

## SEMI-OPEN SETS IN BICLOSURE SPACES

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### Abstract

The aim of this paper is to introduce and study semi-open sets in biclosure spaces. We define semi-continuous maps and semi-irresolute maps and investigate their behavior. Moreover, we introduce pre-semi-open maps in biclosure spaces and study some of their properties.

**Keywords:** closure operator, biclosure space, semi-open set, semi-continuous map, semi-irresolute map, pre semi-open map.

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### 1. INTRODUCTION

In 1963, bitopological spaces were introduced by J.C. Kelly [10] as triples  $(X, \tau_1, \tau_2)$  where  $X$  is a set and  $\tau_1$  and  $\tau_2$  are topologies defined on  $X$ . After that, a larger number of papers have been written to generalize the topological concept to a bitopological setting, see for instance, [1, 7] and [8]. Closure spaces were introduced by E. Čech in [3] and then studied by many mathematicians, see e.g. [4, 5, 6] and [12]. The concept of biclosure spaces was introduced and studied in [2]. In 1966, N. Levine [11] introduced semi-open sets and semi-continuous maps in a topological space. If  $(X, \tau)$  is a topological space and  $A \subseteq X$ , then  $A$  is semi-open if there exists  $G \in \tau$  such that  $G \subseteq A \subseteq \bar{G}$  where  $\bar{G}$  denotes the closure of  $G$  in  $(X, \tau)$ . The concepts of semi-open sets and semi-continuous maps in closure spaces were introduced in [9]. In this paper, we introduce semi-open sets in biclosure spaces and investigate some of their fundamental properties.

Then we use semi-open sets to define semi-open maps, semi-continuous maps, semi-irresolute maps and pre-semi-open maps. We obtain certain properties of semi-openness, semi-continuity, semi-irresoluteness and pre-semi-openness in biclosure spaces.

## 2. PRELIMINARIES

In this section, we recall some basic definitions concerning closure spaces and biclosure spaces.

A map  $u : P(X) \rightarrow P(X)$  defined on the power set  $P(X)$  of a set  $X$  is called a *closure operator* on  $X$  and the pair  $(X, u)$  is called a *closure space* if the following axioms are satisfied:

- (A1)  $u\emptyset = \emptyset$ ,
- (A2)  $A \subseteq uA$  for every  $A \subseteq X$ ,
- (A3)  $A \subseteq B \Rightarrow uA \subseteq uB$  for all  $A, B \subseteq X$ .

A closure operator  $u$  on a set  $X$  is called *idempotent* if  $A \subseteq X \Rightarrow uuA = uA$ . A subset  $A \subseteq X$  is *closed* in the closure space  $(X, u)$  if  $uA = A$  and it is *open* if its complement in  $X$  is closed. The empty set and the whole space are both open and closed.

A closure space  $(Y, v)$  is said to be a *subspace* of  $(X, u)$  if  $Y \subseteq X$  and  $vA = uA \cap Y$  for each subset  $A \subseteq Y$ .

A subset  $A$  of a closure space  $(X, u)$  is called *semi-open* if there exists an open set  $G$  in  $(X, u)$  such that  $G \subseteq A \subseteq uG$ . A subset  $A \subseteq X$  is called *semi-closed* if its complement is semi-open.

If  $(X, u)$  and  $(Y, v)$  are closure spaces, then a map  $f : (X, u) \rightarrow (Y, v)$  is called:

- (i) *open* (respectively, *closed*) if the image of each open (respectively, closed) set in  $(X, u)$  is open (respectively, closed) in  $(Y, v)$ .
- (ii) *continuous* if  $f(uA) \subseteq vf(A)$  for every subset  $A \subseteq X$ . One can see that, if  $f$  is continuous, then the inverse image under  $f$  of each open set in  $(Y, v)$  is open in  $(X, u)$ .

A *biclosure space* is a triple  $(X, u_1, u_2)$  where  $X$  is a set and  $u_1, u_2$  are two closure operators on  $X$ . A subset  $A$  of a biclosure space  $(X, u_1, u_2)$  is called *closed* if  $u_1u_2A = A$ . The complement of closed set is called *open*.

Let  $(X, u_1, u_2)$  be a biclosure space. A biclosure space  $(Y, v_1, v_2)$  is called a *subspace* of  $(X, u_1, u_2)$  if  $Y \subseteq X$  and  $v_i A = u_i A \cap Y$  for all  $i \in \{1, 2\}$  and every subset  $A$  of  $Y$ .

Let  $(X, u_1, u_2)$  and  $(Y, v_1, v_2)$  be biclosure spaces and let  $i \in \{1, 2\}$ . Then a map  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  is called:

- (i) *i-open* (respectively, *i-closed*) if the map  $f : (X, u_i) \rightarrow (Y, v_i)$  is open (respectively, closed).
- (ii) *open* (respectively, *closed*) if  $f$  is *i-open* (respectively, *i-closed*) for all  $i \in \{1, 2\}$ .
- (iii) *biopen* (respectively, *biclosed*) if the map  $f : (X, u_1) \rightarrow (Y, v_2)$  is open (respectively, closed).
- (iv) *i-continuous* if the map  $f : (X, u_i) \rightarrow (Y, v_i)$  is continuous for all  $i \in \{1, 2\}$ .
- (v) *continuous* if  $f$  is *i-continuous* for all  $i \in \{1, 2\}$ .
- (vi) *bi-continuous* if the map  $f : (X, u_1) \rightarrow (Y, v_2)$  is continuous.

**Remark 2.1.** Let  $A$  be a subset of a biclosure space  $(X, u_1, u_2)$ .

- (i)  $A$  is open in  $(X, u_1, u_2)$  if and only if  $A$  is open in both  $(X, u_1)$  and  $(X, u_2)$
- (ii) If  $A$  is an open set in  $(X, u_1, u_2)$ , then  $u_1 u_2(X - A) = u_2 u_1(X - A)$ .

The converse of the statement (ii) in Remark 2.1 need not be true as can be seen from the following example.

**Example 2.2.** Let  $X = \{1, 2, 3\}$  and define a closure operator  $u_1$  on  $X$  by  $u_1 \emptyset = \emptyset$ ,  $u_1 \{1\} = \{1\}$ ,  $u_1 \{2\} = \{2\}$ ,  $u_1 \{3\} = \{3\}$ ,  $u_1 \{1, 3\} = \{1, 3\}$  and  $u_1 \{1, 2\} = u_1 \{2, 3\} = u_1 X = X$ . Define a closure operator  $u_2$  on  $X$  by  $u_2 \emptyset = \emptyset$ ,  $u_2 \{1\} = \{1, 3\}$ ,  $u_2 \{2\} = \{2\}$ ,  $u_2 \{3\} = \{3\}$  and  $u_2 \{1, 2\} = u_2 \{1, 3\} = u_2 \{2, 3\} = u_2 X = X$ . We can see that  $u_1 u_2(X - \{1\}) = u_2 u_1(X - \{1\}) = X$  but  $\{1\}$  is not open in  $(X, u_1, u_2)$ .

**Proposition 2.3.** *Let  $\{A_\alpha\}_{\alpha \in J}$  be a collection of open sets in a biclosure space  $(X, u_1, u_2)$ . Then  $\cup_{\alpha \in J} A_\alpha$  is an open set.*

**Proof.** Let  $A_\alpha$  be open in  $(X, u_1, u_2)$  for each  $\alpha \in J$ , then  $X - A_\alpha$  is closed for all  $\alpha \in J$ . Since  $\cap_{\alpha \in J} (X - A_\alpha) \subseteq X - A_\alpha$  for all  $\alpha \in J$ ,  $u_1 u_2 \cap_{\alpha \in J} (X - A_\alpha) \subseteq u_1 u_2 (X - A_\alpha)$  for each  $\alpha \in J$ . But  $X - A_\alpha = u_1 u_2 (X - A_\alpha)$  for all  $\alpha \in J$ , hence  $u_1 u_2 \cap_{\alpha \in J} (X - A_\alpha) \subseteq X - A_\alpha$  for each  $\alpha \in J$ . Consequently,  $u_1 u_2 \cap_{\alpha \in J} (X - A_\alpha) \subseteq \cap_{\alpha \in J} (X - A_\alpha) \subseteq u_1 u_2 \cap_{\alpha \in J} (X - A_\alpha)$ , i.e.  $u_1 u_2 \cap_{\alpha \in J} (X - A_\alpha) = \cap_{\alpha \in J} (X - A_\alpha)$ . Thus,  $\cap_{\alpha \in J} (X - A_\alpha) = X - \cup_{\alpha \in J} A_\alpha$  is closed in  $(X, u_1, u_2)$ . Therefore,  $\cup_{\alpha \in J} A_\alpha$  is open. ■

The intersection of two open sets in a biclosure space  $(X, u_1, u_2)$  need not be an open set as can be seen from Example 2.2 where  $\{1, 2\}$  and  $\{1, 3\}$  are open in  $(X, u_1, u_2)$  but  $\{1, 2\} \cap \{1, 3\}$  is not open.

