permutator" in [2]) **C** and the constant generator **K** within a set-theoretical setting.

Likewise, the same $\boldsymbol{p}$ will denote the projection generator **CI**, which for several A and Y will define the functions $\boldsymbol{p}:Y\to A^{A^Y}$ such that

$$\boldsymbol{p}_y(M) = M(y) \quad \text{for all } y\in Y \text{ and } M:Y\to A \ . \tag{6}$$

Then, $A=\emptyset$ implies that $\boldsymbol{p}_y=\emptyset$ for all $y\in Y$. Moreover, for all A and Y,

$$\boldsymbol{c}\boldsymbol{p} = \boldsymbol{i}_{A^Y} \ , \tag{7}$$

since $[\boldsymbol{c}\boldsymbol{p}(M)]_y \overset{(5)}{=} \boldsymbol{p}_y(M) \overset{(6)}{=} M(y) = [\boldsymbol{i}_{A^Y}(M)]_y$ for all $y\in Y$ and $M:Y\to A$.

In 0.1 we introduced our widely used $\boldsymbol{i}$ as a set-theoretical function: the one (proper class) that provides every set S with $\boldsymbol{i}_S$, the identity on it. We did not as a combinator (but we boldfaced it) in order to allow the contexts to specify the functions denoted by combinators. Rewriting (7) as $\boldsymbol{c}\boldsymbol{p}=\boldsymbol{i}$ makes the one of $\boldsymbol{p}$ (and hence of $\boldsymbol{c}$) unspecified.

0.3. Algebras
Here, the rank S of any operation $f\colon A^S \to A$ can be any set (see 2.5). Then, in an indexed algebra $\alpha \in \prod_{i\in I} A^{A^{r(i)}}$ the type is *any* function r with domain I. Yet, the algebras of our main concern merely are sets $\mathcal{O}$ of such operations. On A we will also consider operations outside $\mathcal{O}$.

$\mathcal{O}$ determines A. Hence, when all operations in $\mathcal{O}$ are nullary, contrary to conventional definitions, A cannot be larger than the set of their values.

When we have to consider conventional finitary algebras, we assume to replace their operations with the previous *set-ary ones.* E.g., an $f\colon A^2 \to A$ will replaces an $f'\colon A \times A \to A$ by the natural map for $A \times A \simeq A^2$, while f might keep the possible infix notation of f'.

We also assume $\mathcal{O} \neq \emptyset$, since an $\mathcal{O} = \emptyset$ gives the class of all sets as its carrier, according to some definitions of a carrier. Since the case $A = \emptyset$ mainly concerns the initial settings in computer implementations, we will allow uninterested readers to skip it by putting the observations relevant to it into square brackets. As this is the only case where an operation (or indexed algebra) does not determine its rank (or type), outside such brackets "a rank/type of" will become "the rank/type of".

Sometimes, we will consider *based algebras,* which are defined also by one of their bases as in 1.1 in addition to a set or indexing of operations. There, the *choices of a basis* superimpose possibly different "analytic spaces" on the same (free) algebra, see **3.6** of [16]. Then, while we can expect that the few next notions about flocks are relevant to all of them, future developments, like the generalization of flock ranks [1], might not be.

1. ELEMENTARY FUNCTIONS

1.0. Definitions
The *(set-ary) composition of an indexing* $L\colon S \to A^{A^Y}$ *of* Y*–ranked operations with an operation* $g\colon A^S \to A$ is the function $\ell = g \cdot \boldsymbol{c}_L$. (All such operations can also stay outside $\mathcal{O}$.) Then, $\ell\colon A^Y \to A$, where, for a nullary g, by (3) and (4) $S = \emptyset$ implies $\ell = \boldsymbol{k}_{g(\emptyset)}$. [Notice that, when $A = \emptyset$, g cannot be nullary, while L requires $Y \neq \emptyset$.] When S and Y are natural numbers, this composition replaces (finitary) superpositions.

Also, given any set Y and the algebra in 0.3, $\mathcal{L}_Y$ will denote the set of its Y*-ary elementary functions,* viz. the functions we get by such compositions with its operations from all the *projections* in (6). Formally, $\mathcal{L}_Y = \bigcap\{\mathcal{F} \subseteq A^{A^Y} \mid \boldsymbol{p}\colon Y \to \mathcal{F}$ and $(f \in \mathcal{O},\ f\colon A^S \to A$ and $L\colon S \to \mathcal{F}$ imply $f \cdot \boldsymbol{c}_L \in \mathcal{F})\ \}$.

As usual, an indexing $U:X\to A$ is a *generator* of $\mathcal{O}$ when, for every $a\in A$, $a=\ell(U)$ for some $\ell\in\mathcal{L}_X$. We prefer "elementary functions" to "term operations", since we will not use terms to index them. As the arity Y is fixed, when Y is a natural number, $\mathcal{L}_Y$ corresponds only to a proper subset of the algebra clone [3, 4].

When $Y=\emptyset$, the only compositions involved are the ones with an indexing of nullary constants from the empty set of projections. Without any nullary f this gives $\mathcal{L}_\emptyset=\emptyset$. In general, $\mathcal{L}_\emptyset$ is the set of nullary constants that corresponds to the subalgebra closure of the empty set. [When the carrier is trivial and $Y\neq\emptyset$, all projections are empty and $\mathcal{L}_Y=\{\emptyset\}$.]

1.1. Analytic representations
concern all free algebras, viz. the ones that have bases. They will define bases through the set of all *endomorphisms* of an algebra $\mathcal{O}$: $\mathbb{E}_\mathcal{O}=\{h:A\to A \mid h(f(a))=f(h\cdot a)$ for all $a:S\to A$ and $f:A^S\to A$in $\mathcal{O}\}$. The proof of the equivalence with the conventional basis definition (by freedom and carrier generation) is in 0.9 of [13], see also 0.5 of [12].

Given a generator $U:X\to A$, consider the function $\boldsymbol{r}_U:\mathbb{E}_\mathcal{O}\to A^X$, defined by $\boldsymbol{r}_U(h)=h\cdot U$, for $h\in\mathbb{E}_\mathcal{O}$, namely $\boldsymbol{r}_U$ "samples each h at" U by providing each $x\in X$ with the value $h(U(x))$. If this sampling serves to represent every endomorphism by any sample and conversely, namely if

$$\boldsymbol{r}_U:\mathbb{E}_\mathcal{O}\rightarrowtail\!\!\!\twoheadrightarrow A^X\ , \tag{8}$$

then every structure on $\mathbb{E}_\mathcal{O}$ defines another on A^X and we say that

(A) $\boldsymbol{r}_U$ is an *analytic representation of* $\mathbb{E}_\mathcal{O}$*,* while X is its *dimension set,*

(B) its inverse $\eta=\boldsymbol{r}_U^{-1}:A^X\rightarrowtail\!\!\!\twoheadrightarrow\mathbb{E}_\mathcal{O}$, which extends any sample assignment $M\cdot U^{-1}$ onto the endomorphism $h=\eta_M$ with $h(U_x)=M_x$ for all $x\in X$, is the *(sample) extension* function *from* U,

(C) A^X is the set of the *(square universal) matrices of* the algebra *with respect to* U, while every value $M(x)$ of a matrix $M:X\to A$ is its *column at* $x\in X$,

(D) U is a *basis* of the algebra, while its columns $U(x)$ are *reference elements* that form the *basis set* $B\subseteq A$ for $U:X\twoheadrightarrow B$,

(E) the *algebra of the conjugate functions derived from* α with respect to U is the indexing $\chi:A\to A^{A^X}$, defined by (8) from the functional application of endomorphisms as $\chi_a(\boldsymbol{r}_U(h)) = h(a)$ for $h \in \mathbb{E}_{\mathcal{O}} \subseteq A^A$ and $a \in A$, namely $\chi = \boldsymbol{c}_\eta$ since by (B)

$$\chi_a(M) = \eta_M(a)\ , \quad \text{for all } M\colon X \to A \text{ and } a \in A\ , \tag{9}$$

(F) while the value $\chi_a\colon A^X \to A$ of this indexing at the element $a \in A$ is the *function conjugate of a with respect to U.*

Notice also that for a singleton A, viz. for a *trivial algebra,* every set X satisfies (8) [whereas $A = \emptyset$ by (8) implies $X = \emptyset$]. When the algebra is not trivial, $X = \emptyset$ if and only if $\mathbb{E}_{\mathcal{O}} = \{\boldsymbol{i}_A\}$. It does when all algebra elements are constants. By (9) this also implies that $\chi\colon A \mapsto\!\!\!\twoheadrightarrow A^1$ merely is the generator of singleton constants: $X = \emptyset$ if and only if $\chi_a(M) = a$ for all $M\colon X \to A$ and $a \in A$.

