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# TWO CONSTRUCTIONS OF DE MORGAN ALGEBRAS AND DE MORGAN QUASIRINGS

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*Dedicated to Klaus Denecke on the occasion of his 65th birthday*

**Abstract**

De Morgan quasirings are connected to De Morgan algebras in the same way as Boolean rings are connected to Boolean algebras. The aim of the paper is to establish a common axiom system for both De Morgan quasirings and De Morgan algebras and to show how an interval of a De Morgan algebra (or De Morgan quasiring) can be viewed as a De Morgan algebra (or De Morgan quasiring, respectively).

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The concept of a (Boolean) quasiring was introduced by D. Dorninger, H. Länger and M. Mączyński [7] in order to get a ring-like counterpart of an orthomodular lattice similarly as it was done by G. Birkhoff [1] for Boolean rings and Boolean algebras. The motivation of this approach is its application in logics of quantum mechanics, see e.g. [8] for details. Various types of such quasirings were described and compared by the authors in [2]. A similar way was applied when ring-like structures were assigned to De Morgan algebras in [3]. The resulting ring-like structures were called De Morgan quasirings. Presently, De Morgan algebras are studied in multi-valued model checking in computer-science, see e.g. [9] and the references given there.

The aim of the present paper is twofold: At first we find an axiom system which determines both De Morgan algebras and De Morgan quasirings (in dependence of the value of the algebraic constant $1+1$). The second aim is to show when an interval of a De Morgan algebra (resp. De Morgan quasiring) $\mathcal{A}$ can be equipped by operations such that the resulting algebra is a De Morgan algebra (resp. De Morgan quasiring) again and the operations are polynomials over $\mathcal{A}$.

For the reader's convenience, we recall the necessary concepts.

A *De Morgan algebra* (see [1]) is an algebra $\mathcal{A} = (A; \vee, \wedge, {}', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(A; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the unary operation $'$ is an antitone involution on $A$, i.e., $x'' = x$ and $x \leq y$ implies $y' \geq x'$.

A *De Morgan quasiring* (see [3]) is an algebra $\mathcal{A} = (A; +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ satisfying the following identities:

(Q1) $x \cdot x = x$

(Q2) $x \cdot y = y \cdot x$

(Q3) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

(Q4) $0 \cdot x = 0$

(Q5) $1 \cdot x = x$

(Q6) $1 + 0 = 1, \quad 1 + 1 = 0$

(Q7) $x \cdot (1 + (1 + y) \cdot (1 + z)) = 1 + (1 + x \cdot y) \cdot (1 + x \cdot z)$.

We say that a De Morgan quasiring $\mathcal{A}$ satisfies the *correspondence identity* if $x + y = 1 + (1 + x \cdot (1 + y)) \cdot (1 + y \cdot (1 + x))$ holds in $\mathcal{A}$.

As usual, we will replace the multiplicative operation "$\cdot$" by juxtaposition.

The following key result was proved in [3]:

**Proposition 1.**

(a) *Let* $\mathcal{R} = (R; +, \cdot, 0, 1)$ *be a De Morgan quasiring. Define* $x \vee y = 1 + (1 + x)(1 + y)$, $x \wedge y = xy$ *and* $x' = 1 + x$. *Then* $\mathcal{A}(\mathcal{R}) = (R; \vee, \wedge, {}', 0, 1)$ *is a De Morgan algebra.*

(b) *Let* $\mathcal{A} = (A; \vee, \wedge, {}', 0, 1)$ *be a De Morgan algebra. Define* $x + y = (x' \wedge y) \vee (x \wedge y')$, $xy = x \wedge y$. *Then* $\mathcal{R}(\mathcal{A}) = (A; +, \cdot, 0, 1)$ *is a De Morgan quasiring satisfying the correspondence identity.*

(c) *If* $\mathcal{R}$ *is a De Morgan quasiring satisfying the correspondence identity and* $\mathcal{A}$ *is a De Morgan algebra, then the assignments* $\mathcal{A} \mapsto \mathcal{R}(\mathcal{A})$ *and* $\mathcal{R} \mapsto \mathcal{A}(\mathcal{R})$ *are invers to each other, i.e.,* $\mathcal{A}(\mathcal{R}(\mathcal{A})) = \mathcal{A}$ *and* $\mathcal{R}(\mathcal{A}(\mathcal{R})) = \mathcal{R}$.