**Proposition 2.4.** *If  $\{A_\alpha\}_{\alpha \in J}$  is a collection of subsets in a biclosure space  $(X, u_1, u_2)$ , then  $u_1 u_2 \cap_{\alpha \in J} A_\alpha \subseteq \cap_{\alpha \in J} u_1 u_2 A_\alpha$ .*

By Example 2.2,  $u_1 u_2 \{1, 2\} \cap u_1 u_2 \{1, 3\}$  is not contained in  $u_1 u_2 (\{1, 2\} \cap \{1, 3\})$ , i.e. the inclusion of Proposition 2.4 cannot be replaced by equality in general.

**Proposition 2.5.** *If  $\{A_\alpha\}_{\alpha \in J}$  is a collection of closed subsets in a biclosure space  $(X, u_1, u_2)$ , then  $u_1 u_2 \cap_{\alpha \in J} A_\alpha = \cap_{\alpha \in J} u_1 u_2 A_\alpha$ .*

**Proof.** Let  $A_\alpha$  be closed in  $(X, u_1, u_2)$  for all  $\alpha \in J$ . Then  $X - A_\alpha$  is open and  $A_\alpha = u_1 u_2 A_\alpha$  for each  $\alpha \in J$ . By Proposition 2.3,  $\cup_{\alpha \in J} (X - A_\alpha)$  is open. But  $\cup_{\alpha \in J} (X - A_\alpha) = X - \cap_{\alpha \in J} A_\alpha$ , hence  $\cap_{\alpha \in J} A_\alpha$  is closed in  $(X, u_1, u_2)$ . Therefore,  $u_1 u_2 \cap_{\alpha \in J} A_\alpha = \cap_{\alpha \in J} A_\alpha = \cap_{\alpha \in J} u_1 u_2 A_\alpha$ . ■

The converse of Proposition 2.5 is not true in general as shown in the following example.

**Example 2.6.** Let  $X = \{1, 2, 3\}$  and define a closure operator  $u_1$  on  $X$  by  $u_1 \emptyset = \emptyset$ ,  $u_1 \{2\} = u_1 \{3\} = u_1 \{2, 3\} = \{2, 3\}$  and  $u_1 \{1\} = u_1 \{1, 2\} = u_1 \{1, 3\} = u_1 X = X$ . Define a closure operator  $u_2$  on  $X$  by  $u_2 \emptyset = \emptyset$ ,  $u_2 \{1\} = u_2 \{2\} = u_2 \{1, 2\} = \{1, 2\}$  and  $u_2 \{3\} = u_2 \{1, 3\} = u_2 \{2, 3\} = u_2 X = X$ . It is easy to see that  $u_1 u_2 (\{1, 2\} \cap \{1, 3\}) = u_1 u_2 \{1, 2\} \cap u_1 u_2 \{1, 3\}$  but neither  $\{1, 2\}$  nor  $\{1, 3\}$  is closed in  $(X, u_1, u_2)$ .

**Proposition 2.7.** *Let  $(X, u_1, u_2)$  be a biclosure space. If  $G$  is a subset of  $X$ , then  $u_1u_2G - G$  has no nonempty open subset of  $(X, u_1, u_2)$ .*

**Proof.** Let  $G$  be a subset of  $X$  and  $H$  be a nonempty open subset of  $(X, u_1, u_2)$  such that  $H \subseteq u_1u_2G - G$ . Since  $H$  is nonempty, there is  $x \in H \subseteq u_1u_2G - G$ , i.e.  $x \notin X - H$ . Thus,  $u_1u_2G$  is not contained in  $X - H$ . Since  $H \subseteq u_1u_2G - G$ ,  $G \subseteq u_1u_2G - H \subseteq X - H$ . It follows that  $u_1u_2G \subseteq u_1u_2(X - H)$ . But  $H$  is open in  $(X, u_1, u_2)$ ,  $u_1u_2(X - H) = X - H$ . Consequently,  $u_1u_2G \subseteq X - H$ , which is a contradiction. Therefore,  $u_1u_2G - G$  contains no nonempty open set of  $(X, u_1, u_2)$ . ■

**Remark 2.8.** The following statement is equivalent to Proposition 2.7:

Let  $(X, u_1, u_2)$  be a biclosure space and  $G$  be a subset of  $X$ . If  $H$  is an open subset of  $(X, u_1, u_2)$  with  $H \subseteq u_1u_2G - G$ , then  $H$  is an empty set.

Moreover, if the subset  $H$  is an open subset of  $(X, u_1)$  but not open in  $(X, u_2)$ , then  $H$  need not be empty. And if the subset  $H$  is an open subset of  $(X, u_2)$  but not open in  $(X, u_1)$ , then  $H$  need not be empty. By Example 2.6,  $\{2\}$  is a subset of  $X$  such that  $\{1\}$  and  $\{3\}$  are nonempty subsets of  $u_1u_2\{2\} - \{2\}$ . We can see that  $\{1\}$  is open in  $(X, u_1)$  but not open in  $(X, u_2)$ , and  $\{3\}$  is an open subset of  $(X, u_2)$  but not open in  $(X, u_1)$ .

**Proposition 2.9.** *If  $(Y, v_1, v_2)$  is a biclosure subspace of  $(X, u_1, u_2)$ , then for every open subset  $G$  of  $(X, u_1, u_2)$ ,  $G \cap Y$  is an open set in  $(Y, v_1, v_2)$ .*

**Proof.** Let  $G$  be an open set in  $(X, u_1, u_2)$ . By Remark 2.1 (i),  $G$  is open in both  $(X, u_1)$  and  $(X, u_2)$ . Thus,  $v_i(Y - (G \cap Y)) = u_i(Y - (G \cap Y)) \cap Y \subseteq u_i(X - G) \cap Y = (X - G) \cap Y = Y - (G \cap Y)$  for each  $i \in \{1, 2\}$ . Consequently,  $G \cap Y$  is open in both  $(Y, v_1)$  and  $(Y, v_2)$ . Therefore,  $G \cap Y$  is open in  $(Y, v_1, v_2)$ . ■

**Remark 2.10.** By Proposition 2.9, if  $E \subseteq Y$  and  $E = G \cap Y$  for some open subset  $G$  of  $(X, u_1, u_2)$ , then  $E$  is an open set in  $(Y, v_1, v_2)$ . The converse is not true as can be seen from the following example.

**Example 2.11.** Let  $X = \{1, 2, 3\}$  and define a closure operator  $u_1$  on  $X$  by  $u_1\emptyset = \emptyset$ ,  $u_1\{1\} = \{1, 3\}$ ,  $u_1\{2\} = u_1\{2, 3\} = \{2, 3\}$ ,  $u_1\{3\} = \{3\}$  and  $u_1\{1, 2\} = u_1\{1, 3\} = u_1X = X$ . Define a closure operator  $u_2$  on  $X$  by

$u_2\emptyset = \emptyset$ ,  $u_2\{1\} = \{1, 2\}$ ,  $u_2\{2\} = \{2, 3\}$ ,  $u_2\{3\} = \{3\}$  and  $u_2\{1, 2\} = u_2\{1, 3\} = u_2\{2, 3\} = u_2X = X$ . Thus, there are only three open subset of  $(X, u_1, u_2)$ , namely  $\emptyset$ ,  $\{1, 2\}$  and  $X$ . Let  $Y = \{1, 2\}$  and  $(Y, v_1, v_2)$  be a biclosure subspace of  $(X, u_1, u_2)$ . Then  $v_1\emptyset = \emptyset$ ,  $v_1\{1\} = \{1\}$ ,  $v_1\{2\} = \{2\}$ ,  $v_1Y = Y$ ,  $v_2\emptyset = \emptyset$ ,  $v_2\{2\} = \{2\}$  and  $v_2\{1\} = v_2Y = Y$ . We can see that  $\{1\}$  is an open subset of  $(Y, v_1, v_2)$  but there is no any open subset  $G$  of  $(X, u_1, u_2)$  such that  $\{1\} = G \cap Y$ .