1.2. Example

Take A as the set of the usual n–tuples of elements of a field and consider any endomorphism h of their vector space on the same field. If the reference elements are the ones forming the Kronecker matrix, then their endomorphic images $h(U_x)$ for $x \in X = n$ are the column vectors of the usual matrix identifying this endomorphism and $\chi_a(M)$ is the product of vector a times the matrix M.

Therefore, the conjugate function χ_a of a vector a is similar to its linear form. The only difference is that the former acts on vectors, while the latter does on field numbers. (It follows that the conjugate functions in a vector space, as well as in any based universal algebra, form another algebra that always is isomorphic to the starting one, as in 6.6 of [8], whereas linear forms merely form the adjoint space as in II.3 of [1].)

On the contrary, when such a representation of endomorphisms of a based vector space concerns an arbitrary A and/or an arbitrary basis, it gives different matrices, products and conjugate functions. Then, even in vector spaces a (universal) matrix might not be a two-dimensional array. (See 0.4 of [10] for several examples outside vector spaces.)

1.3. Recalled property

In every based algebra as in 1.1 *the conjugate functions are its X-ary.* (Proved in 0.9 [13].)

1.4. Definitions

A case of set-ary composition involves elementary functions only: when $g : A^Y \to A$ is a Y-ary elementary function and $L : Y \to A^{A^G}$ is any indexing of G-ary elementary functions, we consider the set-ary composition $g \cdot \boldsymbol{c}_L : A^G \to A$, which by 1.5 (C) is G-ary elementary. [When $A = \emptyset$, $G \neq \emptyset$ and $g = \emptyset = g \cdot \boldsymbol{c}_L$, since $Y \neq \emptyset$ because g cannot be nullary.]

In the elementary functions we will use, the ranks can be dimension sets, $Y = X$ as in 1.3, and even carriers, $G = A$. This prevents to use any finitary superposition. For instance, in general vector spaces X can have any cardinality, in spite of their finitary operations.

In addition to set-ary composition we define two other operations for elementary functions. Given any $M : Y \twoheadrightarrow G$, we define $\varrho : \mathcal{L}_Y \to A^{A^G}$ by

$$\varrho_\ell(a) = \ell(a \cdot M) \quad \text{for all } \ell \in \mathcal{L}_Y \text{ and } a : G \to A \ . \tag{10}$$

[When $A = \emptyset$, either $\varrho = \emptyset$ for $Y = \emptyset$ or $\varrho : \{\emptyset\} \to \{\emptyset\}$ otherwise.] By 1.5 (A), $\varrho : \mathcal{L}_Y \to \mathcal{L}_G$. Then, we call ϱ_ℓ the *M-condensation of ℓ.* Dually, given any $J : G \rightarrowtail Y$, we define $\varsigma : \mathcal{L}_G \to A^{A^Y}$ by

$$\varsigma_\ell(a) = \ell(a \cdot J) \quad \text{for all } \ell \in \mathcal{L}_G \text{ and } a : Y \to A \ . \tag{11}$$

[When $A = \emptyset$, either $\varsigma = \emptyset$ for $G = \emptyset$ or $\varsigma : \{\emptyset\} \to \{\emptyset\}$ otherwise.] By 1.5 (B), $\varsigma : \mathcal{L}_G \to \mathcal{L}_Y$. Then, we call ς_ℓ the *J-expansion of ℓ onto Y.*

1.5. Lemmata

(A) *Every M-condensation of a Y-ary elementary function with $M : Y \twoheadrightarrow G$ is a G-ary elementary function and conversely: $\varrho : \mathcal{L}_Y \twoheadrightarrow \mathcal{L}_G$.*

(B) *Every J-expansion of a G-ary elementary function onto Y is a Y-ary elementary function and, given any left inverse $M : Y \twoheadrightarrow G$ of J*

$$M \cdot J \;=\; \boldsymbol{i}_G \ , \tag{12}$$

ς is a right inverse of the above-defined ϱ:

$$\varrho \cdot \varsigma \;=\; \boldsymbol{i}_{\mathcal{L}_G} \ . \tag{13}$$

Hence, $\varsigma:\mathcal{L}_G \mapsto\!\!\!\!\to \mathcal{L}_Y$*.*

(C) *If* $g:A^Y\to A$ *is* Y*-ary elementary and* $L:Y\to A^{A^G}$ *is any indexing of* G*-ary elementary functions, then* $g\cdot \boldsymbol{c}_L : A^G \to A$ *is* G*-ary elementary.*

Proofs. [When $A=\emptyset$, all three proofs immediately follows from the definitions in 1.4.]

(A) When $Y=\emptyset$, $G=M=\emptyset$ and $\varrho_\ell \overset{(10)}{=} \ell$ for all $\ell\in\mathcal{L}_Y=\mathcal{L}_G=\mathcal{L}_\emptyset$. Then, we consider $Y\neq\emptyset$.

(Is) If $\ell=\boldsymbol{p}_y : A^Y\to A$ for any $y\in Y$, then $\varrho_\ell(a) \overset{(10)}{=} \boldsymbol{p}_y(a\cdot M) \overset{(6)}{=} (a\cdot M)(y) \overset{(0)}{=} a(M_y) \overset{(6)}{=} \boldsymbol{p}_{M(y)}(a)$ for all $a\in A^G$, viz. $\varrho_\ell\in\mathcal{L}_G$ as in 1.0, since $\varrho_\ell=\boldsymbol{p}_{M(y)} : A^G\to A$. Then, consider $f\in\mathcal{O}$, $f:A^S\to A$ and $L:S\to\mathcal{L}_Y$, such that $\varrho_{L(s)}\in\mathcal{L}_G$ for all $s\in S$, i.e. $\varrho\cdot L:S\to\mathcal{L}_G$.

When $S=\emptyset$, $\varrho_{f\cdot\boldsymbol{c}_L}\in\mathcal{L}_G$, because for all $a\in A^G$ $\varrho_{f\cdot\boldsymbol{c}_L}(a) \overset{1.0}{=} \varrho_{\boldsymbol{k}_{f(\emptyset)}}(a) \overset{(10)}{=} \boldsymbol{k}_{f(\emptyset)}(a\cdot M) \overset{(1)}{=} f(\emptyset) \overset{(1)}{=} \boldsymbol{k}_{f(\emptyset)}(a)$, where $\boldsymbol{k}_{f(\emptyset)}\in\mathcal{L}_G$ as in 1.0. When $S\neq\emptyset$, we get $\varrho_{f\cdot\boldsymbol{c}_L} = f\cdot\boldsymbol{c}_{\varrho\cdot L} \overset{1.0}{\in} \mathcal{L}_G$. In fact, for all $a\in A^G$, since $[\boldsymbol{c}_L(a\cdot M)]_s \overset{(5)}{=} L_s(a\cdot M) \overset{(10)}{=} \varrho_{L(s)}(a) \overset{(0)}{=} [\varrho\cdot L]_s(a) \overset{(5)}{=} [\boldsymbol{c}_{\varrho\cdot L}(a)]_s$ for all $s\in S$, we get $\varrho_{f\cdot\boldsymbol{c}_L}(a) \overset{(10)}{=} (f\cdot\boldsymbol{c}_L)(a\cdot M) \overset{(0)}{=} f(\boldsymbol{c}_L(a\cdot M)) = f(\boldsymbol{c}_{\varrho\cdot L}(a)) \overset{(0)}{=} (f\cdot\boldsymbol{c}_{\varrho\cdot L})(a)$.

