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KNOWLEDGE BASES AND AUTOMORPHIC EQUIVALENCE OF MULTI-MODELS VERSUS LINEAR SPACES AND GRAPHS

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Abstract

The paper considers an algebraic notion of automorphic equivalence of models and of multi-models. It is applied to the solution of the problem of informational equivalence of knowledge bases. We show that in the case of linear subjects of knowledge the problem can be reduced to the well-known in computational group theory problems about isomorphism and conjugacy of subgroups of a general linear group.

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1. Introduction, motivation

The key problem considered in this paper came from the areas of knowledge bases and knowledge mining. At the moment we don't need to formulate what the term "knowledge base" precisely means and thus can stay on descriptive positions. It is clear that knowledge bases obtain, keep and process information according to some, may be very sophisticated, rules and algorithms.

Moreover, a knowledge base is not a static collection of information but a dynamic resource that may itself have the capacity to learn, as part of an artificial intelligence component. These kinds of knowledge bases can suggest solutions to problems sometimes based on feedback provided by the user, and are capable of learning from experience (like an expert system). Knowledge representation, automated reasoning, argumentation and other areas of artificial intelligence are tightly connected with knowledge bases.

Suppose now that we have two knowledge bases. *The major problem under consideration is to find out whether these knowledge bases are equivalent.*

There are many ways to define equivalence relations on the set of all knowledge bases because one can consider such complicated systems with respect to their different features.

We study knowledge bases equivalence from the point of view of their informational abilities. In other words, our goal is to investigate the notion of *informational equivalence of knowledge bases*. Following the common sense intuition, two knowledge bases are informationally equivalent if all the information that can be retrieved from one of them could be also obtained from the other one and vice versa. It is clear that this goal cannot be achieved basing on intuition only. In order to describe precisely the objects we are dealing with one need to put the objects in question on some formal basis. This is done in [9]–[11] and in Section 2. The notion of informational equivalence is also worked out in detail there.

It should be noticed that *a priori* it is not clear whether the problem of informational equivalence verification is *algorithmically decidable*. If we concentrate on finite objects then the reasonable answer is yes, we can build the step-by-step procedure that solves the problem. But when we consider infinite objects it may be problematic. By many reasons, knowledge bases provide examples of infinite objects. Finiteness of the subject of knowledge is a quite natural restriction. Despite this restriction the corresponding knowledge bases can be actually infinite. Using an analogy with semigroup theory one can notice that there exist very small finite semigroups such that the corresponding knowledge turn to be infinite and cannot be reduced to finite (see [15] and the bibliography therein).

A way to show that the problem of informational equivalence of knowledge bases is algorithmically solvable is to build some system of finite objects (invariants), such that the equivalence of those objects would imply the equivalence of the corresponding knowledge bases. This idea gave rise to the notion of *automorphic equivalence of multi-models*.

It was proved that *two knowledge bases with finite multi-models are informationally equivalent if and only if the corresponding multi-models are automorphically equivalent* [9]. The proof of the theorem is essentially grounded on the Galois theory of relations developed by M. Krasner [5].

This paper deals with the properties of automorphically equivalent multi-models. To make it self-contained we provide the reader with all necessary definitions. We also recall some properties of automorphically equivalent multi-models using the examples of multi-models built on graphs. The necessary algebraic background can be found in [7, 8, 11].

The main emphasis of the paper is placed on the situation when subjects of knowledge possess the structure of finite dimensional linear spaces over a finite field. The linearity condition allows us to reduce a general algorithm of automorphic equivalence verification to conjugation condition for two matrix groups. In the paper we prove the corresponding theorem and outline the ways how to use the existing algorithms from computational algebra in order to check the informational equivalence of linear knowledge bases.

2. DEFINITIONS

Definition 2.1. We define a model as a triple (D, Φ, f), where D is a data domain, that is, an algebra in a variety of algebras Θ, Φ is a set of symbols of relations, f is an interpretation of these symbols as relations in D, i. e., if $\varphi \in \Phi$ is an n-ary relation in Φ, then $f(\varphi)$ is a subset of the Cartesian product D^n. Moreover, D may be a multi-sorted set, i.e. , $D = \{D_i, i \in \Gamma\}$, where Γ is a set of sorts [7, 14].

Definition 2.2. A multi-model is a triple (D, Φ, F), where D is a data domain (an algebra), Φ is a set of symbols of relations, F is a set of interpretations of Φ on D [9].