**Proposition 2.12.** *Let  $(X, u_1, u_2)$ ,  $(Y, v_1, v_2)$  and  $(Z, w_1, w_2)$  be biclosure spaces, let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  and  $g : (Y, v_1, v_2) \rightarrow (Z, w_1, w_2)$  be maps.*

- (i) *If  $f$  is 1-open and  $g$  is biopen, then  $g \circ f$  is biopen.*
- (ii) *If  $f$  is biopen and  $g$  is 2-open, then  $g \circ f$  is biopen.*

**Proof.**

- (i) Let  $G$  be an open set in  $(X, u_1)$ . Since  $f$  is 1-open,  $f(G)$  is open in  $(Y, v_1)$ . As  $g$  is biopen,  $g(f(G)) = g \circ f(G)$  is open in  $(Z, w_2)$ . Thus,  $g \circ f$  is biopen.
- (ii) Let  $G$  be an open set in  $(X, u_1)$ . Since  $f$  is biopen,  $f(G)$  is open in  $(Y, v_2)$ . And since  $g$  is 2-open,  $g(f(G)) = g \circ f(G)$  is open in  $(Z, w_2)$ . Thus,  $g \circ f$  is biopen. ■

The composition of two biopen maps need not be a biopen map as can be seen from the following example.

**Example 2.13.** Let  $X = Y = Z = \{1, 2\}$  and define a closure operator  $u_1$  on  $X$  by  $u_1\emptyset = \emptyset$ ,  $u_1\{2\} = \{2\}$ , and  $u_1\{1\} = u_1X = X$ . Define a closure operator  $u_2$  on  $X$  by  $u_2\emptyset = \emptyset$  and  $u_2\{1\} = u_2\{2\} = u_2X = X$ . Define a closure operator  $v_1$  on  $Y$  by  $v_1\emptyset = \emptyset$ ,  $v_1\{1\} = \{1\}$  and  $v_1\{2\} = v_1Y = Y$  and define a closure operator  $v_2$  on  $Y$  by  $v_2\emptyset = \emptyset$ ,  $v_2\{1\} = \{1\}$ ,  $v_2\{2\} = \{2\}$  and  $v_2Y = Y$ . Define a closure operator  $w_1$  on  $Z$  by  $w_1\emptyset = \emptyset$  and  $w_1\{1\} = w_1\{2\} = w_1Z = Z$  and define a closure operator  $w_2$  on  $Z$  by  $w_2\emptyset = \emptyset$ ,  $w_2\{1\} = \{1\}$  and  $w_2\{2\} = w_2Z = Z$ . Let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  and  $g : (Y, v_1, v_2) \rightarrow (Z, w_1, w_2)$  be identity maps. We can see that  $f$  and  $g$  are biopen. But  $g \circ f$  is not biopen because  $\{1\}$  is open in  $(X, u_1)$  but  $g \circ f(\{1\})$  is not open in  $(Z, w_2)$ .

**Proposition 2.14.** *Let  $(X, u_1, u_2)$ ,  $(Y, v_1, v_2)$  and  $(Z, w_1, w_2)$  be biclosure spaces and let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  and  $g : (Y, v_1, v_2) \rightarrow (Z, w_1, w_2)$  be maps.*

- (i) *If  $g \circ f$  is biopen and  $f$  is a 1-continuous surjection, then  $g$  is biopen.*
- (ii) *If  $g \circ f$  is biopen and  $g$  is a 2-continuous injection, then  $f$  is biopen.*

**Proof.**

- (i) Let  $H$  be an open set in  $(Y, v_1)$ . Since  $f$  is 1-continuous,  $f^{-1}(H)$  is open in  $(X, u_1)$ . But  $g \circ f$  is biopen, hence  $g \circ f(f^{-1}(H))$  is open in  $(Z, w_2)$ . As  $f$  is a surjection,  $g \circ f(f^{-1}(H)) = g(H)$ . Therefore,  $g$  is biopen.
- (ii) Let  $G$  be an open set in  $(X, u_1)$ . Since  $g \circ f$  is biopen,  $g \circ f(G)$  is open in  $(Z, w_2)$ . But  $g$  is 2-continuous, hence  $g^{-1}(g \circ f(G))$  is open in  $(Y, v_2)$ . As  $g$  is an injection,  $g^{-1}(g \circ f(G)) = f(G)$ . Therefore,  $f$  is biopen. ■

**Proposition 2.15.** *Let  $(X, u_1, u_2)$  and  $(Y, v_1, v_2)$  be biclosure spaces and let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  be a map. If  $f$  is open, then  $f(G)$  is open in  $(Y, v_1, v_2)$  for every open subset  $G$  of  $(X, u_1, u_2)$ .*

**Proof.** Let  $G$  be an open subset of  $(X, u_1, u_2)$ . By Remark 2.1 (i),  $G$  is open in both  $(X, u_1)$  and  $(X, u_2)$ . Since  $f$  is open,  $f$  is both 1-open and 2-open. Hence,  $f(G)$  is open in both  $(Y, v_1)$  and  $(Y, v_2)$ . Consequently,  $f(G)$  is open in  $(Y, v_1, v_2)$  by Remark 2.1 (i). ■

The converse of Proposition 2.15 is not true in general as can be seen from the following example.

**Example 2.16.** Let  $X = \{1, 2\} = Y$  and define a closure operator  $u_1$  on  $X$  by  $u_1\emptyset = \emptyset$ ,  $u_1\{2\} = \{2\}$  and  $u_1\{1\} = u_1X = X$ . Define a closure operator  $u_2$  on  $X$  by  $u_2\emptyset = \emptyset$  and  $u_2\{1\} = u_2\{2\} = u_2X = X$ . Define a closure operator  $v_1$  on  $Y$  by  $v_1\emptyset = \emptyset$  and  $v_1\{1\} = v_1\{2\} = v_1Y = Y$  and define a closure operator  $v_2$  on  $Y$  by  $v_2\emptyset = \emptyset$ ,  $v_2\{1\} = \{1\}$  and  $v_2\{2\} = v_2Y = Y$ . Let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  be an identity map. It is easy to see that  $f(G)$  is open in  $(Y, v_1, v_2)$  for every open subset  $G$  of  $(X, u_1, u_2)$ . But  $f$  is not 1-open because  $f(\{1\})$  is not open in  $(Y, v_1)$  while  $\{1\}$  is open in  $(X, u_1)$ . Consequently,  $f$  is not open.

**Proposition 2.17.** *Let  $(X, u_1, u_2)$  and  $(Y, v_1, v_2)$  be biclosure spaces and let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  be a map. If  $f$  is continuous, then  $f^{-1}(H)$  is open in  $(X, u_1, u_2)$  for every open subset  $H$  of  $(Y, v_1, v_2)$ .*

**Proof.** Let  $H$  be an open subset of  $(Y, v_1, v_2)$ . By Remark 2.1 (i),  $H$  is open in both  $(Y, v_1)$  and  $(Y, v_2)$ . Since  $f$  is continuous,  $f$  is both 1-continuous and 2-continuous. It follows that  $f^{-1}(H)$  is open in both  $(X, u_1)$  and  $(X, u_2)$ . Therefore,  $f^{-1}(H)$  is open in  $(X, u_1, u_2)$  by Remark 2.1 (i). ■

The converse of Proposition 2.17 need not be true in general as can be seen from the following example.

**Example 2.18.** In Example 2.16,  $f^{-1}(H)$  is open in  $(X, u_1, u_2)$  for every open subset  $H$  of  $(Y, v_1, v_2)$ . But the map  $f$  is not 2-continuous because  $f^{-1}\{2\}$  is not open in  $(X, u_2)$  while  $\{2\}$  is open in  $(Y, v_2)$ . Consequently,  $f$  is not continuous.

**Definition 2.19.** A map  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$ , where  $(X, u_1, u_2)$  and  $(Y, v_1, v_2)$  are biclosure spaces, is called a *homeomorphism* if  $f$  is bijective, continuous and open.

**Proposition 2.20.** *Let  $(X, u_1, u_2)$  and  $(Y, v_1, v_2)$  be biclosure spaces and let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  be a map. If  $f$  is a bijective continuous map, then the following statements are equivalent:*

- (i)  $f$  is a homeomorphism,
- (ii)  $f$  is a closed map,
- (iii)  $f$  is an open map.

**Proof.** (i)  $\rightarrow$  (ii) Since  $f$  is a homeomorphism, i.e.  $f$  is open and bijective. It follows that  $f$  is both 1-open and 2-open. Let  $i \in \{1, 2\}$  and let  $F_i$  be a closed subset of  $(X, u_i)$ . Then  $f(X - F_i) = Y - f(F_i)$  is open in  $(Y, v_i)$ . Hence,  $f(F_i)$  is closed in  $(Y, v_i)$ . Thus,  $f$  is both 1-closed and 2-closed. Therefore,  $f$  is closed.