(Conversely) Choose any right inverse $j:G \mapsto\!\!\!\!\to Y$ of M, $M\cdot j=\boldsymbol{i}_G$. Given $g\in\mathcal{L}_G$, define $\ell:A^Y\to A$ by $\ell(a)=g(a\cdot j)$ for all $a:Y\to A$, which implies that $\ell(a'\cdot M)=g(a')$ for all $a':G\to A$, because for each a' there is an a such that $a'\cdot M=a$ and $a\cdot j=a'$. Also, $\ell\in\mathcal{L}_Y$. In fact, in the (is) part we can reverse the passages to get that ℓ is any $\boldsymbol{p}_y : A^Y\to A$ for $g=\boldsymbol{p}_{M(y)} : A^G\to A$ and that, under the set-ary composition of $\mathcal{L}_G$, ℓ stays Y-ary elementary.

(For instance, when $S\neq\emptyset$, to get the ℓ that corresponds to $g=f\cdot\boldsymbol{c}_{L'}\in\mathcal{L}_G$, where $L':S\to\mathcal{L}_G$, we start from an indexing $L:S\to\mathcal{L}_Y$ that corresponds to L': $\varrho\cdot L=L'$. Then, we set $\ell=f\cdot\boldsymbol{c}_L\in\mathcal{L}_Y$ and check that $\varrho_\ell(a)=f(\boldsymbol{c}_L(a\cdot M))=f(\boldsymbol{c}_{\varrho\cdot L}(a))=(f\cdot\boldsymbol{c}_{L'})(a)=g(a)$ as above.)

(B) (Is) Consider the dual of the preceding proof. If $\ell=\boldsymbol{p}_x : A^G\to A$ for any $x\in G$, then $\varsigma_\ell(a) \overset{(11)}{=} \boldsymbol{p}_x(a\cdot J) \overset{(6)}{=} a(J_x)$ for all $a\in A^Y$, viz. $\varsigma_\ell\in\mathcal{L}_Y$, since $\varsigma_\ell=\boldsymbol{p}_{J(x)} : A^Y\to A$.

Then, consider $f\in\mathcal{O}$, $f:A^S\to A$ and $L:S\to\mathcal{L}_G$, such that $\varsigma_{L(s)}\in\mathcal{L}_Y$

for all $s \in S$. Again, we get $\varsigma_{f\cdot\boldsymbol{c}_L} = f \cdot \boldsymbol{c}_{\varsigma\cdot L} \in \mathcal{L}_Y$. In fact, for all $a \in A^Y$, $\varsigma_{f\cdot\boldsymbol{c}_L}(a) \overset{(11)}{=} (f \cdot \boldsymbol{c}_L)(a \cdot J) \overset{(0)}{=} f(\boldsymbol{c}_L(a \cdot J)) = f(\boldsymbol{c}_{\varsigma\cdot L}(a)) \overset{(0)}{=} (f \cdot \boldsymbol{c}_{\varsigma\cdot L})(a)$, since $\boldsymbol{c}_L(a \cdot J) = \boldsymbol{c}_{\varsigma\cdot L}(a)$ either by (4) or, when $S \neq \emptyset$, because for all $s \in S$ $[\boldsymbol{c}_L(a \cdot J)]_s \overset{(5)}{=} L_s(a \cdot J) \overset{(11)}{=} \varsigma_{L(s)}(a) \overset{(0)}{=} [\varsigma \cdot L]_s(a) \overset{(5)}{=} [\boldsymbol{c}_{\varsigma\cdot L}(a)]_s$. (The case $S = \emptyset$, which is the starting one for $G = \emptyset$, merely changes each nullary constant in $\mathcal{L}_\emptyset$ into a Y-ary one in $\mathcal{L}_Y$, according to 1.0.)

(Inverse) In fact, $[(\varrho \cdot \varsigma)(\ell)]_a \overset{(0)}{=} \varrho_{\varsigma(\ell)}(a) \overset{(10)}{=} \varsigma_\ell(a \cdot M) \overset{(11)}{=} \ell(a \cdot M \cdot J) \overset{(12)}{=} \ell(a) = [\boldsymbol{i}_{\mathcal{L}_G}(\ell)]_a$ for all $\ell \in \mathcal{L}_G$ and $a : G \to A$.

(C) When $g = \boldsymbol{p}_y$ for some $y \in Y$, this follows from (6). When $g = f \cdot \boldsymbol{c}_M$ where $f \in \mathcal{O}$, $f : A^S \to A$ and $M : S \to A^{A^Y}$ is an indexing of Y-ary elementary functions such that $M_s \cdot \boldsymbol{c}_L \in \mathcal{L}_G$ for all $s \in S$ and $L : Y \to \mathcal{L}_G$, if $S = \emptyset$, we get $g \cdot \boldsymbol{c}_L = (f \cdot \boldsymbol{c}_\emptyset) \cdot \boldsymbol{c}_L \overset{(4)}{=} (f \cdot \boldsymbol{k}_\emptyset) \cdot \boldsymbol{c}_L \overset{(3)}{=} \boldsymbol{k}_{f(\emptyset)} \cdot \boldsymbol{c}_L \overset{(2)}{=} \boldsymbol{k}_{f(\emptyset)} \overset{1.0}{\in} \mathcal{L}_G$ (replace Y with G in 1.0).

Otherwise, (by the same replacement) we get that $g \cdot \boldsymbol{c}_L = (f \cdot \boldsymbol{c}_M) \cdot \boldsymbol{c}_L = f \cdot (\boldsymbol{c}_M \cdot \boldsymbol{c}_L)$ is in $\mathcal{L}_G$ by setting $L'_s = M_s \cdot \boldsymbol{c}_L$ for all $s \in S \neq \emptyset$ to get an $L' : S \to \mathcal{L}_G$ such that $\boldsymbol{c}_M \cdot \boldsymbol{c}_L = \boldsymbol{c}_{L'}$. In fact, $[(\boldsymbol{c}_M \cdot \boldsymbol{c}_L)(a)]_s \overset{(0)}{=} [\boldsymbol{c}_M(\boldsymbol{c}_L(a))]_s \overset{(5)}{=} M_s(\boldsymbol{c}_L(a)) \overset{(0)}{=} (M_s \cdot \boldsymbol{c}_L)(a) = L'_s(a) \overset{(5)}{=} [\boldsymbol{c}_{L'}(a)]_s$ for all $a : G \to A$. ∎

1.6. Corollaries

(A) *An algebra endomorphism $h \in \mathbb{E}_\mathcal{O}$ is an endomorphism of every Y-ary elementary function:*

$$h(\ell(a)) = \ell(h \cdot a) \quad \text{for all } a : Y \to A \text{ and } \ell \in \mathcal{L}_Y\ . \tag{14}$$

(B) *Subalgebras are closed under elementary functions.*

Proofs. Same routine checks as for the finitary case. (E.g., for (14) $h(\boldsymbol{p}_y(a)) \overset{(6)}{=} h(a(y)) \overset{(0)}{=} (h \cdot a)(y) \overset{(6)}{=} \boldsymbol{p}_y(h \cdot a)$, while (4) or $h(L_s(a)) = L_s(h \cdot a)$, viz. by (5) and (0) $(h \cdot \boldsymbol{c}_L(a))(s) = (\boldsymbol{c}_L(h \cdot a))(s)$, for all $s \in S$ imply $h(\ell(a)) \overset{1.0}{=} h((f \cdot \boldsymbol{c}_L)(a)) \overset{(0)}{=} h(f(\boldsymbol{c}_L(a)) \overset{1.1}{=} f(h \cdot \boldsymbol{c}_L(a)) = f(\boldsymbol{c}_L(h \cdot a)) \overset{(0)}{=} (f \cdot \boldsymbol{c}_L)(h \cdot a) \overset{1.0}{=} \ell(h \cdot a)$ for all $f \in \mathcal{O}$ of rank S.) ∎

2. Elementary interpolators

2.0. Definitions

Given a based algebra as in 1.1, according to **2.0** of [15] we say that $c \in A$ is a *flock combiner of* χ or *of the algebra with respect to* U, when $\chi_c(\boldsymbol{k}_a) = a$ for all $a \in A$. Then, the element of a singleton A is a flock combiner. Hence, for $X = \emptyset$, $c \in A$ is a flock combiner if and only if $A = \{c\}$, since $\boldsymbol{k}_a = \emptyset$.