A model (D, Φ, f) is a particular case of a multi-model (D, Φ, F). The definition of multi-model takes into account the fact that instances f can change, for example under some circumstances or according to some rules. All these f constitute the set F. In general multi-models may be infinite but we consider only the finite ones.

Now we are going to relate knowledge bases and multi-models. We assume that every knowledge under consideration is represented by three components:

1) *The description of knowledge.* It is a syntactical part of knowledge, written out in the language of the given (usually First Order) logic.

2) *The subject of knowledge* which is an object in the given applied field, i.e., an object for which we determine knowledge.

3) *The content of knowledge* (its semantics).

Subject of knowledge is represented by a model (D, Φ, f) where D is an algebra (a set with a system of necessary algebraic operations), Φ is a set of symbols of relations naturally reflecting the problem in question, and f is a possible interpretation of each symbol f from Φ in the given algebra D. Interpretation f depends on the state of the subject in the given moment. Since the states may change, the multi-model (D, Φ, F) where F is a set of various interpretations f is considered. *A knowledge base over the given multi-model (D, Φ, F) is denoted by $KB(D, \Phi, F)$.*

Let now T be a set of formulas describing the knowledge from some field or on some topic. Denote by $T^f = A$ the content of knowledge in the state f.

Consider a category of logical knowledge description which we denote by $L_{\Phi\Theta}$. Objects of this category have the form (X, T),where X is a finite set of variables and T is a set of First Order formulas written in the variables from X. Morphisms in $L_{\Phi\Theta}$ are defined by the means of algebraic logic (see [11]).

Consider also the categories $K_{\Phi\Theta}(f)$ of knowledge content, where f runs the set of interpretations Φ. Their objects have the form (X, A), where A is a subset in an affine space over the given model and morphisms are naturally defined.

A knowledge base $KB = KB(D, \Phi, F)$ consists of the category of knowledge description $L_{\Phi\Theta}$, and the categories of knowledge content $K_{\Phi\Theta}(f)$. They are related by the contra-variant functors

$$Ct_f : L_{\Phi\Theta} \to K_{\Phi\Theta}(f).$$

These functors Ct_f transform knowledge description to content of knowledge.

We view the description T as a *query* to a knowledge base, and T^f as a *reply to this query.*

Let $\mathrm{KB}(D_1, \Phi_1, F_1)$ and $\mathrm{KB}(D_2, \Phi_2, F_2)$ be two knowledge bases. In order to define the informational equivalence of knowledge bases, consider two diagrams:

$$\begin{tikzcd} L_{\Phi_1\Theta} \arrow[r, "\beta"] \arrow[d, "Ct_f"'] & L_{\Phi_2\Theta} \arrow[d, "Ct_{f^\alpha}"] \\ K_{\Phi_1\Theta}(f) \arrow[r, "\gamma"] & K_{\Phi_2\Theta}(f^\alpha) \end{tikzcd} \qquad \begin{tikzcd} L_{\Phi_1\Theta} \arrow[d, "Ct_f"'] & L_{\Phi_2\Theta} \arrow[l, "\beta'"'] \arrow[d, "Ct_{f^\alpha}"] \\ K_{\Phi_1\Theta}(f) & K_{\Phi_2\Theta}(f^\alpha) \arrow[l, "\gamma^{-1}"'] \end{tikzcd}$$

where $\alpha : F_1 \to F_2$ is a bijection, β, β' are functors of the categories $L_{\Phi\Theta}$, γ is an isomorphism of the categories $K_{\Phi\Theta}(f)$.

Definition 2.3. Knowledge bases $KB_1 = KB(D_1, \Phi_1, F_1)$ and $KB_2 = KB(D_2, \Phi_2, F_2)$ are called informationally equivalent if it is possible to choose α,β, β' and γ such that they match the commutative diagrams above.

For the given model (D, Φ, f) we have a group $Aut(f)$ consisting of all bijections $s : D \to D$ compatible with the interpretation of symbols of relations. This means that for every n-ary relation $\varphi \in \Phi$ and every element $(a_1, a_2, ..., a_n) \in f(\varphi)$ the element $(sa_1, sa_2, ..., sa_n)$ belongs to $f(\varphi)$ as well.

Recall that two models (D_1, Φ_1, f_1) and (D_2, Φ_2, f_2) are called isomorphic if the sets Φ_1 and Φ_2 coincide and there is a bijection $\sigma : D_1 \to D_2$, which is an isomorphism of algebras, and for any n-ary relation $\varphi \in \Phi$ we have $(a_1, a_2, ..., a_n) \in f_1(\varphi)$ if and only if $(\sigma a_1, \sigma a_2, ..., \sigma a_n) \in f_2(\varphi)$.