(ii) $\rightarrow$ (iii) Let  $i \in \{1, 2\}$  and let  $G_i$  be an open subset of  $(X, u_i)$ . Then  $X - G_i$  is closed in  $(X, u_i)$ . By the assumption,  $f$  is both closed and bijective. It follows that  $f$  is both 1-closed and 2-closed. Consequently,  $f(X - G_i) = Y - f(G_i)$  is closed in  $(Y, v_i)$ . Hence,  $f(G_i)$  is open in  $(Y, v_i)$ . Thus,  $f$  is both 1-open and 2-open. Therefore,  $f$  is open.

(iii) $\rightarrow$ (i) By the assumption,  $f$  is a homeomorphism. ■



## 3. SEMI-OPEN SETS IN BICLOSURE SPACES

In this section, we introduce a new type of open sets in biclosure spaces and study some of their properties.

**Definition 3.1.** A subset  $A$  of a biclosure space  $(X, u_1, u_2)$  is called *semi-open*, if there exists an open subset  $G$  of  $(X, u_1)$  such that  $G \subseteq A \subseteq u_2G$ . The complement of a semi-open set in  $X$  is called *semi-closed*.

Clearly, if  $(X, u_1, u_2)$  is a biclosure space and  $A$  is open (respectively, closed) in  $(X, u_1)$ , then  $A$  is semi-open (respectively, semi-closed) in  $(X, u_1, u_2)$ . The converse is not true as can be seen from the following example.

**Example 3.2.** Let  $X = \{1, 2, 3\}$  and define a closure operator  $u_1$  on  $X$  by  $u_1\emptyset = \emptyset$ ,  $u_1\{1\} = u_1\{3\} = u_1\{1, 3\} = \{1, 3\}$ ,  $u_1\{2\} = \{2, 3\}$  and  $u_1\{1, 2\} = u_1\{2, 3\} = u_1X = X$ . Define a closure operator  $u_2$  on  $X$  by  $u_2\emptyset = \emptyset$ ,  $u_2\{3\} = \{3\}$  and  $u_2\{1\} = u_2\{2\} = u_2\{1, 2\} = u_2\{1, 3\}, u_2\{2, 3\} = u_2X = X$ . It follows that  $\{2, 3\}$  is semi-open in  $(X, u_1, u_2)$  but  $\{2, 3\}$  is open in neither  $(X, u_1)$  nor  $(X, u_2)$ . Moreover,  $\{1\}$  is semi-closed in  $(X, u_1, u_2)$  but  $\{1\}$  is closed in neither  $(X, u_1)$  nor  $(X, u_2)$ .

**Proposition 3.3.** Let  $(X, u_1, u_2)$  be a biclosure space and let  $A \subseteq X$ . Then  $A$  is semi-closed if and only if there exists a closed subset  $F$  of  $(X, u_1)$  such that  $X - u_2(X - F) \subseteq A \subseteq F$ .

**Proof.** Let  $A$  be semi-closed in  $(X, u_1, u_2)$ . Then there exists an open set  $G$  in  $(X, u_1)$  such that  $G \subseteq X - A \subseteq u_2G$ . Thus, there exists a closed subset  $F$  of  $(X, u_1)$  such that  $G = X - F$  and  $X - F \subseteq X - A \subseteq u_2(X - F)$ . Therefore,  $X - u_2(X - F) \subseteq A \subseteq F$ .

Conversely, by the assumption, there is a closed subset  $F$  of  $(X, u_1)$  such that  $X - u_2(X - F) \subseteq A \subseteq F$ . Thus, there exists an open set  $G$  in  $(X, u_1)$  such that  $F = X - G$  and  $X - u_2G \subseteq A \subseteq X - G$ . It follows that  $G \subseteq X - A \subseteq u_2G$ . Therefore,  $A$  is semi-closed in  $(X, u_1, u_2)$ . ■

**Proposition 3.4.** Let  $\{A_\alpha\}_{\alpha \in J}$  be a collection of semi-open sets in a biclosure space  $(X, u_1, u_2)$ . Then  $\cup_{\alpha \in J} A_\alpha$  is a semi-open set in  $(X, u_1, u_2)$ .

**Proof.** Let  $A_\alpha$  be semi-open in  $(X, u_1, u_2)$  for all  $\alpha \in J$ . Hence, for each  $\alpha \in J$ , we have an open set  $G_\alpha$  in  $(X, u_1)$  such that  $G_\alpha \subseteq A_\alpha \subseteq u_2G_\alpha$ .

Thus,  $\cup_{\alpha \in J} G_\alpha \subseteq \cup_{\alpha \in J} A_\alpha \subseteq \cup_{\alpha \in J} u_2 G_\alpha$ . Since  $G_\alpha \subseteq \cup_{\alpha \in J} G_\alpha$  for each  $\alpha \in J$ ,  $u_2 G_\alpha \subseteq u_2 \cup_{\alpha \in J} G_\alpha$  for all  $\alpha \in J$ . Thus,  $\cup_{\alpha \in J} u_2 G_\alpha \subseteq u_2 \cup_{\alpha \in J} G_\alpha$ . Consequently,  $\cup_{\alpha \in J} G_\alpha \subseteq \cup_{\alpha \in J} A_\alpha \subseteq u_2 \cup_{\alpha \in J} G_\alpha$ . As  $G_\alpha$  is open in  $(X, u_1)$  for all  $\alpha \in J$ ,  $u_1 \cap_{\alpha \in J} (X - G_\alpha) \subseteq u_1(X - G_\alpha) = X - G_\alpha$  for each  $\alpha \in J$ . Thus,  $u_1 \cap_{\alpha \in J} (X - G_\alpha) \subseteq \cap_{\alpha \in J} (X - G_\alpha)$ . It follows that  $\cap_{\alpha \in J} (X - G_\alpha)$  is closed in  $(X, u_1)$ , i.e.  $\cup_{\alpha \in J} G_\alpha$  is open in  $(X, u_1)$ . Therefore,  $\cup_{\alpha \in J} A_\alpha$  is semi-open in  $(X, u_1, u_2)$ . ■

If  $\{A_\alpha\}_{\alpha \in J}$  is a collection of semi-open sets in a biclosure space  $(X, u_1, u_2)$ , then  $\cap_{\alpha \in J} A_\alpha$  need not be a semi-open set in  $(X, u_1, u_2)$  as shown in the following example.

**Example 3.5.** In the biclosure space  $(X, u_1, u_2)$  from Example 2.2, we can see that  $\{1, 2\}$  and  $\{1, 3\}$  are semi-open but  $\{1, 2\} \cap \{1, 3\}$  is not semi-open.

**Proposition 3.6.** *Let  $\{A_\alpha\}_{\alpha \in J}$  be a collection of semi-closed sets in a biclosure space  $(X, u_1, u_2)$ . Then  $\cap_{\alpha \in J} A_\alpha$  is semi-closed.*

**Proof.** Clearly, the complement of  $\cap_{\alpha \in J} A_\alpha$  is  $\cup_{\alpha \in J} (X - A_\alpha)$ . Since  $A_\alpha$  is semi-closed in  $(X, u_1, u_2)$  for each  $\alpha \in J$ ,  $X - A_\alpha$  is semi-open for all  $\alpha \in J$ . But  $\cup_{\alpha \in J} (X - A_\alpha)$  is a semi-open set by Proposition 3.4. Therefore,  $\cap_{\alpha \in J} A_\alpha$  is semi-closed in  $(X, u_1, u_2)$ . ■

If  $\{A_\alpha\}_{\alpha \in J}$  is a collection of semi-closed sets in a biclosure space  $(X, u_1, u_2)$ , then  $\cup_{\alpha \in J} A_\alpha$  need not be a semi-closed set as shown in the following example.

**Example 3.7.** In the biclosure space  $(X, u_1, u_2)$  from Example 2.2, we can see that  $\{2\}$  and  $\{3\}$  are semi-closed but  $\{2\} \cup \{3\}$  is not semi-closed.