In the vector space example 1.2 the matrix $\boldsymbol{k}_a$ has all the columns equal to $a = (a_0, a_1, \ldots, a_{n-1})$, where this n-tuple denotes a function $a : n \to A$. Then, $c = (c_0, c_1, \ldots, c_{n-1})$ is a flock combiner if and only if $\sum_{i \in n} c_i a_j = a_j$ for all $j \in n$ and $a \in A$, namely if and only if $\sum_{i \in n} c_i = 1$.

When we deal with an algebra disregarding any basis, we define the Y*-ary (elementary) interpolators* as the idempotent Y-ary elementary functions, viz. the left inverses $g : A^Y \to A$ in $\mathcal{L}_Y$ of the constant generator $\boldsymbol{k} : A \to A^Y$:

$$g \cdot \boldsymbol{k} = \boldsymbol{i}_A \ . \tag{15}$$

We will denote the set of the Y-ary interpolators by $\mathcal{I}_Y \subseteq \mathcal{L}_Y \subseteq A^{A^Y}$. Again, for $Y = \emptyset$ and $A \neq \emptyset$, $g \in \mathcal{L}_\emptyset$ satisfies (15) if and only if A is singleton.

[When $A = \emptyset$, by 1.0 either $\mathcal{L}_Y = \emptyset$ for $Y = \emptyset$ or $\mathcal{L}_Y = \{\emptyset\}$ for $Y \neq \emptyset$. Then, $\mathcal{I}_Y = \emptyset$ for $Y = \emptyset$ or by (15) $\mathcal{I}_Y = \{\emptyset\}$ for $Y \neq \emptyset$.]

In the mentioned example, when Y is a finite set of size m, we deal with linear functions g similar to the linear forms of 1.2, but with m variables. When Y is any set and the vector space general, the (finite) number of such variables depends on each g. In both cases, (15) is equivalent to a normalization condition like $\sum_i c_i = 1$ as above.

2.1. Corollaries

(A) *When the algebra has a basis $U : X \to A$, $c \in A$ is a flock combiner if and only if its conjugate function χ_c is an X-ary interpolator. Conversely, any X-ary interpolator is such a χ_c.*

(B) *When $Y = \emptyset$, $g : 1 \to A$ is an interpolator if and only if $g(\emptyset)$ is the value of a (single) nullary operation in a trivial algebra: $g \in \mathcal{O}$ and $A = \{g(\emptyset)\}$.*

(C) *When Y is singleton, $g : A^Y \to A$ is an interpolator if and only if g is the unary projection in* (6).

(D) *For all Y and $y \in Y$, the projection $\boldsymbol{p}_y : A^Y \to A$ is an interpolator.*

Proofs.

(A) The condition $\chi_c(\boldsymbol{k}_a) = a$ for all $a \in A$ in 2.0 by 1.3 and (0) is equivalent to replace g with χ_c in (15). [When $A = \emptyset$, $\mathcal{I}_X = \emptyset$ by the remarks in 1.1 and 2.0.]

(B) (If) Such a g is in $\mathcal{L}_\emptyset = \{g\}$ and satisfies (15), because $\boldsymbol{k}_a = \emptyset$ and A is singleton. (Only if) Any other $\emptyset$-ary interpolator should increase $\mathcal{L}_\emptyset$, hence A itself, which prevents to satisfy (15).

(C) (Only if) When $Y = \{y\}$, for all $M : Y \to A$, $M = \boldsymbol{k}_{M(y)}$. Then, $g(M) = g(\boldsymbol{k}_{M(y)}) \overset{(0)}{=} (g \cdot \boldsymbol{k})(M(y)) \overset{(15)}{=} M(y) \overset{(6)}{=} \boldsymbol{p}_y(M)$. (If) See (D).

(D) When $Y = \emptyset$, we have a false premise. When $Y \neq \emptyset$, for all $y \in Y$ we get (15): $(\boldsymbol{p}_y \cdot \boldsymbol{k})(a) \overset{(0)}{=} \boldsymbol{p}_y(\boldsymbol{k}_a) \overset{(6)}{=} \boldsymbol{k}_a(y) \overset{(1)}{=} a = \boldsymbol{i}_A(a)$ for all $a \in A$. [When $A = \emptyset$, $\boldsymbol{p}_y = \emptyset \in \{\emptyset\} = \mathcal{I}_Y$]. ∎

2.2. Dilatations

We are qualifying the interpolators as “idempotent” in order to conform to the algebraic jargon. However, this idempotency merely is an instance of a wider *use of the constant generator,* which recently [16] served to define invariance in the based universal algebras of 1.1.

If we generalize (15) by only requiring that $g \cdot \boldsymbol{k} \in \mathbb{E}_\mathcal{O}$, then this $\delta = g \cdot \boldsymbol{k}$ with $g \in \mathcal{L}_Y$ and $Y = X$ by 1.3 becomes a *dilatation* as in 1.0 of [14] or **2.2** of [15] in our based algebra. In fact, δ there was defined with respect to U by setting $g = \chi_d$ for certain $d \in A$, which were called *dilatation indicators.* Then, flock combiners are the indicators of the identical dilatation.

Dilatation indicators served to quantify the “amount” of a dilatation by a carrier element, its indicator, that depends on the basis. The same dependency occurred for another dilatation definition, the one by the “scalars” of a based algebra, which is equivalent to the former by **2.4** ibid., but disregards any indicator. On the contrary, in a general universal algebra we can also disregard bases by calling *dilatation* any such a $\delta \in \mathbb{E}_\mathcal{O}$ with any Y.

Scalars, together with flock combiners, allowed the semi-linear transformations for vector spaces to generalize into the “Segre transformations” between based algebras (**3.3** of [16]), which were able to define invariance, contrary to all isomorphism-like notions. However, the basis dependent scalars occurred in the latter transformations only through the dilatations they define, as the vector space scalars do in the former.

Therefore, the general dilatations might replace the scalars in a possible further generalization (for general algebras) of Segre transformations, provided that we also replace flock combiners. As 3.6 (B) will detail, the interpolators partly do it, because they contain the idempotent conjugate functions as in 2.1 (A).

2.3. Theorems

(A) $g' : A^G \to A$ *is the* M*-condensation with* $M : Y \twoheadrightarrow G$ *of a* Y*-ary interpolator* g*,* $g' = \varrho_g$*, if and only if it is a* G*-ary interpolator,* namely any such an M gives a condensation restriction $\varrho' : \mathcal{I}_Y \twoheadrightarrow \mathcal{I}_G$. *Hence,*

(B) *given* $J : G \rightarrowtail Y$*, the* J*-expansion* $g : A^Y \to A$ *of a* G*-ary elementary function* g'*,* $g = \varsigma_{g'}$*, onto* Y *is an interpolator if and only if* g' *is. Then, any such a* J *gives an expansion restriction* $\varsigma' : \mathcal{I}_G \rightarrowtail \mathcal{I}_Y$*.*

(C) *Set-ary composition preserves interpolators: if* $g : A^Y \to A$ *is a* Y*-ary interpolator and* $L : Y \to A^{A^G}$ *is any indexing of* G*-ary interpolators, then* $g \cdot \boldsymbol{c}_L : A^G \to A$ *is a* G*-ary interpolator.*

Proofs.

(A) After 1.5 (A), we only have to prove that (15) is equivalent to $\varrho_g \cdot \boldsymbol{k} = \boldsymbol{i}_A$ with $\boldsymbol{k} : A \to A^G$. (Only if) For all $a \in A$, $(\varrho_g \cdot \boldsymbol{k})(a) \overset{(0)}{=} \varrho_g(\boldsymbol{k}_a) \overset{(10)}{=} g(\boldsymbol{k}_a \cdot M) \overset{(2)}{=} g(\boldsymbol{k}_a) \overset{(0)}{=} (g \cdot \boldsymbol{k})(a) \overset{(15)}{=} \boldsymbol{i}_A(a)$. (If) From this chain: $g \cdot \boldsymbol{k} = \varrho_g \cdot \boldsymbol{k}$, which implies (15). (Recall the remark about the two $\boldsymbol{k}$ in (2).)