Let us note that this correspondence is the same as for Boolean algebras and Boolean rings [1], for orthomodular lattices and Booelan quasirings [8].

## 1. THE UNIFORM AXIOM SYSTEM

H. Dobbertin [6] has shown that if $1+1 = 1$ holds in an associative Newman algebra $\mathcal{A}$ then it becomes a Boolean algebra, and if $1+1 = 0$ then it becomes a Boolean ring. For ortholattices and Boolean quasirings, a similar algebra (called N-algebra) was introduced in [5] by the first author and H. Länger such that again for $1 + 1 = 1$ it is an ortholattice, and for $1 + 1 = 0$ it is a Boolean quasiring. This motivated us to find an appropriate algebra with this property also for De Morgan algebras and De Morgan quasirings. Hence, we define the following

**Definition**. A *D-algebra* is an algebra $\mathcal{A} = (A; +, \cdot, {}', 0, 1)$ of type $(2, 2, 1, 0, 0)$ satisfying the following identities:

(D1) $xx = x$

(D2) $xy = yx$

(D3) $x(yz) = (xy)z$

(D4) $0x = 0$

(D5) $1x = x$

(D6) $0' = 1$

(D7) $((xy)'x')' = x$

(D8) $x + y = ((((1+1)'x)'y)'(((1+1)'y)'x)')'$

(D9) $x(y'z')' = ((xy)'(xz)')'$.

**Lemma 1.** *Every D-algebra satisfies*

(a) $x'' = x$,

(b) $(x'y')'x = x$,

(c) *if* $1 + 1 = 0$ *then* $x' = 1 + x$.

***Proof.***

(a) Putting $y = 0$ in (D7) and applying (D2), (D4), (D6) and (D5), we immediately get $x'' = x$.

(b) This follows directly from (D7) and (a): Putting in (D7) $x'$ instead of $x$ and $y'$ instead of $y$, we get $((x'y')'x'')' = x'$, and application of (a) yields the result.

(c) By (D2), (D8), (D5) and (a), we compute $1 + x = (((1+1)x)'((1+1)'x)'')'$. Hence, if $1 + 1 = 0$ we get — using (D2), (D6) and (D5) — $1 + x = ((0x)'(0'x))' = (1x)' = x'$.

■

We are going to show that D-algebras can serve as a uniform axiomatization of both De Morgan algebras and De Morgan quasirings. In this context, De Morgan quasirings are considered as algebras $(A; +, \cdot, {}', 0, 1)$ with the additional operation $x' = 1 + x$.

**Theorem 1.** *Let* $\mathcal{A} = (A; +, \cdot, ', 0, 1)$ *be an algebra of type* $(2, 2, 1, 0, 0)$*. Then* $\mathcal{A}$ *is a De Morgan algebra if and only if* $\mathcal{A}$ *is a D-algebra with* $1+1 = 1$*.*

***Proof.*** Assume that $\mathcal{A} = (A; +, \cdot, ', 0, 1)$ is a De Morgan algebra. Then clearly $1 + 1 = 1$. Since the retract $(A; \cdot)$ is a meet-semilattice with 0 and 1, it satisfies (D1)–(D5). The unary operation is an antitone involution thus also (D6) holds. Due to the De Morgan laws we have $x + y = (x'y')'$ thus, using the absorption law, we infer $((xy)'x')' = xy + x = x$ proving (D7).

Further,

$$((((1 + 1)'x)'y)'(((1 + 1)'y)'x)')'$$

$$= ((0'y)'(0'x)')' = (y'x')' = (x'y')' = x + y$$

proving (D8). Using distributivity in $\mathcal{A}$, we obtain $x(y'z')' = x(y + z) = xy + xz = ((xy)'(xz)')'$ proving (D9).

Conversely, let $\mathcal{A} = (A; +, \cdot, ', 0, 1)$ be a D-algebra satisfying $1+1 = 1$. Applying (D1)–(D5), we recognize that $(A; \cdot)$ is a meet-semilattice with 0 and 1. Using (D8), (D6), Lemma 1(a), (D4) and (D5), we derive

$$x + y = ((((1 + 1)'x)'y)'(((1 + 1)'y)'x)')' = ((0'y)'(0'x)')' = (y'x')' =$$

$$= (x'y')',$$

i.e.,

$$(*) \quad x + y = (x'y')'$$

which is the De Morgan law. This implies — by use of (D1)—(D5) and Lemma 1(a) – that $(A; +)$ is a join-semilattice. By Lemma 1(b), we get

$$x = (x'y')'x = (x + y)x$$

which is the first absorption law. By (D7) we derive the second one

$$xy + x = ((xy)'x')' = x.$$

Thus $(A; +, \cdot, 0, 1)$ is a bounded lattice. Denote by $\leq$ its induced order.

