

## SEMIGROUP OF CONTRACTIONS OF WREATH PRODUCTS OF METRIC SPACES\*

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### Abstract

In this paper semigroups of contractions of metric spaces are considered. The semigroup of contractions of the wreath product of metric spaces is calculated.

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## 1. Introduction

In articles [1, 2] F. Harary and G. Sabidussi introduced a new construction of composition of graphs. Later this construction was called the wreath product of graphs.

A notion of the wreath product of metric spaces was introduced in [4] analogously to the Sabidussi's and Harary's one.

It is known [4] that the isometry group of the wreath product of metric spaces  $X$  and  $Y$  is isomorphic as a permutation group to the wreath product of isometry groups of spaces  $X$  and  $Y$ .

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With every metric space we can associate a few transformations semigroups: the semigroup of partial isometries, the semigroups of 1-Lipschitzian transformations (semigroup of contractions) and the semigroup of partial 1-Lipschitzian transformations. We shall consider semigroups of contractions of metric spaces.

The main result of this report is the following one

**Theorem 1.** *The semigroup of contractions of wreath product of metric spaces  $X$  and  $Y$  is isomorphic as a transformation semigroup to the wreath product of semigroups of contractions of spaces  $X$  and  $Y$*

$$Ctr(X wr Y) \simeq Ctr X \wr Ctr Y.$$

## 2. Preliminaries

Let  $(X, d_X)$  be a metric space. A contraction (or an 1-Lipschitzian transformation) of  $X$  is a mapping  $f : X \rightarrow X$  such that for arbitrary  $a, b \in X$  the inequality

$$d_X(f(a), f(b)) \leq d_X(a, b)$$

holds.

**Example 1.** Let  $z$  be some point in  $X$ . It is clear that a mapping  $f : X \rightarrow X$  such that  $f(x) = z$  for every point  $x \in X$  is a contraction of  $(X, d_X)$ .

**Example 2.** Assume that there exists

$$\min_{u, v \in X, u \neq v} \{d_X(u, v)\} = q,$$

then  $q > 0$ . Let  $a, b$  be points of  $X$  such that  $d_X(a, b) = q$ . Define a mapping  $f : X \rightarrow X$  by the rule:

$$f(a) = a, \quad f(b) = b$$

and  $f(x) \in \{a, b\}$  for other points  $x \in X$ . Then  $f$  is a contraction of  $(X, d_X)$ .

The set of all contractions of the space  $(X, d_X)$  forms a semigroup under composition. We call it the *semigroup of contractions* of metric space  $(X, d_X)$  and denote by  $Ctr X$ .

**Example 3.** Let  $(X, d_X)$  be a metric space with equidistant metric i.e. there exists some positive  $c$  such that  $d_X(a, b) = c$  for all distinct  $a, b \in X$ . Then the semigroup of contractions of  $(X, d_X)$  is the full transformations semigroup  $T_X$ .

Observe, that the isometry group  $IsX$  of the space  $X$  is the subgroup of the semigroup of contractions  $CtrX$  of this space.

**Proposition 1.** *Let  $f$  be a contraction of metric space  $(X, d_X)$ . If  $f$  is one-to-one and the inverse mapping  $f^{-1}$  is also a contraction then  $f$  is an isometry of the space  $(X, d_X)$ .*

The proof of this proposition is straightforward.

Metric spaces  $(X, d_X)$  and  $(Y, d_Y)$  are called isomorphic ([3]) if there exists a scale, that is a strictly increasing continuous function  $s : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ ,  $s(0) = 0$ , such that  $d_X = s(d_Y)$ .

It is easy to see that if metric spaces  $(X, d_X)$  and  $(Y, d_Y)$  are isomorphic then their semigroups of contraction  $CtrX$  and  $CtrY$  are isomorphic.

Assume that there exists a positive number  $r$ , such that for arbitrary points  $x_1, x_2 \in X$ ,  $x_1 \neq x_2$ , the inequality  $d_X(x_1, x_2) \geq r$  holds. Additionally assume that the diameter  $diamY$  of the space  $(Y, d_Y)$  is finite. Then fix a scale  $s(x)$  such that

$$(1) \quad diam(s(Y)) < r.$$

Define a metric on the cartesian product  $X \times Y$  by the rule:

$$(2) \quad \rho_s((x_1, y_1), (x_2, y_2)) = \begin{cases} d_X(x_1, x_2), & \text{if } x_1 \neq x_2 \\ s(d_Y(y_1, y_2)), & \text{if } x_1 = x_2 \end{cases}.$$

We call  $(X \times Y, \rho_s)$  the wreath product of metric spaces  $X$  and  $Y$  with scale  $s$  and denote it by  $Xwr_sY$ .