**Proposition 3.8.** *Let  $(X, u_1, u_2)$  be a biclosure space and  $u_2$  be idempotent. If  $A$  is semi-open in  $(X, u_1, u_2)$  and  $A \subseteq B \subseteq u_2 A$ , then  $B$  is semi-open.*

**Proof.** Let  $A$  be semi-open in  $(X, u_1, u_2)$ . Then there exists an open set  $G$  in  $(X, u_1)$  such that  $G \subseteq A \subseteq u_2 G$ , hence  $u_2 A \subseteq u_2 u_2 G$ . Since  $u_2$  is idempotent,  $u_2 A \subseteq u_2 G$ . Thus,  $G \subseteq A \subseteq B \subseteq u_2 A \subseteq u_2 G$ . Therefore,  $B$  is semi-open. ■

**Proposition 3.9.** *Let  $(Y, v_1, v_2)$  be a biclosure subspace of  $(X, u_1, u_2)$  and  $A \subseteq Y$ . If  $A$  is semi-open in  $(X, u_1, u_2)$ , then  $A$  is semi-open in  $(Y, v_1, v_2)$ .*

**Proof.** Let  $A$  be semi-open in  $(X, u_1, u_2)$ . Then there exists an open set  $G$  in  $(X, u_1)$  such that  $G \subseteq A \subseteq u_2G$ . It follows that  $A \cap Y \subseteq u_2G \cap Y$ . But  $A = A \cap Y$ , hence  $G \subseteq A = A \cap Y \subseteq u_2G \cap Y = v_2G$ . Since  $G$  is open in  $(X, u_1)$ ,  $v_1(Y - G) = u_1(Y - G) \cap Y \subseteq u_1(X - G) \cap Y = (X - G) \cap Y = Y - G$ . Thus,  $Y - G$  is closed in  $(Y, v_1)$ , i.e.  $G$  is open in  $(Y, v_1)$ . Therefore,  $A$  is semi-open in  $(Y, v_1, v_2)$ . ■

The converse of Proposition 3.9 need not be true as can be seen from the following example.

**Example 3.10.** In the biclosure spaces  $(X, u_1, u_2)$  and  $(Y, v_1, v_2)$  from Example 2.11, we can see that  $\{2\} \subseteq Y$  and  $\{2\}$  is semi-open in  $(Y, v_1, v_2)$  but  $\{2\}$  is not semi-open in  $(X, u_1, u_2)$ .

**Definition 3.11.** Let  $(X, u_1, u_2)$  and  $(Y, v_1, v_2)$  be biclosure spaces. A map  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  is called *semi-open* (respectively, *semi-closed*) if  $f(A)$  is semi-open (respectively, semi-closed) in  $(Y, v_1, v_2)$  for every open (respectively, closed) subset  $A$  of  $(X, u_1, u_2)$ .

Clearly, if  $f$  is open (respectively, closed), then  $f$  is semi-open (respectively, semi-closed). The converse need not be true in general as can be seen from the following example.

**Example 3.12.** Let  $X = \{1, 2\} = Y$  and define a closure operator  $u_1$  on  $X$  by  $u_1\emptyset = \emptyset$ ,  $u_1\{1\} = \{1\}$  and  $u_1\{2\} = u_1X = X$ . Define a closure operator  $u_2$  on  $X$  by  $u_2\emptyset = \emptyset$ ,  $u_2\{1\} = \{1\}$ ,  $u_2\{2\} = \{2\}$  and  $u_2X = X$ . Define a closure operator  $v_1$  on  $Y$  by  $v_1\emptyset = \emptyset$ ,  $v_1\{1\} = \{1\}$  and  $v_1\{2\} = v_1Y = Y$  and define a closure operator  $v_2$  on  $Y$  by  $v_2\emptyset = \emptyset$  and  $v_2\{1\} = v_2\{2\} = v_2Y = Y$ . Let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  be an identity map. It is easy to see that  $f$  is semi-open but not open because  $f(\{2\})$  is not open in  $(Y, v_1, v_2)$  while  $\{2\}$  is open in  $(X, u_1, u_2)$ . Moreover, we can see that  $f$  is semi-closed but not closed because  $f(\{1\})$  is not closed in  $(Y, v_1, v_2)$  while  $\{1\}$  is closed in  $(X, u_1, u_2)$ .

**Proposition 3.13.** Let  $(X, u_1, u_2)$ ,  $(Y, v_1, v_2)$  and  $(Z, w_1, w_2)$  be biclosure spaces. Let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  and  $g : (Y, v_1, v_2) \rightarrow (Z, w_1, w_2)$  be maps. Then  $g \circ f$  is semi-open if  $f$  is open and  $g$  is semi-open.

**Proof.** Let  $G$  be an open subset of  $(X, u_1, u_2)$  and let  $f$  be open. By Proposition 2.15,  $f(G)$  is open in  $(Y, v_1, v_2)$ . As  $g$  is semi-open,  $g(f(G)) = g \circ f(G)$  is semi-open in  $(Z, w_1, w_2)$ . Therefore,  $g \circ f$  is semi-open. ■

**Proposition 3.14.** *Let  $(X, u_1, u_2)$ ,  $(Y, v_1, v_2)$  and  $(Z, w_1, w_2)$  be biclosure spaces. Let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  and  $g : (Y, v_1, v_2) \rightarrow (Z, w_1, w_2)$  be maps. If  $g \circ f$  is semi-open and  $f$  is a continuous surjection, then  $g$  is semi-open.*

**Proof.** Let  $H$  be an open set in  $(Y, v_1, v_2)$  and let  $f$  be continuous. By Proposition 2.17,  $f^{-1}(H)$  is open in  $(X, u_1, u_2)$ . Since  $g \circ f$  is semi-open,  $g \circ f(f^{-1}(H))$  is semi-open in  $(Z, w_1, w_2)$ . But  $f$  is a surjection, hence  $g \circ f(f^{-1}(H)) = g(H)$ . Thus,  $g(H)$  is semi-open in  $(Z, w_1, w_2)$ . Therefore,  $g$  is semi-open. ■

#### 4. SEMI-CONTINUOUS MAPS IN BICLOSURE SPACES

In this section, we study the concept of semi-continuous maps obtained by using semi-open sets.

**Definition 4.1.** Let  $(X, u_1, u_2)$  and  $(Y, v_1, v_2)$  be biclosure spaces. A map  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  is called *semi-continuous* if  $f^{-1}(G)$  is a semi-open subset of  $(X, u_1, u_2)$  for every open subset  $G$  of  $(Y, v_1, v_2)$ .

Clearly, if  $f$  is continuous, then  $f$  is semi-continuous. The converse need not be true as can be seen from the following example.

**Example 4.2.** Let  $X = \{1, 2\} = Y$  and define a closure operator  $u_1$  on  $X$  by  $u_1\emptyset = \emptyset$ ,  $u_1\{1\} = \{1\}$ ,  $u_1\{2\} = u_1X = X$ . Define a closure operator  $u_2$  on  $X$  by  $u_2\emptyset = \emptyset$  and  $u_2\{1\} = u_2\{2\} = u_2X = X$ . Define a closure operator  $v_1$  on  $Y$  by  $v_1\emptyset = \emptyset$ ,  $v_1\{1\} = \{1\}$ ,  $v_1\{2\} = \{2\}$ ,  $v_1Y = Y$  and define a closure operator  $v_2$  on  $Y$  by  $v_2\emptyset = \emptyset$ ,  $v_2\{1\} = \{1\}$  and  $v_2\{2\} = v_2Y = Y$ . Let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  be an identity map. It is easy to see that  $f$  is semi-continuous but not continuous because  $f^{-1}(\{2\})$  is not open in  $(X, u_1, u_2)$  while  $\{2\}$  is open in  $(Y, v_1, v_2)$ .

**Proposition 4.3.** *Let  $(X, u_1, u_2)$  and  $(Y, v_1, v_2)$  be biclosure spaces. A map  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  is semi-continuous if and only if  $f^{-1}(F)$  is a semi-closed subset of  $(X, u_1, u_2)$  for every closed subset  $F$  of  $(Y, v_1, v_2)$ .*

**Proposition 4.4.** *Let  $(X, u_1, u_2)$ ,  $(Y, v_1, v_2)$  and  $(Z, w_1, w_2)$  be biclosure spaces and let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  and  $g : (Y, v_1, v_2) \rightarrow (Z, w_1, w_2)$  be maps. If  $g$  is continuous and  $f$  is semi-continuous, then  $g \circ f$  is semi-continuous.*

**Proof.** Let  $H$  be an open subset of  $(Z, w_1, w_2)$  and let  $g$  be continuous. By Proposition 2.17,  $g^{-1}(H)$  is open in  $(Y, v_1, v_2)$ . As  $f$  is semi-continuous,  $f^{-1}(g^{-1}(H)) = (g \circ f)^{-1}(H)$  is semi-open in  $(X, u_1, u_2)$ . Therefore,  $g \circ f$  is semi-continuous. ■

**Definition 4.5.** A biclosure space  $(X, u_1, u_2)$  is said to be a  $T_s$ -space if every semi-open set in  $(X, u_1, u_2)$  is open in  $(X, u_1, u_2)$ . Clearly, the closure space  $(X, u_1, u_2)$  in Example 3.12 is a  $T_s$ -space.

**Proposition 4.6.** Let  $(X, u_1, u_2)$  and  $(Z, w_1, w_2)$  be biclosure spaces and  $(Y, v_1, v_2)$  be a  $T_s$ -space and let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  and  $g : (Y, v_1, v_2) \rightarrow (Z, w_1, w_2)$  be maps. If  $f$  and  $g$  are semi-continuous, then  $g \circ f$  is semi-continuous.

