

**THE MAXIMAL SUBSEMIGROUPS
OF THE IDEALS OF SOME SEMIGROUPS
OF PARTIAL INJECTIONS**

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Abstract

We study the structure of the ideals of the semigroup IO_n of all isotone (order-preserving) partial injections as well as of the semigroup IM_n of all monotone (order-preserving or order-reversing) partial injections on an n -element set. The main result is the characterization of the maximal subsemigroups of the ideals of IO_n and IM_n .

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1. INTRODUCTION

Let $X_n = \{1, 2, \dots, n\}$ be an n - element set ordered in the usual way. The monoid PT_n of all partial transformations of X_n is a very interesting object. In this paper we will multiply transformations from the right to the left and use the corresponding notation for the right to the left composition of transformations: $x(\alpha\beta) = (x\alpha)\beta$, for $x \in X_n$. We say that a transformation $\alpha \in PT_n$ is *isotone* (*order-preserving*) if $x \leq y \implies x\alpha \leq y\alpha$ for all x, y from the domain of α , *antitone* (*order-reversing*) if $x \leq y \implies y\alpha \leq x\alpha$ for all x, y from the domain of α and *monotone* if it is isotone or antitone.

In the present paper, we study the structure of the semigroups IO_n of all isotone partial injections and IM_n of all monotone partial injections of X_n . From the definition of monotone transformations, it is clear that $IO_n \subseteq IM_n$.

Some semigroups of transformations have been studied since the sixties. In fact, presentations of the semigroup O_n of all isotone transformations and of the semigroup PO_n of all isotone partial transformations (excluding the permutation in both cases) were established by Aizenštat ([1]) in 1962 and by Popova ([16]), respectively, in the same year. Some years later (1971), Howie ([14]) studied some combinatorial and algebraic properties of O_n and, in 1992, Gomes and Howie ([13]) established some more properties of O_n , namely its rank and idempotent rank. In recent years it has been studied in different aspects by several authors (for example [4, 15, 17, 18]). The monoid IO_n of all isotone partial injections of X_n has been the object of study since 1997 by Fernandes in various papers ([7, 8, 9]). Some basic properties of IO_n , in particular, a description of Green's relations, congruences and a presentation, were obtained in [2]. Ganyushkin and Mazorchuk ([12]) studied some properties of IO_n as describe ideals, systems of generators, maximal subsemigroups and maximal inverse subsemigroups of IO_n .

In [10], Fernandes, Gomes and Jesus gave a presentation of both the semigroups M_n of all monotone transformations of X_n and the semigroup PM_n of all monotone partial transformations. Dimitrova and Koppitz ([4]) considered the maximal subsemigroups of M_n and its ideals. Delgado and Fernandes ([3]) have computed the abelian kernels of the semigroup IM_n . Fernandes, Gomes and Jesus ([11]) exhibited some properties as well as a presentation for the semigroup IM_n . Dimitrova and Koppitz ([5]) characterized the maximal subsemigroups of IM_n .

In this paper we consider the ideals of the semigroups IO_n and IM_n . In Section 2 we describe the maximal subsemigroups of the ideals of the semigroup IO_n . Each of the considered ideals has exactly $2^{\binom{n}{r}} - 2$ maximal subsemigroups. In Section 3 we characterize the maximal subsemigroups of the ideals of the semigroup IM_n . It happens that each of the considered ideals has exactly $2^{\binom{n}{r}+1} - 3$ maximal subsemigroups.