Definition 2.4. Let two models (D_1, Φ_1, f_1) and (D_2, Φ_2, f_2) be given. Assume that D_1 and D_2 are the algebras with the same operations, defined in the variety of algebras Θ. The models are called automorphically equivalent, if there is an isomorphism of algebras $\mu : D_1 \to D_2$ such that the groups of automorphisms are conjugated by μ, i.e.,

$$Aut(f_2) = \mu Aut(f_1)\mu^{-1}.$$

Definition 2.5. Two multi-models (D_1, Φ_1, F_1) and (D_2, Φ_2, F_2) are called automorphically equivalent, if there is a bijection $\alpha : F_1 \to F_2$, such that the models (D_1, Φ_1, f) and (D_2, Φ_2, f^α) are automorphically equivalent for every $f \in F_1$.

This means that it is possible to correlate the instances of these multi-models in such a way that the corresponding models turn to be automorphically equivalent.

If the models are isomorphic, then they are automorphically equivalent as well. But the converse statement is far from being true (see [4] where this question was studied for graphs). In particular, if we have a graph which is a tree, then an automorphically equivalent graph need not be a tree while an isomorphic graph is necessarily a tree since isomorphism preserves all the properties. There is also another example of two automorphically equivalent graphs where the first one is connected while the second is not. In the case of isomorphic graphs it is impossible since isomorphism preserves the connectedness property.

Example 2.6 (see [4]). Consider two graphs G_1 and G_2 with the same set of vertices $V = \{1, 2, 3, 4\}$.

The graph G_1 is a rooted tree with the root 1 and the sons 2,3,4, while G_2 is not connected and consists of two components, where the first one is a triangle with the vertices 1,2,3 and the second component is an isolated vertex 4.

The set of edges of G_1 is $E_1 = \{e_1^1 = (1,2), e_1^2 = (2,1),\ e_1^3 = (1,3), e_1^4 = (3,1),\ e_1^5 = (1,4), e_1^6 = (4,1)\}$. The automorphisms group consists of all permutations on the set $\{2, 3, 4\}$.

For graph G_2 we have the set of edges $E_2 = \{e_2^1 = (1,2), e_2^2 = (2,1),\ e_2^3 = (1,3), e_2^4 = (3,1),\ e_2^5 = (2,3), e_2^6 = (3,2)\}$. The automorphisms group consists of all permutations on the set $\{1, 2, 3\}$.

It is easy to see that

1. There exists a bijection $\alpha : E_1 \to E_2$

$$\alpha = \begin{pmatrix} e_1^1\, e_1^2\, e_1^3\, e_1^4\, e_1^5 e_1^6 \\ e_2^1\, e_2^2\, e_2^3\, e_2^4\, e_2^5\, e_2^6 \end{pmatrix}$$

2. There exists a bijection μ, written explicitly in [4].

Groups of automorphisms are conjugated by the bijection μ. Therefore, graphs are automorphically equivalent. This example illustrates that automorphic equivalence of two graphs is not as strict as isomorphism and does not preserve the basic characteristics of the graphs, like a tree structure or connectedness.

Two main problems constitute the core of the whole theory:

Problem 2.7. To what extent the automorphic equivalence relation is wider than isomorphism.

Problem 2.8. How automorphic equivalence of two finite multi-models can be verified.

The first problem is algebraic, while the second one has applications in the knowledge bases theory in the problem of verification of the informational equivalence of knowledge bases.

Evidently, the wideness of the relation of automorphic equivalence is its advantage. We will see that for informationally equivalent knowledge bases the subjects of knowledge are not necessarily isomorphic. They may be very different, as it was mentioned in the graph example.

3. LINEAR KNOWLEDGE BASES

Let the knowledge bases $KB(D_1, \Phi_1, F_1)$ and $KB(D_2, \Phi_2, F_2)$ correspond to multi-models (D_1, Φ_1, F_1) and (D_2, Φ_2, F_2), respectively. For the sake of simplicity consider the case of models $F_1 = f_1$ and $F_2 = f_2$. From now on assume that D_1 and D_2 are finite dimensional vector spaces over a field P. Let $dim\ D_1 = dim\ D_2 = n$. We will see that to each knowledge base of such type correspond a subgroup in the group $GL_n(P)$.