**Proof.** Let  $H$  be open in  $(Z, w_1, w_2)$ . Since  $g$  is semi-continuous,  $g^{-1}(H)$  is semi-open in  $(Y, v_1, v_2)$ . But  $(Y, v_1, v_2)$  is a  $T_s$ -space, hence  $g^{-1}(H)$  is open in  $(Y, v_1, v_2)$ . As  $f$  is semi-continuous,  $f^{-1}(g^{-1}(H)) = (g \circ f)^{-1}(H)$  is semi-open in  $(X, u_1, u_2)$ . Therefore,  $g \circ f$  is semi-continuous. ■

**Proposition 4.7.** Let  $(X, u_1, u_2)$ ,  $(Y, v_1, v_2)$  and  $(Z, w_1, w_2)$  be biclosure spaces, and let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  and  $g : (Y, v_1, v_2) \rightarrow (Z, w_1, w_2)$  be maps.

- (i) If  $f$  is a semi-open surjection and  $g \circ f$  is continuous, then  $g$  is semi-continuous.
- (ii) If  $g$  is a semi-continuous injection and  $g \circ f$  is open, then  $f$  is semi-open.
- (iii) If  $g$  is an open injection and  $g \circ f$  is semi-continuous, then  $f$  is semi-continuous.

**Proof.**

- (i) Let  $H$  be an open subset of  $(Z, w_1, w_2)$  and let  $g \circ f$  be continuous. By Proposition 2.17,  $(g \circ f)^{-1}(H)$  is open in  $(X, u_1, u_2)$ . Since  $f$  is a semi-open map,  $f((g \circ f)^{-1}(H)) = f(f^{-1}(g^{-1}(H)))$  is semi-open in  $(Y, v_1, v_2)$ . But  $f$  is a surjection, thus  $f(f^{-1}(g^{-1}(H))) = g^{-1}(H)$ . Therefore,  $g$  is semi-continuous.
- (ii) Let  $G$  be an open subset of  $(X, u_1, u_2)$  and let  $g \circ f$  be open. By Proposition 2.15,  $g \circ f(G)$  is open in  $(Z, w_1, w_2)$ . Since  $g$  is semi-continuous,

$g^{-1}(g \circ f(G))$  is semi-open in  $(Y, v_1, v_2)$ . But  $g$  is an injection, hence  $g^{-1}(g \circ f(G)) = f(G)$ . Therefore,  $f$  is semi-open.

- (iii) Let  $H$  be an open subset of  $(Y, v_1, v_2)$  and let  $g$  is open. By Proposition 2.15,  $g(H)$  is open in  $(Z, w_1, w_2)$ . Since  $g \circ f$  is semi-continuous,  $(g \circ f)^{-1}(g(H))$  is semi-open in  $(X, u_1, u_2)$ . But  $g$  is an injection, it follows that  $(g \circ f)^{-1}(g(H)) = f^{-1}(g^{-1}(g(H))) = f^{-1}(H)$ . Therefore,  $f$  is semi-continuous. ■

## 5. SEMI-IRRESOLUTE MAPS IN BICLOSURE SPACES

In this section, we introduce semi-irresolute maps in biclosure spaces obtained by using semi-open sets. We then study some of their basic properties.

**Definition 5.1.** Let  $(X, u_1, u_2)$  and  $(Y, v_1, v_2)$  be biclosure spaces. A map  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  is called *semi-irresolute* if  $f^{-1}(G)$  is semi-open in  $(X, u_1, u_2)$  for every semi-open set  $G$  in  $(Y, v_1, v_2)$ .

It is easy to show that the composition of two semi-irresolute maps of biclosure spaces is again a semi-irresolute map.

**Remark 5.2.** If a map  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  is semi-irresolute, then  $f$  is semi-continuous. The converse need not be true as shown in the following example.

**Example 5.3.** Let  $X = \{1, 2\} = Y$  and define a closure operator  $u_1$  on  $X$  by  $u_1\emptyset = \emptyset$  and  $u_1\{1\} = u_1\{2\} = u_1X = X$ . Define a closure operator  $u_2$  on  $X$  by  $u_2\emptyset = \emptyset$  and  $u_2\{1\} = u_2\{2\} = u_2X = X$ . Define a closure operator  $v_1$  on  $Y$  by  $v_1\emptyset = \emptyset$ ,  $v_1\{1\} = \{1\}$ ,  $v_1\{2\} = v_1Y = Y$  and define a closure operator  $v_2$  on  $Y$  by  $v_2\emptyset = \emptyset$  and  $v_2\{1\} = v_2\{2\} = v_2Y = Y$ . Let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  be an identity map. It is easy to see that there are only two open sets in  $(Y, v_1, v_2)$ , namely  $\emptyset$  and  $Y$ , and their inverse images are semi-open in  $(X, u_1, u_2)$ . Thus,  $f$  is semi-continuous. But  $f$  is not semi-irresolute because  $f^{-1}(\{2\})$  is not semi-open in  $(X, u_1, u_2)$  while  $\{2\}$  is semi-open in  $(Y, v_1, v_2)$ .

**Proposition 5.4.** Let  $(X, u_1, u_2)$  and  $(Y, v_1, v_2)$  be biclosure spaces and let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  be a map. Then  $f$  is semi-irresolute if and only if  $f^{-1}(B)$  is semi-closed in  $(X, u_1, u_2)$ , whenever  $B$  is semi-closed in  $(Y, v_1, v_2)$ .

**Proposition 5.5.** *Let  $(X, u_1, u_2)$  and  $(Y, v_1, v_2)$  be biclosure spaces and let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  be an open, semi-irresolute and surjective map. Then  $(Y, v_1, v_2)$  is a  $T_s$ -space if  $(X, u_1, u_2)$  is a  $T_s$ -space.*

**Proof.** Let  $(X, u_1, u_2)$  be a  $T_s$ -space and let  $B$  be a semi-open subset of  $(Y, v_1, v_2)$ . Since  $f$  is semi-irresolute,  $f^{-1}(B)$  is semi-open in  $(X, u_1, u_2)$ . As  $(X, u_1, u_2)$  is a  $T_s$ -space,  $f^{-1}(B)$  is open in  $(X, u_1, u_2)$ . Since  $f$  is open,  $f(f^{-1}(B))$  is open in  $(Y, v_1, v_2)$  by Proposition 2.15. But  $f$  is a surjection, hence  $f(f^{-1}(B)) = B$ . Thus,  $B$  is open in  $(Y, v_1, v_2)$ . Therefore,  $(Y, v_1, v_2)$  is a  $T_s$ -space. ■

**Proposition 5.6.** *Let  $(X, u_1, u_2)$ ,  $(Y, v_1, v_2)$  and  $(Z, w_1, w_2)$  be biclosure spaces, and let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  and  $g : (Y, v_1, v_2) \rightarrow (Z, w_1, w_2)$  be maps. If  $f$  is semi-irresolute and  $g$  is semi-continuous, then  $g \circ f$  is semi-continuous.*

**Proposition 5.7.** *Let  $(X, u_1, u_2)$  and  $(Y, v_1, v_2)$  be biclosure spaces and let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  be a bijective map.*

- (i) *If  $f$  is 1-continuous and  $f^{-1}$  is 2-continuous, then  $f$  is semi-irresolute.*
- (ii) *If  $f$  is 2-continuous and  $f^{-1}$  is 1-continuous, then  $f^{-1}$  is semi-irresolute.*

**Proof.**

- (i) Let  $B$  be semi-open in  $(Y, v_1, v_2)$ . Then there exists an open subset  $H$  of  $(Y, v_1)$  such that  $H \subseteq B \subseteq v_2H$ . Since  $f^{-1}$  is 2-continuous,  $f^{-1} : (Y, v_2) \rightarrow (X, u_2)$  is continuous. Thus,  $f^{-1}(v_2(H)) \subseteq u_2f^{-1}(H)$ , i.e.  $f^{-1}(H) \subseteq f^{-1}(B) \subseteq u_2f^{-1}(H)$ . As  $f$  is 1-continuous,  $f : (X, u_1) \rightarrow (Y, v_1)$  is continuous, hence  $f^{-1}(H)$  is open in  $(X, u_1)$ . Consequently,  $f^{-1}(B)$  is semi-open in  $(X, u_1, u_2)$ . Therefore,  $f$  is semi-irresolute.
- (ii) Let  $A$  be semi-open in  $(X, u_1, u_2)$ . Then there exists an open set  $G$  of  $(X, u_1)$  such that  $G \subseteq A \subseteq u_2G$ . Since  $f$  is 2-continuous,  $f : (X, u_2) \rightarrow (Y, v_2)$  is continuous. Thus,  $f(u_2G) \subseteq v_2f(G)$ , i.e.  $f(G) \subseteq f(A) \subseteq v_2f(G)$ . But  $f^{-1}$  is 1-continuous, hence  $f^{-1} : (Y, v_1) \rightarrow (X, u_1)$  is continuous. Since  $f(G)$  is the inverse image of  $G$  under  $f^{-1}$ ,  $f(G)$  is open in  $(Y, v_1)$ . Consequently,  $f(A)$  is semi-open in  $(Y, v_1, v_2)$ . But  $f(A)$  is the inverse image of  $A$  under  $f^{-1}$ , thus  $f^{-1}$  is semi-irresolute. ■

## 6. PRE-SEMI-OPEN MAPS IN BICLOSURE SPACES

In this section, we introduce pre-semi-open maps obtained by using semi-open sets. We study some of their properties.