By (D9) and $(*)$ we infer

$$x(y + z) = x(y'z')' = ((xy)'(xz)')' = xy + xz,$$

i.e., this lattice is distributive.

By $(*)$ and Lemma 1(a), we obtain $(xy)' = x' + y'$. Assume $x \leq y$. Then $xy = x$ and hence $x' = x' + y'$, i.e. $y' \leq x'$. Thus the mapping $x \mapsto x'$ is antitone and, by (a) of Lemma 1, it is an antitone involution of $A$. We have shown that $\mathcal{A} = (A; +, \cdot, ', 0, 1)$ is a De Morgan algebra. ∎

**Theorem 2.** *Let* $\mathcal{A} = (A; +, \cdot, ', 0, 1)$ *be an algebra of type* $(2, 2, 1, 0, 0)$. *Then* $\mathcal{A}$ *is a De Morgan quasiring satisfying the correspondence identity if and only if* $\mathcal{A}$ *is a D-algebra with* $1 + 1 = 0$.

***Proof.*** Suppose that $\mathcal{A}$ is a De Morgan quasiring with the correspondence identity. Then it satisfies (D1)–(D5). Since $x' = 1 + x$, by (Q6) we infer $1 = 1 + 0 = 0'$ which is (D6). By (Q7) we get

$$x(y'z')' = x(1 + (1 + y)(1 + z)) = 1 + (1 + xy)(1 + xz) = ((xy)'(xz)')'$$

proving (D9). By Lemma 1(i) in [3], we have $((xy)'x')' = 1+(1+xy)(1+x) = 1 + (1 + x)(1 + xy) = x$ proving (D7). By (Q6) we have $1 + 1 = 0$. It remains to show (D8). Using the correspondence identity, (Q2), $1 + 1 = 0$, $1 = 0'$ and (Q5), we infer

$$x + y = 1 + (1 + x(1 + y))(1 + y(1 + x)) = ((x'y)'(xy')')'$$

$$= ((((1 + 1)'x)'y)'(((1 + 1)'y)'x)')'$$

thus $\mathcal{A}$ is a D-algebra satisfying $1 + 1 = 0$.

Conversely, let $\mathcal{A} = (A; +, \cdot, ', 0, 1)$ be a D-algebra satisfying $1 + 1 = 0$. From (D1)–(D5) we have (Q1)–(Q5). Since — by Lemma 1(c) — $x' = 1 + x$, using (D6) we get $1 + 0 = 0' = 1$. Together with our assumption $1 + 1 = 0$, we obtain (Q6).

By (D9) we have

$$x(1 + (1 + y)(1 + z)) = x(y'z')' = ((xy)'(xz)')' = 1 + (1 + xy)(1 + xz)$$

which is (Q7). Hence, $\mathcal{A}$ is a De Morgan quasiring. We need to prove the correspondence identity. For this, we put 0 instead of $1 + 1$ in (D8) and apply (D6), (D5) and (D2). Then we obtain

$$x + y = ((x'y)'(y'x)')' = 1 + (1 + y(1 + x))(1 + x(1 + y))$$

$$= 1 + (1 + x(1 + y))(1 + y(1 + x)).$$

∎

**Remark 1.** An example of a D-algebra which is neither a De Morgan algebra nor a De Morgan quasiring (i.e., $1+1 \notin \{0,1\}$) is a direct product of a De Morgan algebra and a De Morgan quasiring (both with more than one element).

**Remark 2.** As pointed out by Martin Goldstern, the following modification of our Theorems 1 and 2 holds (which is much more general, but uses different – and larger – sets of laws than our results):

Let $V_0$ and $V_1$ be varieties in the language $\{+,*,0,1\}$ of type $(2,2,0,0)$ such that

- $x*1=x,\ x*0=0$ and $x+0=x$ hold in both varieties,
- $V_1$ satisfies $x+1=1$,
- $V_0$ satisfies $1+1=0$.