(B) When $G \neq \emptyset$, J has a left inverse M as in (12). Then, use (13) and (A). (The injectivity of ς' follows from the one of ς in 1.5 (B).)

When $G = \emptyset$, by 2.1 (B) such an interpolator g' can only occur in a trivial algebra and by (11) g is the (single) Y-ary constant, which satisfy (15). Conversely, such an interpolator g by (11) has to be a Y-ary constant, which can satisfy (15) only when A is singleton. Since g' is $\emptyset$-ary elementary, $g' \in \mathcal{O}$ and by 2.1 (B) it is an interpolator.

(C) After 1.5 (C) we only have to prove (15), which is trivial when $Y = \emptyset$, since by 2.1 (B) A is singleton and, as remarked in 1.0, $g \cdot \boldsymbol{c}_L = \boldsymbol{k}_{g(\emptyset)} : A^G \to A$. When $Y \neq \emptyset$, we have to prove that $g \cdot \boldsymbol{c}_L \cdot \boldsymbol{k} = \boldsymbol{i}_A$ knowing that $g \cdot \boldsymbol{k} = \boldsymbol{i}_A$ and $L_y \cdot \boldsymbol{k} = \boldsymbol{i}_A$ for all $y \in Y$. This follows immediately after rewriting the last premise as $\boldsymbol{c}_L \cdot \boldsymbol{k} = \boldsymbol{k}$, which comes from $[(\boldsymbol{c}_L \cdot \boldsymbol{k})(a)]_y \overset{(0)}{=} [\boldsymbol{c}_L(\boldsymbol{k}(a))]_y \overset{(5)}{=} L_y(\boldsymbol{k}(a)) \overset{(0)}{=} (L_y \cdot \boldsymbol{k})(a) = \boldsymbol{i}_A(a) = a \overset{(1)}{=} \boldsymbol{k}_a(y)$ for all $a \in A$ and $y \in Y$. ∎

2.4. Examples

(A) Consider $\mathcal{L}_Y$ for nontrivial Boolean algebras, when Y is a finite set of size $m > 0$. We get 2^{2^m} elementary functions, which correspond to the 2^k subsets of the $k = 2^m$ meet terms on m variables or negated variables under their one to one canonical representation as joins of some of such k minterms.

Among them the interpolators g correspond to the subsets which contain the *"affirmative" minterm,* namely the minterm without negations, together with any number of *mix minterms,* namely the minterms with at least one negated variable and one without negation. In fact, according to 2.0, each g has to become an identity after that $\boldsymbol{k}$ in (15) equalizes the variables. This occurs if and only if the affirmative minterm, which leads to the identity, comes either alone or with mix minterms, which lead to the null constant.

Therefore, the number of Y-ary interpolators is the number of the subsets of mix minterms. As these terms are $k - 2$, we get $2^{2^m-2} = \frac{1}{4}2^{2^m}$ interpolators: *one of four elementary functions is an interpolator.* Such a density is intermediate between one of three and one of five, the densities for interpolators of the familiar vector spaces *with** GF(3) and GF(5) respectively.

Notice also that there only are m projections, which work as trivial interpolators by 2.1 (D), against the non trivial ones that are $\frac{1}{4}2^{2^m} - m = i_m$ and the $2^{2^m} = e_m$ elementary functions:

m	1	2	3	4	5	...
i_m	0	2	61	16,380	1,073,741,819	...
e_m	1	16	256	65,536	4,294,967,296	...

(B) While (A) concerns universal algebras with fair endowments of interpolators, now we mention based ones with the smallest sets of interpolators that 2.1 (D) allows them. Consider the catenation monoid on the set A of words from an alphabet I, where $U : I \to A$ (the "canonical" basis) provides each letter $i \in I$ with the word $i' = U(i)$ made of the single letter i.

Given Y, we also get the set B of words from Y. It allows us to determine any Y-ary elementary function $\ell : A^Y \to A$ by a word $w \in B$ as $\ell = \boldsymbol{c}_\gamma(w)$ through the catenation homomorphisms $\gamma_M : B \to A$, for all $M : Y \to A$, such

*We avoid to say that a vector space has an "underlying" field, because recent simpler characterizations of such spaces hint the opposite, see 1.7 of [14].

that $\gamma_M(y') = M(y)$ for all $y \in Y$, where again y' is the singleton word of y. (In fact, $\ell(M) = \gamma_M(w) = [\boldsymbol{c}_\gamma(w)]_M$ by (5).)

Then, by (15), (0) and (5) $\ell \in \mathcal{I}_Y$ if and only if for all $a \in A$ $\gamma_{\boldsymbol{k}_a}(w) = a$, which occurs if and only if ℓ comes from a singleton word: $w = y'$ for some $y \in Y$. Hence, *the only interpolators are the* ℓ such that, for all $M \in A^Y$ and any $y \in Y$, $\ell(M) \overset{(5)}{=} \gamma_M(y') = M(y) \overset{(6)}{=} \boldsymbol{p}_y(M)$, namely the projections.

Likewise, consider the term algebra of the (constant) type $r : I \to 1$ for unary indexed algebras. These terms essentially are the previous words, but our unary operations α_i merely append letter i to each argument word. Since the set-ary composition of 1.0 for a singleton indexing corresponds to functional composition, we determine every elementary function by any word $w \in A$ and any projection index $y \in Y$ as the function $\ell : A^Y \to A$ with the values $\ell(a)$ that we get for all $a : Y \to A$ by catenating a_y and w.

Then, by (15) $\ell \in \mathcal{I}_Y$ if and only if this catenation, with any $a_y = \boldsymbol{k}_v(y) \overset{(1)}{=} v$ for all $v \in A$, does not change it. This can only happen when $w = \emptyset$, which again corresponds only to the projection $\ell = \boldsymbol{p}_y$, for all $y \in Y$. (It should easy to see, but by more formalism, that any term algebra of whatever type has this triviality of interpolators.)

2.5. Set-ary ranks

In 1.4 we only gave a cardinality motivation for choosing sets as ranks for elementary functions (and algebra operations), instead of the finite ordinals of most algebraic treatments or the seldom used general ordinals. Yet, there is something else.

Indexing arguments by ordinals might look natural as it recalls their left to right conventional writing. However, while this can be meaningless for the indexed objects, it can well conceal structural features.

For instance, consider the finite Boolean case of 2.4 (A), when the atoms are let us say the "objects" of the meet semi-concept algebra for a Formal Concept lattice (**2** of [18]). Then, the natural rank Y of an interpolator of "prototypic subsets of objects" is an index *set made of subsets* of "attributes", a complex structure that is not an order. Conversely, consider the terms for syntax trees [17], where ordinals do serve (to define "frontiers"). If we always assume that ranks are ordinals, their rôle appears weaker.

In general, we would avoid unnecessary conditions. Requiring an order is an additional condition, which should serve some mathematical purpose, not the ink-theoretical one of recalling how one might write arguments.

An instance where ordinals always serve mathematical purposes is the one of the unknowns for the terms of an indexed algebra. If we choose ordinals as unknowns (see **2.1** in [9]), then we satisfy the General Recursion Principle (**13.1** in [6]), which allows terms to define other entities consistently, whereas free choices can lead to inconsistencies (see **7.2**–**4** in [8] or **1.5** of [11]).

Interestingly, conventional Algebra does not use ordinals as unknowns.

2.6. Lemma
If an algebra has a subset $\mathcal{O}' \neq \emptyset$ of idempotent operations, then $\mathcal{O}'$ defines a reduct such that all its elementary functions are interpolators of the former algebra.

Proof. Notice that, according to 0.3, A is the carrier also for $\mathcal{O}'$ even when $\mathcal{O}'$ only has nullary operations. In fact, no nullary operation can be idempotent in a non trivial algebra, while in a trivial one $\mathcal{O}' \neq \emptyset$ prevents a smaller carrier.