**Proposition 2** ([4]). *Let  $s_1$  and  $s_2$  be scales such that the inequality (1) holds. Then spaces  $(X \times Y, \rho_{s_1})$  and  $(X \times Y, \rho_{s_2})$  are isomorphic.*

Since the wreath product of metric spaces is unique up to isomorphism we assume in the sequel that the corresponding scale is fixed. Denote the wreath product of metric spaces  $X$  and  $Y$  by  $XwrY$ .

For the definition of the wreath product of transformation semigroups see [5].

### 3. Proof of the main theorem

At first let us prove that an arbitrary element

$$\varphi = [g, h(x)] \in CtrX \wr CtrY$$

defines a contraction of  $XwrY$ . The definition of the wreath product of transformation semigroup ([5]) implies that  $\varphi$  acts on  $X \times Y$ . We shall see that  $\varphi$  does not increase the metric  $\rho_s$ . Indeed,

$$\begin{aligned} \rho_s(\varphi(x_1, y_1), \varphi(x_2, y_2)) &= \\ &= \rho_s((x_1^g, y_1^{h(x_1)}), (x_2^g, y_2^{h(x_2)})) = \begin{cases} d_X(x_1^g, x_2^g), & \text{if } x_1^g \neq x_2^g \\ s(d_Y(y_1^{h(x_1)}, y_2^{h(x_2)})), & \text{if } x_1^g = x_2^g. \end{cases} \end{aligned}$$

Since  $g \in CtrX$ , it follows that  $d_X(x_1^g, x_2^g) \leq d_X(x_1, x_2)$ . Therefore, if  $x_1 \neq x_2$  and  $x_1^g \neq x_2^g$  then

$$\rho_s(\varphi(x_1, y_1), \varphi(x_2, y_2)) = d_X(x_1^g, x_2^g) \leq d_X(x_1, x_2) = \rho_s((x_1, y_1), (x_2, y_2)).$$

Using (1) and (2), we get that if  $x_1 \neq x_2$  and  $x_1^g = x_2^g$  then

$$\begin{aligned} \rho_s(\varphi(x_1, y_1), \varphi(x_2, y_2)) &= \\ &= s(d_Y(y_1^{h(x_1)}, y_2^{h(x_2)})) \leq r \leq d_X(x_1, x_2) = \rho_s((x_1, y_1), (x_2, y_2)). \end{aligned}$$

For  $g \in CtrX$  the equality  $x_1 = x_2$  implies  $x_1^g = x_2^g$ . Note that  $t$  is a contraction of  $Y$  iff  $t$  is a contraction of  $s(Y)$ . Then  $h(x_1) = h(x_2)$ , that is  $h(x_1)$  and  $h(x_2)$  define the same contraction  $t$  of  $Y$ . Hence,

$$\begin{aligned} s(d_Y(y_1^{h(x_1)}, y_2^{h(x_2)})) &= \\ &= s(d_Y(y_1^{h(x_1)}, y_2^{h(x_1)})) \leq s(d_Y(y_1, y_2)) = \rho_s((x_1, y_1), (x_2, y_2)). \end{aligned}$$

Therefore we have

$$\rho_s(\varphi(x_1, y_1), \varphi(x_2, y_2)) \leq \rho_s((x_1, y_1), (x_2, y_2)).$$

This means that  $\varphi$  defines a contraction of  $XwrY$ .

Now let us prove that for any contraction  $\varphi$  of  $XwrY$  there exist  $g \in \text{Ctr}X$  and  $h(x) \in \text{Ctr}Y^X$  such that  $[g, h(x)]$  acts on  $X \times Y$  as  $\varphi$  does. Let the function  $\varphi$  map some point  $(x_1, y_1)$  to a point  $(x_2, y_2)$ . Using (1) and (2) we obtain that the function  $\varphi$  maps any point of the form  $(x_1, \star)$  to a point of the form  $(x_2, \star)$ . It follows that  $\varphi$  acts as a contraction on each isometric copy  $s(Y)_x$ ,  $x \in X$ . In each copy  $s(Y)_x$  chooses a point  $y_x$ . Then  $\varphi$  is a contraction on  $\{y_x, x \in X\}$ . This implies that there exist  $g \in \text{Ctr}X$  and  $h(x) \in \text{Ctr}s(Y)^X$ , where  $[g, h(x)]$  acts on  $X \times Y$  as  $\varphi$  does. Since  $\text{Ctr}(s(Y)) \simeq \text{Ctr}Y$ , it follows that we can consider  $[g, h(x)]$  as an element of  $\text{Ctr}X \wr \text{Ctr}Y$ . This completes the proof of the theorem.