Let $E_0$ and $E_1$ be laws defining $V_0$ and $V_1$, respectively. Let $W$ be the variety defined by the equations

$$s+(1+1)=t+(1+1) \text{ for } (s=t)\in E_0$$

and

$$s*(1+1)=t*(1+1) \text{ for } (s=t)\in E_1.$$

Then

1. $V_0 \subseteq W$,
2. $V_1 \subseteq W$,
3. $V_0 = W \cap [1+1=0]$ (where $[1+1=0]$ is the variety of all algebras satisfying $1+1=0$),
4. $V_1 = W \cap [1+1=1]$ (where $[1+1=1]$ is the variety of all algebras satisfying $1+1=1$).

***Proof.*** Easy calculation. ■

## 2. Interval algebras

If $\mathcal{A} = (A; \wedge, \vee, ', 0, 1)$ is a Boolean algebra and $a, b \in A$ with $a \leq b$ then the interval $[a, b]$ can be made into a Boolean algebra $([a, b]; \vee, \wedge, ^*, a, b)$ in such a way that $x^* = (x' \vee a) \wedge b = (x' \wedge b) \vee a$, i.e., $x^*$ is a polynomial of the original algebra $\mathcal{A}$. This new algebra is called an interval Boolean algebra. Similarly, one can establish interval MV-algebras (see [4]) or interval residuated lattices or BL-algebras (see [10]). The aim of this section is to show under what conditions this construction can be made for De Morgan algebras and, due to Proposition 1, also for De Morgan quasirings.

Let $\mathcal{A} = (A; \vee, \wedge, ', 0, 1)$ be a De Morgan algebra. An element $a \in A$ is called *Boolean* if $a \wedge a' = 0$ (or, equivalenty, $a \vee a' = 1$). It is evident that the set $\mathrm{B}(\mathcal{A})$ of all Boolean elements of $\mathcal{A}$ forms a Boolean algebra which is a subalgebra of $\mathcal{A}$. We are going to show that Boolean elements play some role for our construction.

**Theorem 3.** *Let $\mathcal{A} = (A; \vee, \wedge, ', 0, 1)$ be a De Morgan algebra, $a, b \in A$ with $a \leq b$, and put*

$$x^* = (x' \vee a) \wedge b = (x' \wedge b) \vee a.$$

*Then $([a, b]; \vee, \wedge, ^*, a, b)$ is a De Morgan algebra if and only if $b \leq a \vee a'$ and $b \wedge b' \leq a$.*

***Proof.*** If $([a, b]; \vee, \wedge, ^*, a, b)$ is a De Morgan algebra, we must have $a^* = b$ and $b^* = a$. Hence $(a' \vee a) \wedge b = b$ and $(b' \wedge b) \vee a = a$, thus $b \leq a \vee a'$ and $b \wedge b' \leq a$.

Conversely, suppose that $b \leq a \vee a'$ and $b \wedge b' \leq a$. Clearly, the mapping $x \mapsto x^*$ is antitone and maps $[a, b]$ into $[a, b]$. So all we have to show is that it is an involution. Let $x \in [a, b]$ then, by using the De Morgan laws and distributivity, we have

$$\begin{aligned} x^{**} &= (((x' \vee a) \wedge b)' \wedge b) \vee a = (((x \wedge a') \vee b') \wedge b) \vee a \\ &= (x \wedge a' \wedge b) \vee (b' \wedge b) \vee a = (x \wedge a' \wedge b) \vee a \\ &= (x \vee a) \wedge (a' \vee a) \wedge (b \vee a) = x \wedge (a' \vee a) \wedge b \\ &= x \wedge b = x. \end{aligned}$$

∎

**Corollary 1.** *If $a, b$ are Boolean elements of $\mathcal{A}$, then $([a, b]; \vee, \wedge, ^*, a, b)$ is a De Morgan algebra.*

**Corollary 2.** *Let $a \in A$ and put $x_a = x' \vee a$, $x^a = x' \wedge a$. Then the following are equivalent:*

(i) *$a$ is a Boolean element of $\mathcal{A}$.*

(ii) *$([a, 1]; \vee, \wedge, {}_a, a, 1)$ is a De Morgan algebra.*

(iii) *$([0, a]; \vee, \wedge, {}^a, 0, a)$ is a De Morgan algebra.*

We say that a De Morgan algebra is a *De Morgan chain* if it is a chain with respect to the induced order.