Given any G, let $\mathcal{L}'_G$ denote the set of the G-ary elementary functions of $\mathcal{O}'$. Clearly, $\mathcal{L}'_G \subseteq \mathcal{L}_G$. Then, we only have to prove that any $\ell \in \mathcal{L}'_G$ satisfies (15). This is trivial for the G-ary projections, while the induction step for composed ℓ comes from 2.3 (C) where $g \in \mathcal{O}'$ satisfies (15) by idempotency and because $g = g \cdot \boldsymbol{i}_{A^Y} = g \cdot \boldsymbol{cp} \in \mathcal{L}_Y$ by (7). ■

3. Flocks

3.0. Definitions
Given an algebra and any set Y as in 2.0, for each $M : Y \to A$ we call *M–flock in the algebra* the set $\Phi_M = \{g(M) \mid g \in \mathcal{I}_Y\} \subseteq A$ and we say that Φ_M is *the flock of M*. We will also say that such flocks are the *abstract flocks* of the algebra. If all subsets of the carrier are flocks, i.e. it has a "discrete geometry", then we say that the algebra is *flock trivial.* If for all Y the only Y-ary interpolators are the projections as in 2.1 (D), $\boldsymbol{p}: Y \twoheadrightarrow \mathcal{I}_Y$, then we say that the algebra is *interpolator trivial.*

When M is an identity on $Y = G \subseteq A$, viz. $M = \boldsymbol{i}_G: G \hookrightarrow A$, its domain G determines Φ_M. Then, we call it *the flock closure of* G (for "affine hull" see 3.7) and we denote Φ_M by $[\,G]$. Hence, by (15)

(16) $$[\,G] = \{g(\boldsymbol{i}_G) \mid g \in \mathcal{L}_G \text{ such that } g(\boldsymbol{k}_a) = a \text{ for all } a \in A\} .$$

According to [15], when the algebra has a basis U as in 1.1 and $L:X\to A$, we say that $\Phi'_L = \{\chi_c(L) \mid c$ is a flock combiner$\}$ is the *L–flock with respect to U*. Here we will also call such flocks *inner flocks* of the based algebra.

In the example 1.2 of based vector spaces with $n = 3$, according to 2.0, when $Y = 2$ and $M:Y\to A$ indexes the first two reference vectors, $M = U\cdot \boldsymbol{i}_2$, we get the line joining them: $\Phi_M = \{g(M) \mid g \in \mathcal{I}_2\} = \{c_0U_0 + c_1U_1 \mid c_0 + c_1 = 1\}$. When $Y = 3$ and the three vectors in M are not collinear, Φ_M is the plane spanning them, e.g. for $M = U$ the plane of the reference vectors. With bigger Y, we can get the whole space.

3.1. Corollaries

In every algebra the following sets are abstract flocks:

(A) *the carrier A and*

(B) *the singletons $\{a\}$ for all $a \in A$.*

(C) *In every non trivial algebra $\emptyset$ is an abstract flock: $\emptyset = [\emptyset]$.*

(D) *In every based algebra the inner flocks are abstract flocks of the corresponding (free) algebra.*

(E) *Endomorphisms preserve all abstract flocks: for all $L:Y\to A$ if $h \in \mathbb{E}_{\mathcal{O}}$, $\{h(a) \mid a \in \Phi_L\} = \Phi_M$ where $M = h\cdot L:Y\to A$.*

Proofs.

(A) Set $Y = A$ in the projection generator, $\boldsymbol{p}:A\to A^{A^A}$, to get $\boldsymbol{p}_a \in \mathcal{I}_A$ for all $a \in A$ by 2.1 (D). Then, $A = \Phi_{\boldsymbol{i}_A} = [A]$, since $\boldsymbol{p}_a(\boldsymbol{i}_A) \overset{(6)}{=} a$.

(B) As above, but set $Y = 1$ and use 2.1 (C).

(C) Take $Y = M = \emptyset$ in 3.0, then $\Phi_M = \emptyset = [\emptyset]$, since by 2.1 (B) $\mathcal{I}_\emptyset \neq \emptyset$ only for a singleton A.

(D) For all $L:X\to A$ by 2.1 (A), $a \in \Phi'_L$ if and only if $a = \chi_c(L)$ and $\chi_c \in \mathcal{I}_X$, i.e. if and only if $a \in \Phi_L$.

(E) $\{h(a) \mid a \in \Phi_L\} = \{h(g(L)) \mid g \in \mathcal{I}_Y\} \overset{(14)}{=} \{g(h\cdot L)) \mid g \in \mathcal{I}_Y\} = \Phi_M$. ■

3.2. Lemma

Given any $M:Y \twoheadrightarrow G$ with $G \subseteq A$, $\Phi_{\boldsymbol{i}_G} = \Phi_M$.

Proof. ($\Phi_{\boldsymbol{i}_G} \subseteq \Phi_M$) Let $g'(\boldsymbol{i}_G) \in \Phi_{\boldsymbol{i}_G}$, where $g' \in \mathcal{I}_G$. By 2.3 (A) $g' = \varrho_g$ for some $g \in \mathcal{I}_Y$. Hence, $g'(\boldsymbol{i}_G) = \varrho_g(\boldsymbol{i}_G) \overset{(10)}{=} g(\boldsymbol{i}_G\cdot M) = g(M) \in \Phi_M$.

($\Phi_{\boldsymbol{i}_G} \supseteq \Phi_M$) Let $g(M) \in \Phi_M$, where $g \in \mathcal{I}_Y$. By 2.3 (A) $\varrho_g = g'$ for some $g' \in \mathcal{I}_G$. Hence, by the previous passages $g(M) = g'(\boldsymbol{i}_G) \in \Phi_{\boldsymbol{i}_G}$. ■

3.3. Theorems

(A) *Abstract flocks are all and only the flock closures: for all Y and $M \in A^Y$, $\Phi_M = [G]$, where $G \subseteq A$ is the range of M.*

(B) *Flock closures form a closure system: for all $G, Y \subseteq A$*

$$G \subseteq [G]\,, \tag{17}$$

$$G \subseteq Y \quad \text{implies} \quad [G] \subseteq [Y] \quad \text{and} \tag{18}$$

$$[[G]] = [G]\,. \tag{19}$$

(C) *An algebra is flock trivial if and only if it is interpolator trivial.*

Proofs.

(A) (Only) Given such an M, we have its range G, $M{:}Y \twoheadrightarrow G$, as in 3.2, which by (16) gives the required $[G]$. (All) By definition 3.0.

(B) When $G = \emptyset$, (17) is trivial. When $G \neq \emptyset$, for every $a \in G$, by 2.1 (D) the projection $\boldsymbol{p}_a{:}A^G \to A$ is a G-ary interpolator. Then, $a \overset{(6)}{=} \boldsymbol{p}_a(\boldsymbol{i}_G) \overset{(16)}{\in} [G]$ gets (17). To get (18), notice that its premise implies that, if $J = \boldsymbol{i}_G{:}G \rightarrowtail Y$, then $\boldsymbol{i}_G = \boldsymbol{i}_Y \cdot J$. Hence, for all $g' \in \mathcal{I}_G$, by 2.3 (B) there is $g = \varsigma_{g'} \in \mathcal{I}_Y$ such that $g'(\boldsymbol{i}_G) = g'(\boldsymbol{i}_Y \cdot J) \overset{(11)}{=} \varsigma_{g'}(\boldsymbol{i}_Y) = g(\boldsymbol{i}_Y) \overset{(16)}{\in} [Y]$.

After (17), we get (19) by merely proving $[[G]] \subseteq [G]$. For $G = \emptyset$ this by 3.1 (C) is immediate, when the algebra is not trivial, while even for a trivial algebra with $[\emptyset] \neq \emptyset$ (19) comes from $[G] = A$.

Otherwise, set $Y = [G]$. Then, since for each $g \in Y$ there is $\ell \in \mathcal{I}_G$ such that $\ell(\boldsymbol{i}_G) = y$ as in (16), we can choose an indexing $L{:}Y \to \mathcal{I}_G$ such that $L_y(\boldsymbol{i}_G) = y$ for all $y \in Y$, viz. by (5) $\boldsymbol{c}_L(\boldsymbol{i}_G) = \boldsymbol{i}_Y = \boldsymbol{i}_{[G]}$. Now, for each $a \in [[G]]$ by (16) there exists $g \in \mathcal{I}_{[G]} = \mathcal{I}_Y$ such that $a = g(\boldsymbol{i}_{[G]})$. Hence, there is $\ell \in \mathcal{I}_G$ such that $a = g(\boldsymbol{c}_L(\boldsymbol{i}_G)) \overset{(0)}{=} (g \cdot \boldsymbol{c}_L)(\boldsymbol{i}_G) \overset{2.3(\mathsf{C})}{=} \ell(\boldsymbol{i}_G) \overset{(16)}{\in} [G]$.