**Definition 6.1.** Let  $(X, u_1, u_2)$  and  $(Y, v_1, v_2)$  be biclosure spaces. A map  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  is called *pre-semi-open* (respectively, *pre-semi-closed*) if  $f(A)$  is a semi-open (respectively, semi-closed) subset of  $(Y, v_1, v_2)$  for every semi-open (respectively, semi-closed) subset  $A$  of  $(X, u_1, u_2)$ .

It is easy to show that the composition of two pre-semi-open maps in biclosure spaces is again a pre-semi-open map.

Clearly, if a map  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  is pre-semi-open, then  $f$  is semi-open. The converse need not be true as shown in the following example.

**Example 6.2.** In Example 2.16, the map  $f$  is semi-open but  $f$  is not pre-semi-open because  $\{1\}$  is semi-open in  $(X, u_1, u_2)$  but  $f(\{1\})$  is not semi-open in  $(Y, v_1, v_2)$ .

**Proposition 6.3.** *Let  $(X, u_1, u_2)$  and  $(Y, v_1, v_2)$  be biclosure spaces and let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  be a map. Then the following statements are equivalent:*

- (i)  $f$  is pre-semi-open
- (ii) If  $B \subseteq Y$  and  $C$  is a semi-closed subset of  $(X, u_1, u_2)$  such that  $f^{-1}(B) \subseteq C$ , then  $B \subseteq E$  and  $f^{-1}(E) \subseteq C$  for some semi-closed subset  $E$  of  $(Y, v_1, v_2)$ .

**Proof.** (i)  $\rightarrow$  (ii) Let  $B$  be a subset of  $Y$  and let  $C$  be a semi-closed subset of  $(X, u_1, u_2)$  such that  $f^{-1}(B) \subseteq C$ . Then  $f(X - C)$  is a semi-open subset of  $(Y, v_1, v_2)$ . Put  $E = Y - f(X - C)$ . Then  $E$  is semi-closed in  $(Y, v_1, v_2)$  and  $X - C \subseteq X - f^{-1}(B) = f^{-1}(Y - B)$ . Hence,  $f(X - C) \subseteq f(f^{-1}(Y - B)) \subseteq Y - B$ . Thus,  $Y - (Y - B) \subseteq Y - f(X - C)$ , i.e.  $B \subseteq E$  and  $f^{-1}(E) = f^{-1}(Y - f(X - C)) = X - f^{-1}(f(X - C)) \subseteq X - (X - C) = C$ . Therefore,  $E$  is a semi-closed subset of  $(Y, v_1, v_2)$  such that  $B \subseteq E$  and  $f^{-1}(E) \subseteq C$ .

(ii)  $\rightarrow$  (i) Let  $A$  be a semi-open subset of  $(X, u_1, u_2)$ . Then  $X - A$  is semi-closed in  $(X, u_1, u_2)$  and  $f^{-1}(Y - f(A)) = X - f^{-1}(f(A)) \subseteq X - A$  where  $Y - f(A)$  is a subset of  $Y$ . By the assumption, there is a semi-closed



subset  $E$  of  $(Y, v_1, v_2)$  such that  $Y - f(A) \subseteq E$  and  $f^{-1}(E) \subseteq X - A$ . Hence,  $Y - E \subseteq f(A)$  and  $A \subseteq X - f^{-1}(E)$ . It follows that  $Y - E \subseteq f(A) \subseteq f(X - f^{-1}(E)) = f(f^{-1}(Y - E)) \subseteq Y - E$ , i.e.  $f(A) = Y - E$ . Thus,  $f(A)$  is semi-open in  $(Y, v_1, v_2)$ . Therefore,  $f$  is pre-semi-open. ■

**Proposition 6.4.** *Let  $(X, u_1, u_2)$  and  $(Y, v_1, v_2)$  be biclosure spaces and let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  be a map. If  $f$  is pre-semi-open, then for every  $y \in Y$  and every semi-closed subset  $C$  of  $(X, u_1, u_2)$  such that  $f^{-1}(\{y\}) \subseteq C$ , there exists a semi-closed subset  $E$  of  $(Y, v_1, v_2)$  such that  $y \in E$  and  $f^{-1}(E) \subseteq C$ .*

**Proof.** Let  $y \in Y$  and let  $C$  be a semi-closed subset of  $(X, u_1, u_2)$  such that  $f^{-1}(\{y\}) \subseteq C$ . Since  $\{y\} \subseteq Y$ , there exists a semi-closed subset  $E$  of  $(Y, v_1, v_2)$  such that  $y \in E$  and  $f^{-1}(E) \subseteq C$  by Proposition 6.3. ■

The converse of the previous statement is not true in general as can be seen from the following example.

**Example 6.5.** Let  $X = \{1, 2, 3\} = Y$  and define a closure operator  $u_1$  on  $X$  by  $u_1\emptyset = \emptyset$ ,  $u_1\{1\} = u_1\{2\} = u_1\{1, 2\} = \{1, 2\}$  and  $u_1\{3\} = u_1\{1, 3\} = u_1\{2, 3\} = u_1X = X$ . Define a closure operator  $u_2$  on  $X$  by  $u_2\emptyset = \emptyset$ ,  $u_2\{3\} = \{3\}$  and  $u_2\{1\} = u_2\{2\} = u_2\{1, 2\} = u_2\{1, 3\} = u_2\{2, 3\} = u_2X = X$ . Define a closure operator  $v_1$  on  $Y$  by  $v_1\emptyset = \emptyset$ ,  $v_1\{1\} = \{1\}$ ,  $v_1\{2\} = \{2\}$ ,  $v_1\{3\} = v_1\{1, 2\} = v_1\{1, 3\} = v_1\{2, 3\} = v_1Y = Y$  and define a closure operator  $v_2$  on  $Y$  by  $v_2\emptyset = \emptyset$  and  $v_2\{1\} = v_2\{2\} = v_2\{3\} = v_2\{1, 2\} = v_2\{1, 3\} = v_2\{2, 3\} = v_2Y = Y$ . Let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  be an identity map. Then there are only three semi-closed subsets of  $(X, u_1, u_2)$ , namely  $\emptyset$ ,  $\{1, 2\}$  and  $X$ . Moreover, we can see that there are only four semi-closed subsets of  $(Y, v_1, v_2)$ , namely  $\emptyset$ ,  $\{1\}$ ,  $\{2\}$  and  $Y$ . Then for every  $y \in Y$  and every semi-closed subset  $C$  of  $(X, u_1, u_2)$  such that  $f^{-1}(\{y\}) \subseteq C$ , there exists a semi-closed subset  $E$  of  $(Y, v_1, v_2)$  such that  $y \in E$  and  $f^{-1}(E) \subseteq C$ . But  $f$  is not pre-semi-open because  $\{3\}$  is semi-open in  $(X, u_1, u_2)$  but  $f(\{3\})$  is not semi-open in  $(Y, v_1, v_2)$ .

**Proposition 6.6.** *Let  $(X, u_1, u_2)$  and  $(Y, v_1, v_2)$  be biclosure spaces and let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  be a map. Then the following statements are equivalent:*

- (i)  $f$  is pre-semi-closed.

- (ii) If  $D \subseteq Y$  and  $A$  is a semi-open subset of  $(X, u_1, u_2)$  such that  $f^{-1}(D) \subseteq A$ , then  $D \subseteq M$  and  $f^{-1}(M) \subseteq A$  for some semi-open subset  $M$  of  $(Y, v_1, v_2)$ .
- (iii) If  $y \in Y$  and  $A$  is a semi-open subset of  $(X, u_1, u_2)$  such that  $f^{-1}(\{y\}) \subseteq A$ , then  $y \in M$  and  $f^{-1}(M) \subseteq A$  for some semi-open subset  $M$  of  $(Y, v_1, v_2)$ .