## 4. Corollary

Now let  $(X_1, d_1), (X_2, d_2), \dots, (X_n, d_n), n \geq 2$  be a finite sequence of metric spaces. Assume that the diameters of the spaces  $(X_2, d_2), (X_3, d_3), \dots, (X_n, d_n)$  are finite. Additionally assume that there exists a finite sequence of positive numbers  $r_1, r_2, \dots, r_{n-1}$ , such that for arbitrary points  $a, b \in X_i$ ,  $a \neq b$ , the inequalities  $d_i(a, b) \geq r_i$  hold,  $1 \leq i \leq n-1$ .

**Proposition 3** [4]. *Let  $(X_1, d_1)$ ,  $(X_2, d_2)$  and  $(X_3, d_3)$  be metric spaces as above. Then the spaces  $(X_1wrX_2)wrX_3$  and  $X_1wr(X_2wrX_3)$  are isomorphic.*

Using proposition (3) we introduce the  $n$ -iterated wreath product of metric spaces.

First fix a finite sequence of scales  $s_i(x)$ ,  $2 \leq i \leq n$  such that

$$(3) \quad \text{diam}(s_2(X_2)) < r_1,$$

$$\text{diam}(s_3(X_3)) < s_2(r_2),$$

.....

$$\text{diam}(s_n(X_n)) < s_{n-1}(r_{n-1}).$$

Define a metric  $\rho_{s_2, \dots, s_n}$  on the cartesian product  $X_1 \times X_2 \times \dots \times X_n$  by the rule:

$$(4) \quad \rho_{s_2, \dots, s_n}((a_1, \dots, a_n), (b_1, \dots, b_n)) =$$

$$= \begin{cases} d_1(a_1, b_1), & \text{if } a_1 \neq b_1; \\ s_1(d_2(a_2, b_2)), & \text{if } a_1 = b_1 \text{ and } a_2 \neq b_2; \\ s_2(d_3(a_3, b_3)), & \text{if } a_1 = b_1, a_2 = b_2, a_3 \neq b_3; \\ \dots \quad \dots \quad \dots & \\ s_{n-1}(d_n(a_n, b_n)), & \text{if } a_1 = b_1, \dots, a_{n-1} = b_{n-1}. \end{cases}$$

where  $a_1, b_1 \in X_1, \dots, a_n, b_n \in X_n$ .

We call  $(X_1 \times X_2 \times \dots \times X_n, \rho_{s_2, \dots, s_n})$  the *n-iterated wreath product* of the spaces  $(X_1, d_1), (X_2, d_2), \dots, (X_n, d_n)$  with respect to the sequence of scales  $s_i(x)$ ,  $2 \leq i \leq n$ , and denote it by

$$X_1 wr_{s_1} X_2 wr_{s_2} X_3 wr_{s_3} \dots wr_{s_n} X_n.$$

Let  $X_1, X_2, \dots, X_n$  be metric spaces as above. Fix two finite sequences of scales  $s_i$ ,  $1 \leq i \leq n$  and  $g_j$ ,  $1 \leq j \leq n$  such that the inequalities (3) hold for each of this sequences. Then from Propositions 3 and 2 we obtain

**Proposition 4.** *The spaces*

$$X_1 wr_{s_1} X_2 wr_{s_2} X_3 wr_{s_3} \dots wr_{s_n} X_n$$

and

$$X_1 wr_{g_1} X_2 wr_{g_2} X_3 wr_{g_3} \dots wr_{g_n} X_n$$

are isomorphic.