(C) (If) For every $G \subseteq A$ the interpolator triviality gives $G = \{y \mid y \in G\} = \{\boldsymbol{i}_G(y) \mid y \in G\} \overset{(6)}{=} \{\boldsymbol{p}_y(\boldsymbol{i}_G) \mid y \in G\} = \{g(\boldsymbol{i}_G) \mid g \in \mathcal{I}_G\} = [G]$.

(Only if) By contradiction, assume that for some Y there is $g \in \mathcal{I}_Y$ that is not a projection: for some $M{:}Y \to A$, $g(M) \neq M(y)$ for all $y \in Y$.

If G is the range of M, $M{:}Y \twoheadrightarrow G$, then from $G = \{M(y) \mid y \in Y\} \overset{(6)}{=} \{\boldsymbol{p}_y(M) \mid y \in Y\}$ we get $G \subset \{\boldsymbol{p}_y(M) \mid y \in Y\} \cup \{g(M)\} = \{\ell(M) \mid \ell = g \text{ and } \ell = \boldsymbol{p}_y \text{ for some } y \in Y\} \overset{2.1(\mathsf{D})}{\subseteq} \Phi_M \overset{(\mathsf{A})}{=} [\,G]$.

Yet, this proper containment, $[\,G] \supset G$, by (19) implies that, for no $G' \subseteq A$, $G = [\,G']$. Hence, by (A) not all subsets of A are abstract flocks. ∎

3.4. Corollaries

(A) *The flock closure system is invariant under automorphisms: for all* $j \in \mathbb{E}_{\mathcal{O}}$ *with* $j{:}A \rightarrowtail\!\!\twoheadrightarrow A$ *and* $G \subseteq A$, $\{j(d) \mid d \in [G]\} = [\{j(a) \mid a \in G\}]$.

(B) *All subalgebra carriers are flocks.*

Proofs.

(A) $\{j(d) \,|\, d \in [\,G]\} \overset{3.0}{=} \{j(d) \mid d \in \Phi_{\boldsymbol{i}_G}\} \overset{3.1(\mathsf{E})}{=} \Phi_{j\cdot\boldsymbol{i}_G} \overset{3.3(\mathsf{A})}{=} [\,\{j(a) \,|\, a \in G\}]$.

(B) In (16) take G as the carrier of any subalgebra to get by 1.6 (B) that $[\,G] \subseteq G$. Then, use (17). ∎

3.5. Examples

By 3.3 (C) and 3.4 (B) in a universal algebra the sparseness of flocks might vary between the ones of subsets and of subalgebras respectively. In (A) and (B) we show that we can reach such bounds, whereas in (C,D,E) finite Boolean algebras and other similar algebras will provide us with an intermediate case, close to the well-known vector space case.

(A) By 2.4 (B) the word monoid is an interpolator trivial algebra. Therefore, its flocks are all carrier subsets by 3.3 (C).

(B) Consider any (finitary) lattice or also the algebra with the operations $i, s{:}A^A \to A$ that perform the infima and suprema of the ranges of their arguments in any complete lattice on A. Since here we disregard effective computability, we are rewording complete lattices by the set-ary operations of 0.3. (For a presentation of their elementary functions as term functions of indexed algebras with "operations" like as $\bigwedge, \bigvee{:}\mathcal{P}(A) \to A$, see [7].)

These operations are idempotent. Hence, all elementary functions are, as $\mathcal{O}' = \mathcal{O}$ in 2.6. Then, by 3.3 (A) and (16) all flocks are subalgebra carriers.

(C) Consider finite Boolean algebras, e.g. the set algebras on $A = \mathcal{P}(n)$ for some natural number n. There, flocks are closed under union as it clearly follows from the minterm characterization of interpolators in 2.4 (A). (Dually, one might easily get an intersection closure from a maxterm characterization.) Yet, one cannot always get a flock closure $[\,G]$ by unions (nor intersections) from the sets in G.

In fact, in $A = \mathcal{P}(8)$ take $G = \{\{1,3,5,7\},\{2,3,6,7\},\{4,5,6,7\}\}$, which is a basis set, and in $\mathcal{I}_G$, according to 2.4 (A), take the interpolator corresponding to the conventional Boolean expression $(x \cap y \cap z) \cup (x \cap \overline{y}) = (x \cap y \cap z) \cup (x \cap \overline{y} \cap z) \cup (x \cap \overline{y} \cap \overline{z})$, where the variables alphabetically match this listing order of sets in G. Then, (16) gets that $A \supset [\,G] \ni \{1,5,7\}$, a set that cannot come from G without complements. Hence, the "coplanarity" kinship of $\{1,5,7\}$ with the three sets of G is not due to the inclusion lattice.

On the contrary, when G has less than three elements, 2.4 (A) easily shows that all $[\,G]$ comes from such a lattice. For a doubleton G by (16) the "collinear" sets of its two sets are their intersection and union.

Such a G also shows that $[\,G]$ might not have as many elements as $\mathcal{P}(G)$. In fact, $\mathcal{P}(G)$ has four elements corresponding to the Boolean algebra with two atoms, whereas $[\,G]$, when one of the elements of G stays below the other, has two elements corresponding to the single atom Boolean algebra.

(D) The previous example introduces a first property of flocks in general finite Boolean algebras and in the "set-ary Boolean algebras" that correspond to any complemented atomic distributive lattice, which consist of its Boolean finitary operations together with the infimum and supremum operations as in (B) (as for Boolean set algebras).

Given two elements $a, b \in A$, we consider their lattice interval $[a, b\,] = \{c \mid a \le c \le b\}$. Then, *flock closures stay in the intervals between the corresponding infima and suprema:* $[G] \subseteq [\bigwedge G, \bigvee G\,]$ *for all* $G \subseteq A$.

Proof. For an empty G we get the equality, since by 3.1 (C) and 2.1 (B)

$$[\,G] = [\,\emptyset] = \begin{cases} A & \text{when } A \text{ is singleton and} \\ \emptyset & \text{otherwise} \end{cases}, \tag{20}$$

which is the same that occurs to $[\bigwedge G, \bigvee G\,] = [1, 0\,]$. Then, consider $G \neq \emptyset$.

Extend the min/max-term characterization in 2.4 (A) as follows. On A consider the identity and the negation, $\boldsymbol{i}_A, \neg : A \to A$, to get the set $\{\boldsymbol{i}_A, \neg\}^G$ of the "sign" functions $m : G \to \{\boldsymbol{i}_A, \neg\}$, and its subset $\mathcal{M} \subset \{\boldsymbol{i}_A, \neg\}^G$ of non constant $m : G \twoheadrightarrow \{\boldsymbol{i}_A, \neg\}$. ($\mathcal{M} = \emptyset$, when G is at most singleton.) Because of completeness we again get functions t_G with values $t_G(m) = \bigwedge\{m(a) \mid a \in G\}$, which correspond to the minterms, and the maxterm ones with values $T_G(m) = \bigvee\{m(a) \mid a \in G\}$ for all $m \in \{\boldsymbol{i}_A, \neg\}^G$.