Theorem 3.1. *The knowledge bases $KB(D_1, \Phi_1, f_1)$ and $KB(D_2, \Phi_2, f_2)$, where D_1 and D_2 are n-dimensional vector spaces over a finite field P, are informationally equivalent if and only if the corresponding subgroups are conjugated in $GL_n(P)$.*

Proof. Consider the multi-models (D_1, Φ_1, f_1) and (D_2, Φ_2, f_2) corresponding to $KB(D_1, \Phi_1, f_1)$ and $KB(D_2, \Phi_2, f_2)$, respectively.

By Theorem 11 (see [9]) the knowledge bases $KB(D_1, \Phi_1, f_1)$ and $KB(D_2, \Phi_2, f_2)$ are informationally equivalent if and only if the multi-models (D_1, Φ_1, f_1) and (D_2, Φ_2, f_2) are automorphically equivalent. This means that they are equivalent if there exists an isomorphism of algebras $\mu : D_1 \to D_2$ such that

$$Aut(f_2) = \mu Aut(f_1)\mu^{-1}.$$

Since $dim\ D_1 = dim\ D_2 = n$, the spaces D_1 and D_2 are isomorphic to the space P^n of rows of the length n. This means that μ is an automorphism of the linear space P^n and thus an element of the group $GL_n(P)$.

Let $\varphi \in \Phi$ be an m-ary relation on D. Then $f(\varphi) \subseteq D^m$. So $f(\varphi)$ is the set of rows of the form $\bar{a} = (a_1, \ldots, a_m)$, where $a_i \in P^n$, $i = 1, \ldots, m$. By definition, an automorphism $\sigma \in Aut(D)$ belongs to $Aut(f)$ if for every $\varphi \in \Phi$ and $(a_1, \ldots, a_m) \in f(\varphi)$ we have $(\sigma(a_1), \ldots, \sigma(a_m)) \in f(\varphi)$, i.e., $\sigma(\bar{a}) \in f(\varphi)$. This means that $Aut(f_1)$ is a subgroup of $Aut(D_1)$ and $Aut(f_2)$ is a subgroup of $Aut(D_2)$. Taking into account that $Aut(D_1) \backsimeq Aut(D_2) \backsimeq GL_n(P)$ we conclude that $Aut(f_1)$ and $Aut(f_2)$ can be embedded in $GL_n(P)$. Denote the corresponding subgroups by G and H. We constructed the subgroups in $GL_n(P)$ corresponding to our knowledge bases. It is easy to see that

Theorem 3.2 ([3]). *Models (D_1, Φ_1, f_1) and (D_2, Φ_2, f_2) are automorphically equivalent if and only if the subgroups G and H are conjugated.*

Hence, the problem of informational equivalence of the knowledge bases whose domains are finite dimensional vector spaces is reduced to conjugacy problem for given subgroups in the group $GL_n(P)$. ■

Suppose now that we have finite domains for linear knowledge bases $KB(D_1, \Phi_1, f_1)$ and $KB(D_2, \Phi_2, f_2)$. This means that the ground field P is a finite field $P = \mathbb{F}_q$, $q = p^n$, p is prime. Then, according to Theorem 3.1 the question about informational equivalence of these knowledge bases is algorithmically solvable. Hence, the next destination is to find out in which cases we can rely on efficient (in some sense) algorithms.

So, in the case of linear knowledge bases with finite domains the problem of informational equivalence of knowledge bases is reduced to the following question.

Let $GL_n(P)$ be a general linear group of rank n defined over a finite field $P = \mathbb{F}_q$. Let G and H be two subgroups in $GL_n(P)$. *How to check that G and H are isomorphic? If G and H are isomorphic, how to check that they are conjugated in $GL_n(P)$?* These problems are quite popular in computational group theory.

The answers heavily depend on the structure of the subgroups in question.

Example 3.3. Let G and H lie in $GL_n(P)$.

1. The subgroups G and H are maximal subgroups consisting of unipotent elements. Then according to Lie-Kolchin theorem [2] they are conjugated. In order to check if all elements are unipotent it is enough to compute the eigenvalues of the elements.

2. The subgroups G and H are cyclic. Then they are conjugated if and only if their generators have the same Jordan structure.

3. Subgroups G and H are abelian. The isomorphism problem is reduced to computing abelian invariants (see [13]).

4. If the groups are p-groups, there is an algorithm by E. O'Brien checking isomorphism problem (see [6]).

5. There is a general approach described in [1].

6. For conjugation problem there is a practical algorithm described in [12].

Remark 3.4. Most of the algorithms mentioned above are implemented in MAGMA and GAP computational systems.

Remark 3.5. All these algorithms are far from being polynomial. However in many cases they work pretty fast.

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