Since the *n-iterated wreath product* of metric spaces is unique up to isomorphism we assume in the sequel that the corresponding sequence of scales  $s_i$ ,  $1 \leq i \leq n$  is fixed. Denote the *n-iterated wreath product* of metric spaces  $X_1, X_2, \dots, X_n$  by

$$X_1 wr X_2 wr X_3 wr \dots wr X_n.$$

Note, that we can consider the space  $X_1 wr X_2 wr X_3 wr \dots wr X_n$  as the space

$$(\dots(((X_1 wr X_2) wr X_3) wr) \dots wr X_n).$$

From Proposition 3 and Theorem 1 it follows:

**Theorem 2.** *The semigroup of contractions of the  $n$ -iterated wreath product of the metric spaces  $X_1, X_2, \dots, X_n$  is isomorphic as a transformation semigroup to the  $n$ -iterated wreath product of semigroups of contractions of the spaces  $X_1, X_2, \dots, X_n$*

$$Ctr(X_1 wr X_2 wr \dots wr X_n) \simeq Ctr X_1 \wr Ctr X_2 \wr \dots \wr Ctr X_n.$$

## 5. Example

Let  $k_1, k_2, \dots, k_n$  be a finite sequence of natural numbers and let  $(Y_{k_i}, d_i)$  be a finite sequence of metric spaces such that for any  $1 \leq i \leq n$  the following conditions hold:

- $|Y_{k_i}| = k_i$ ,
- $d_i(a, b) = 1$  for distinct points  $a, b \in Y_{k_i}$ .

Fix a real number  $\eta \in (0, 1)$ . Define a finite sequence of scales  $s_i(t) = \eta^{i-1} \cdot t$ ,  $2 \leq i \leq n$ . The functions from this sequence satisfy inequalities (3). Then we can consider the space

$$Y_1 wr_{s_2} Y_2 wr_{s_3} Y_3 wr_{s_4} \dots wr_{s_n} Y_n.$$

This space consists of  $k_1 k_2 \dots k_n$  tuples of the form  $(u_1, \dots, u_n) \in \prod_{i=1}^n Y_{k_i}$ . The distance between distinct points of this space is defined by the following rule:

$$d((u_1, \dots, u_n), (v_1, \dots, v_n)) = \eta^l \text{ if } u_1 = v_1, \dots, u_{l-1} = v_{l-1}, u_l \neq v_l.$$

Denote this space by  $B(k_1, \dots, k_n)$ .

The well-known folklore result says that the isometry group of this space is isomorphic as a permutation group to the wreath product of the symmetric groups  $S_{k_1}, S_{k_2}, \dots, S_{k_n}$ .

Another way to describe the space  $B(k_1, \dots, k_n)$  is as follows.

Recall, that a connected simple (non directed, without loops) graph is a *tree* if it has no cycles. It is easy to see that if a graph  $T$  is a tree then for any two vertices of  $T$  there exists a unique path connecting them. A *rooted tree*  $(T, v_0)$  is a tree with a fixed vertex  $v_0$  named *the root* of the tree. For every nonnegative integer  $l$  the *level number*  $l$  ( $l$ -th level) is the set  $V_l$  of all vertices  $v \in V(T)$  such that the length of the path between  $v$  and  $v_0$  in  $T$  is equal to  $l$ . Respectively, the level number 0 contains only the root  $v_0$ . A homogeneous  $(k_1, k_2, \dots, k_n)$ -tree is a rooted tree such that each vertex of  $(i-1)$ -th level is connected with exactly  $k_i$  vertices of  $i$ -th level,  $1 \leq i \leq n$ .

Let  $(T, v_0)$  be a finite homogeneous  $(k_1, k_2, \dots, k_n)$ -rooted tree. Then all vertices  $v \in V(T)$  have degree 1. We can introduce a natural ultrametric on  $V_n$  putting

$$\rho(v_i, v_j) = \eta^{(s+1)}, v_i \neq v_j$$

and  $\rho(v_i, v_j) = 0, v_i = v_j$ , where  $s$  is the length of the maximal common part of the paths connecting vertices  $v_i$  and  $v_j$  with the root.

- The space  $(V_n, \rho)$  is isometric to the space  $B(k_1, \dots, k_n)$ .

An endomorphism of a homogeneous rooted tree is called elliptic (due to J. Rhodes [7]) if it preserves numbers of levels.

- The semigroup of contractions of  $(V_n, \rho)$  is isomorphic to the semigroup of elliptic endomorphisms of rooted tree  $(T, v_0)$ .

From theorem 2 it follows

- The semigroup of contractions of the space  $B(k_1, \dots, k_n)$  is isomorphic as a transformation semigroup to the  $n$ -iterated wreath product of transformations semigroups  $T_{k_1}, T_{k_2}, \dots, T_{k_n}$ :

$$\text{Ctr}(B(k_1, \dots, k_n)) \simeq T_{k_1} \wr T_{k_2} \wr \dots \wr T_{k_n}.$$

This result immediately implies from [6].

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