Then, use (16) as in 2.4 (A) to characterize $[\,G]$:

$$(21)\quad c \in [\,G] \quad \text{iff} \quad c = \bigwedge G \vee \bigvee\{\, t_G(m) \mid m \in \mathcal{M}'\} \text{ for some } \mathcal{M}' \subseteq \mathcal{M}$$

$$(22)\qquad \text{or} \quad \text{iff} \quad c = \bigvee G \wedge \bigwedge\{T_G(m) \mid m \in \mathcal{M}'\} \text{ for some } \mathcal{M}' \subseteq \mathcal{M}\ .$$

The lower bound for c comes from (21) and the upper one from (22). ■

This property allows us to replace (16) with a specific characterization: *in any set-ary Boolean algebra on A the flock closure of any $G \subseteq A$ is*

$$(23)\qquad [\,G] = \{g(\boldsymbol{i}_G) \mid g \in \mathcal{L}_G \text{ such that } \bigwedge G \le g(\boldsymbol{i}_G) \le \bigvee G\}.$$

Proof. When $G = \emptyset$, we get this equality by (20) according to our $\mathcal{L}_\emptyset$ in 1.0 and to $[\bigwedge G, \bigvee G\,] = [1, 0\,]$. Then, consider $G \neq \emptyset$.

After the previous property we can merely show that, for each $g(\boldsymbol{i}_G)$ in (23), $g(\boldsymbol{i}_G) = g'(\boldsymbol{i}_G)$ for some $g' \in \mathcal{L}_G$ that satisfies (15) as in (16). Since $g \in \mathcal{L}_G$, it has a canonical minterm form: $g(\boldsymbol{i}_G) = \bigvee\{t_G(m) \mid m \in \mathcal{J}\}$ for some $\mathcal{J} \subseteq \{\boldsymbol{i}_A, \neg\}^G$. Hence, according to (21), we show that either $g' = g$, when $\mathcal{J} = \{\boldsymbol{k}_{\boldsymbol{i}_A}\} \cup \mathcal{M}'$ with $\mathcal{M}' \subseteq \mathcal{M}$, or we can get g' with such a $\mathcal{J}$.

$\mathcal{J} \neq \{\boldsymbol{k}_{\boldsymbol{i}_A}\} \cup \mathcal{M}'$ implies that the constant $m' = \boldsymbol{k}_\neg{:}G \to \{\neg\}$ stays in $\mathcal{J}$ or the $m'' = \boldsymbol{k}_{\boldsymbol{i}_A}{:}G \to \{\boldsymbol{i}_A\}$ does not. In the former case, we can merely delete m' from $\mathcal{J}$. In fact, $\neg(\bigvee G) = \bigwedge\{\neg a \mid a \in G\} = t_G(m') \le g(\boldsymbol{i}_G) \le \bigvee G$ by (23). This implies that $\bigvee G = 1$ and $\neg(\bigvee G) = t_G(m') = 0$.

In the latter case, dually, we can add m''. In fact, if $g'(\boldsymbol{i}_G) = g(\boldsymbol{i}_G) \vee t_G(m'')$, then $g'(\boldsymbol{i}_G) = g(\boldsymbol{i}_G)$, since $t_G(m'') = \bigwedge G \le g(\boldsymbol{i}_G)$ by (23). ■

Therefore, "set-ary flocks" are intersections between subalgebra carriers and intervals: for all $G \subseteq A$, $[\,G] = [\![G]\!] \cap [\bigwedge G, \bigvee G\,]$, where $[\![G]\!] = \{g(\boldsymbol{i}_G) \mid g \in \mathcal{L}_G\}$. This characterization easily becomes a necessary condition for the "finitary flocks", the flocks of the (finitary) Boolean algebra of our lattice. In fact, the elementary functions of the latter also are of the former.

Furthermore, when $[\,G] \neq \emptyset$, every such a set-ary flock carries a Boolean algebra, where the two constants are the previous bounds (which coincide for a singleton $[\,G]$). In fact, on $[\,G]$ we can define other *local Boolean operations* $\sqcap$, $\sqcup$ and $\boxminus$, e.g. by respectively restricting $\wedge$ and $\vee$ from A

to $[G]$ and "localizing" $\neg$ as the restriction of the relative complement, $\boxminus c = ((\neg c) \vee \bigwedge G) \wedge \bigvee G = ((\neg c) \wedge \bigvee G) \vee \bigwedge G$ for all $c \in [G]$. Then, we show that *such local Boolean operations are Boolean operations on this flock.*

Proof. The closures of $[G]$ under $\sqcap$ and $\sqcup$, i.e. under $\wedge$ and $\vee$, follow from (21) and (22) as for the set case mentioned in (C). The one under $\boxminus$ from (23) and its (double) definition.

The distributive lattice and absorbtion identities for $\sqcap$ and $\sqcup$ follow from the ones for $\wedge$ and $\vee$. The complement ones from (23): $c \sqcap (\boxminus c) = c \wedge (\boxminus c) = (c \wedge (\neg c \vee \bigwedge G)) \wedge \bigvee G = (0 \vee (c \wedge \bigwedge G)) \wedge \bigvee G = \bigwedge G \wedge \bigvee G = \bigwedge G$ and $c \sqcup (\boxminus c) = c \vee (\boxminus c) = (c \vee (\neg c \vee \bigwedge G)) \wedge (c \vee \bigvee G) = 1 \wedge \bigvee G = \bigvee G$ for all $c \in [G]$. ■

(E) In a general Boolean Algebra, which might lack $\bigwedge G$ or $\bigvee G$, the loss of such local constants for (D) also implies the loss of (local) negation. Still, the lattices on flocks, as we found in the set algebra of (C), persist (even without (21) and (22)). In fact, by (16) and 2.6 with $\mathcal{O}' = \{\wedge, \vee\}$ *every flock in a Boolean algebra is closed under meets and joins.*

3.6. Two open problems

(A) A problem is to characterize the closure systems that are flock systems. In fact, we can represent any such a closure system on A by an algebra on A. As the well-known case of vector spaces and the Boolean one in 3.5 show, this representation can be convenient even in the finite.

(B) Another problem concerns the generalization of Segre transformations (**3.3** of [16]) from based algebras to all universal algebras, as we mentioned in 0.0. When G is the basis set of a based algebra, $U{:}X \twoheadrightarrow G$, the flock $[G]$ by (17) contains all reference elements. By 3.1 (D) and 3.3 (A) this "reference flock" is an inner flock, which any Segre transformation has to preserve into another reference flock (ibid.).

As shown in **3.1** (C) of [16] and **3.4** ibid., this preservation implies the one of all inner flocks. Hence, requiring that a bijection $\sigma'{:}A \rightarrowtail\!\!\!\twoheadrightarrow B$ between the carriers of any two universal algebras preserves all flocks by 2.1 (A) generalizes the reference flock preservation for based algebras. (Such a σ' has to admit of functions $T_{\sigma'}{:}\mathcal{P}(A) \to \mathcal{P}(B)$ such that $a \in [G]$ if and only if $\sigma'(a) \in [T_{\sigma'}(G)]$ for all $G \in \mathcal{P}A$. By (16) this requires a correspondence between the G-ary interpolators and the $T_{\sigma'}(G)$-ary ones.)

Yet, this generalization might be weak, because now we do not require that from a possible reference flock we reach another reference one. Namely, it might not guarantee that transformations between algebras depend on possible bases. Therefore, one might ask *which are the interpolator correspondences that guarantee it* (if any).

Together with the generalization of dilatations, we mentioned in 2.2, such a correspondence might serve to formalize an abstract Segre transformation for all algebras. It might be another step toward replacing the isomorphism-like notions, which fail to formalize sameness and invariance as the counterexample in **3.6** of [15] did show.

3.7. "Semi-affine lattice"

(or "semi-affinity lattice") might be a name for the intersection complete lattice ensuing from the closure system of 3.3 (B). Prefix "semi" should recall that in universal algebras not all nonempty flocks are congruence classes (e.g. 1.2 (B) of [14] shows this in the word monoid of 2.4 (B)), whereas in vector spaces they are.

In Universal Algebra too congruence classes were useful, theoretically for term algebras and even experimentally for the term algebra on words of 2.4 (B), due to applications of Formal Language theory to lexical recognition.

Hence, one might well think that our generalization of Affine Geometry splits its objects into both congruence classes and our flocks, in spite of the lack of a lattice of congruence classes ($\emptyset$ is not one of them, though two of them can be disjoint.) and in spite of Linear Algebra treatments that avoid such classes, see the one in [1].

For the same motivations here we prefer "flock" to "affine subspace".

Acknowledgments

Klaus Denecke provided the author with useful hints. The Italian ministry of universities partly supported this research.

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Received 17 April 2009
Revised 20 December 2009