**Proof.** (i)  $\rightarrow$  (ii) The proof is a minor modification of the proof (i) $\rightarrow$ (ii) in Proposition 6.3.

(ii) $\rightarrow$  (iii) Let  $y \in Y$  and  $A$  be a semi-open subset of  $(X, u_1, u_2)$  such that  $f^{-1}(\{y\}) \subseteq A$ . By (ii), put  $D = \{y\}$ . Then there exists a semi-open subset  $M$  of  $(Y, v_1, v_2)$  such that  $y \in M$  and  $f^{-1}(M) \subseteq A$ .

(iii) $\rightarrow$ (i) Let  $C$  be a semi-closed subset of  $(X, u_1, u_2)$ . Then  $X - C$  is semi-open in  $(X, u_1, u_2)$  and  $f^{-1}(Y - f(C)) = X - f^{-1}(f(C)) \subseteq X - C$ . Let  $y \in Y - f(C) \subseteq Y$  and put  $A = X - C$ . Then  $f^{-1}(\{y\}) \subseteq X - C = A$ . By (iii), there exists a semi-open subset  $M_y$  of  $(Y, v_1, v_2)$  such that  $y \in M_y$  and  $f^{-1}(M_y) \subseteq A = X - C$ , i.e.  $C \subseteq X - f^{-1}(M_y)$ . Hence,  $f(C) \subseteq f(X - f^{-1}(M_y)) = f(f^{-1}(Y - M_y)) \subseteq Y - M_y$ . Thus,  $y \in M_y \subseteq Y - f(C)$  for all  $y \in Y - f(C)$ . It follows that  $Y - f(C) = \cup_{y \in Y - f(C)} M_y$ . By Proposition 3.4,  $\cup_{y \in Y - f(C)} M_y$  is semi-open in  $(Y, v_1, v_2)$ . Consequently,  $f(C)$  is semi-closed in  $(Y, v_1, v_2)$ . Therefore,  $f$  is pre-semi-closed. ■

**Proposition 6.7.** Let  $(X, u_1, u_2)$ ,  $(Y, v_1, v_2)$  and  $(Z, w_1, w_2)$  be biclosure spaces and let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  and  $g : (Y, v_1, v_2) \rightarrow (Z, w_1, w_2)$  be maps.

- (i) If  $f$  is a semi-irresolute surjection and  $g \circ f$  is pre-semi-open, then  $g$  is pre-semi-open.
- (ii) If  $g$  is a semi-irresolute injection and  $g \circ f$  is pre-semi-open, then  $f$  is pre-semi-open.
- (iii) If  $f$  is a pre-semi-open surjection and  $g \circ f$  is semi-irresolute, then  $g$  is semi-irresolute.
- (iv) If  $g$  is a pre-semi-open injection and  $g \circ f$  is semi-irresolute, then  $f$  is semi-irresolute.

**Proof.** (i) Let  $B$  be semi-open in  $(Y, v_1, v_2)$ . Since  $f$  is semi-irresolute,  $f^{-1}(B)$  is semi-open in  $(X, u_1, u_2)$ . But  $g \circ f$  is pre-semi-open and  $f$  is surjective, hence  $g \circ f(f^{-1}(B)) = g(B)$  is semi-open in  $(Z, w_1, w_2)$ . Therefore,  $g$  is pre-semi-open.

The proofs of (ii)-(iv) are minor modifications of that of (i) ■

**Proposition 6.8.** *Let  $(X, u_1, u_2)$  and  $(Y, v_1, v_2)$  be biclosure spaces and let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  be a continuous, pre-semi-open and injective map. Then  $(X, u_1, u_2)$  is a  $T_s$ -space if  $(Y, v_1, v_2)$  is a  $T_s$ -space.*

**Proof.** Let  $(Y, v_1, v_2)$  be a  $T_s$ -space and let  $A$  be a semi-open subset of  $(X, u_1, u_2)$ . Since  $f$  is pre-semi-open,  $f(A)$  is semi-open in  $(Y, v_1, v_2)$ . But  $(Y, v_1, v_2)$  is a  $T_s$ -space, hence  $f(A)$  is open in  $(Y, v_1, v_2)$ . As  $f$  is continuous,  $f^{-1}(f(A))$  is open in  $(X, u_1, u_2)$  by Proposition 2.17. Since  $f$  is injective,  $f^{-1}(f(A)) = A$ . Thus,  $A$  is open in  $(X, u_1, u_2)$ . Therefore,  $(X, u_1, u_2)$  is a  $T_s$ -space. ■

**Proposition 6.9.** *Let  $(X, u_1, u_2)$  and  $(Y, v_1, v_2)$  be biclosure spaces and let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  is a 1-open and 2-continuous map, then  $f$  is pre-semi-open.*

**Proof.** Let  $A$  be semi-open in  $(X, u_1, u_2)$ . Then there exists an open subset  $G$  of  $(X, u_1)$  such that  $G \subseteq A \subseteq u_2G$ . Consequently,  $f(G) \subseteq f(A) \subseteq f(u_2G)$ . Since  $f$  is 2-continuous,  $f : (X, u_2) \rightarrow (Y, v_2)$  is continuous. Hence,  $f(u_2G) \subseteq v_2f(G)$ , i.e.  $f(G) \subseteq f(A) \subseteq v_2f(G)$ . But  $f$  is 1-open, thus  $f : (X, u_1) \rightarrow (Y, v_1)$  is open. It follows that  $f(G)$  is open in  $(Y, v_1)$ . Thus,  $f(A)$  is a semi-open set in  $(Y, v_1, v_2)$ . Therefore,  $f$  is pre-semi-open. ■

**Definition 6.10.** A map  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$ , where  $(X, u_1, u_2)$  and  $(Y, v_1, v_2)$  are biclosure spaces, is called a *semi-homeomorphism* if  $f$  is bijective, semi-irresolute and pre-semi-open.

It is easy to show that the composition of two semi-homeomorphisms of biclosure spaces is again a semi-homeomorphism.

**Remark 6.11.** The concepts of a homeomorphism and a semi-homeomorphism are independent as can be seen from two following examples.

**Example 6.12.** In Example 3.12, the map  $f$  is a semi-homeomorphism but  $f$  is not open. Consequently,  $f$  is not a homeomorphism.

**Example 6.13.** Let  $X = \{1, 2, 3\} = Y$  and define a closure operator  $u_1$  on  $X$  by  $u_1\emptyset = \emptyset$ ,  $u_1\{2\} = u_1\{3\} = u_1\{2, 3\} = \{2, 3\}$  and  $u_1\{1\} = u_1\{1, 2\} = u_1\{1, 3\} = u_1X = X$ . Define a closure operator  $u_2$  on  $X$  by  $u_2\emptyset = \emptyset$ ,  $u_2\{1\} = \{1, 3\}$  and  $u_2\{2\} = u_2\{3\} = u_2\{1, 2\} = u_2\{1, 3\} = u_2\{2, 3\} = u_2X = X$ . Define a closure operator  $v_1$  on  $Y$  by  $v_1\emptyset = \emptyset$ ,  $v_1\{2\} = v_1\{3\} = v_1\{2, 3\} = \{2, 3\}$  and  $v_1\{1\} = v_1\{1, 2\} = v_1\{1, 3\} = v_1Y = Y$ . Define a closure operator  $v_2$  on  $Y$  by  $v_2\emptyset = \emptyset$  and  $v_2\{1\} = v_2\{2\} = v_2\{3\} = v_2\{1, 2\} = v_2\{1, 3\} = v_2\{2, 3\} = v_2Y = Y$ . Let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  be the identity map. Then  $f$  is a homeomorphism but  $f$  is not semi-irresolute because  $f^{-1}(\{1, 2\})$  is not semi-open in  $(X, u_1, u_2)$  while  $\{1, 2\}$  is semi-open in  $(Y, v_1, v_2)$ , i.e.  $f$  is not semi-homeomorphism.

**Proposition 6.14.** *Let  $(X, u_1, u_2)$  and  $(Y, v_1, v_2)$  be biclosure spaces and let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  be a bijective map. Then  $f$  is pre-semi-open if and only if  $f$  is pre-semi-closed.*

As a direct consequence of Proposition 6.14, we have:

**Proposition 6.15.** *Let  $(X, u_1, u_2)$  and  $(Y, v_1, v_2)$  be biclosure spaces and let  $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$  be a bijective semi-irresolute map, then the following statements are equivalent:*

- (i)  $f$  is a semi-homeomorphism,
- (ii)  $f$  is a pre-semi-closed map,
- (iii)  $f$  is a pre-semi-open map.

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