**Corollary 3.** *Let $\mathcal{A} = (A; \vee, \wedge, {}', 0, 1)$ be a De Morgan chain and $a, b \in A$ with $a < b$. Then the following are equivalent:*

(i) *$a' = b$.*

(ii) *$b' = a$.*

(iii) *$x' \in [a, b]$ for every $x \in [a, b]$.*

(iv) *$x^* = x'$.*

(v) *$([a, b]; \vee, \wedge, {}^*, a, b)$ is a De Morgan algebra.*

***Proof.*** Evidently, (i)–(iv) are equivalent, and by Theorem 3, (v) is equivalent to (i) and (ii). ■

**Example 1.** Let $\mathcal{A}$ be a De Morgan algebra whose diagram is visualized in Figure 1.

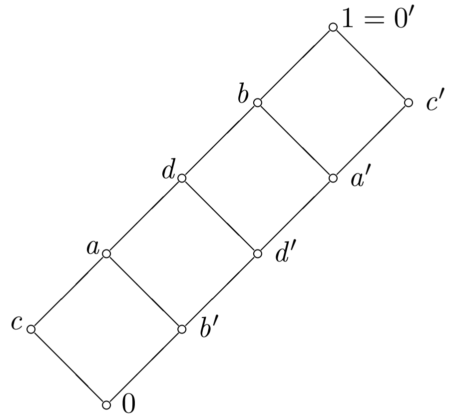


Figure.1

By Theorem 3, the interval $[a, b] = \{a, d, b\}$ is an interval De Morgan algebra $([a, b]; \vee, \wedge, ^*, a, b)$, and we have

$$a^* = (a' \vee a) \wedge b = b \wedge b = b,$$

$$b^* = (b' \vee a) \wedge b = a \wedge b = a,$$

$$d^* = (d' \vee a) \wedge b = d \wedge b = d.$$

Note that the Boolean elements of $\mathcal{A}$ are only $0, 1, c, c'$ hence the converse of Corollary 1 does not hold.

Due to the correspondence given by Proposition 1, we can introduce the induced order in every De Morgan quasiring given by

$$x \leq y \quad \text{if and only if} \quad xy = x.$$

Since $x' = 1 + x$, we call an element $a$ of a De Morgan quasiring $\mathcal{A}$ to be *Boolean* if

$$a(1 + a) = 0.$$

Now, we can state the following result.

**Theorem 4.** *Let $\mathcal{A} = (A; +, \cdot, 0, 1)$ be a De Morgan quasiring satisfying the correspondence identity, $a, b \in A$ and $a \leq b$. Define $x +_{ab} y = (x + y) \vee a$. Then $([a, b]; +_{ab}, \cdot, a, b)$ is a De Morgan quasiring satisfying the correspondence identity if and only if $a(1 + a) \leq 1 + b$ and $b(1 + b) \leq a$.*

***Proof.*** By Theorem 3 and Proposition 1, $([a, b], \vee, \wedge, ^*, a, b)$ is a De Morgan algebra where $x^* = (x' \vee a) \wedge b$ if and only if $a(1+a) = a \wedge a' \leq b' = 1+b$ and $b(1+b) = b \wedge b' \leq a$. Again by Proposition 1, this is equivalent to the fact that $([a, b]; \oplus_{ab}, \cdot, a, b)$ is a De Morgan quasiring satisfying the correspondence identity where $xy = x \wedge y$ and $x \oplus_{ab} y = (x^* \wedge y) \vee (x \wedge y^*)$. We compute for $x, y \in [a, b]$

$$\begin{aligned} x \oplus_{ab} y &= ((x' \vee a) \wedge b \wedge y) \vee (x \wedge (y' \vee a) \wedge b) \\ &= ((x' \vee a) \wedge y) \vee ((y' \vee a) \wedge x) \\ &= (x' \wedge y) \vee a \vee (x \wedge y') \vee a \\ &= (x' \wedge y) \vee (x \wedge y') \vee a = (x + y) \vee a = x +_{ab} y. \end{aligned}$$

■

**Corollary 4.** *Let $\mathcal{A} = (A; +, \cdot, 0, 1)$ be a De Morgan quasiring satisfying the correspondence identity and $a \in A$. Then $([0, a]; +, \cdot, 0, a)$ is a De Morgan quasiring satisfying the correspondence identity if and only if a is a Boolean element. Define $x +_a y = (x + y) \vee a$. Then $([a, 1]; +_a, \cdot, a, 1)$ is a De Morgan quasiring satisfying the correspondence identity if and only if a is a Boolean element.*

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