We will try to keep the standard notation. For every partial transformation α by $\text{dom } \alpha$ and $\text{im } \alpha$ we denote the domain and the image of α , respectively. If α is injective, the number $\text{rank } \alpha := |\text{dom } \alpha| = |\text{im } \alpha|$ is called the rank of α . Clearly, $\text{rank } \alpha\beta \leq \min\{\text{rank } \alpha, \text{rank } \beta\}$ and $\text{im } \beta = \text{im } \alpha\beta$ as well as $\text{dom } \alpha = \text{dom } \alpha\beta$ if $\text{im } \alpha = \text{dom } \beta$. From the definition of isotone and antitone transformation, it follows that every element $\alpha \in IM_n$ is uniquely determined by $\text{dom } \alpha$ and $\text{im } \alpha$ satisfying $|\text{dom } \alpha| = |\text{im } \alpha|$. Moreover, for every $A, B \subset X_n$ of the same cardinality there exists one isotone transformation $\alpha \in IO_n \subseteq IM_n$ and one antitone transformation $\beta \in IM_n$ such that $\text{dom } \alpha = \text{dom } \beta = A$ and $\text{im } \alpha = \text{im } \beta = B$. We will denote by $\alpha_{A,B}$ the unique isotone element $\alpha \in IM_n$ for which $A = \text{dom } \alpha$ and $B = \text{im } \alpha$, and by $\beta_{A,B}$ the unique antitone element $\beta \in IM_n$ for which $A = \text{dom } \beta$ and $B = \text{im } \beta$. The elements $\alpha_{A,A}$, $A \in X_n$, exhaust all idempotents in IO_n as well as in IM_n . For the elements $\beta_{A,A}$, we have $\beta_{A,A}^2 = \alpha_{A,A}$. In case $A = B = X_n$, we will use the notations α_n and β_n instead of α_{X_n, X_n} and β_{X_n, X_n} .

The Green's relations \mathcal{L} , \mathcal{R} , \mathcal{J} and \mathcal{H} on IO_n as well as on IM_n are characterized as follows:

$$\alpha \mathcal{L} \beta \iff \text{im } \alpha = \text{im } \beta$$

$$\alpha \mathcal{R} \beta \iff \text{dom } \alpha = \text{dom } \beta$$

$$\alpha \mathcal{J} \beta \iff \text{rank } \alpha = \text{rank } \beta$$

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

Obviously, every \mathcal{H} -class in IO_n contains exactly one element and every \mathcal{H} -class in $IM_n \setminus \{\alpha \in IM_n : \text{rank } \alpha \leq 1\}$ contains exactly two elements. In the set $\{\alpha \in IM_n : \text{rank } \alpha \leq 1\}$, every \mathcal{H} -class contains exactly one element.

2. MAXIMAL SUBSEMIGROUPS OF THE IDEALS OF IO_n

The semigroup IO_n is the union of the \mathcal{J} -classes J_0, J_1, \dots, J_n , where

$$J_r := \{\alpha \in IO_n : \text{rank } \alpha = r\} \text{ for } r = 0, \dots, n.$$

It is well known that the ideal $I(n, r)$ ($r = 0, \dots, n$) of the semigroup IO_n is the union of \mathcal{J} -classes J_0, J_1, \dots, J_r , i.e.

$$I(n, r) = \{\alpha \in IO_n : \text{rank } \alpha \leq r\}.$$

Every principal factor on IO_n is a Rees quotient $I(n, r)/I(n, r-1)$ ($1 \leq r \leq n$) of which we think as $J_r \cup \{0\}$ (as it is usually convenient), where the product of two elements of J_r is taken to be zero if it falls into $I(n, r-1)$.

Let us denote by Λ_r the collection of all subsets of X_n of cardinality r .

The \mathcal{R} -, \mathcal{L} - and \mathcal{H} - classes in J_r have the following form:

$$R_A := \{\alpha \in I(n, r) : \text{dom } \alpha = A\}, \quad A \in \Lambda_r;$$

$$L_B := \{\alpha \in I(n, r) : \text{im } \alpha = B\}, \quad B \in \Lambda_r;$$

$$H_{A,B} := \{\alpha_{A,B}\} = R_A \cap L_B, \quad A, B \in \Lambda_r.$$

Clearly, each R_A - class (L_A - class), $A \in \Lambda_r$ contains exactly one idempotent $\alpha_{A,A}$. Thus if E_r is the set of all idempotents in the class J_r , then $|E_r| = \binom{n}{r}$.

Since the product $\alpha\beta$ for all $\alpha, \beta \in J_r$ belongs to the class J_r if and only if $\text{im } \alpha = \text{dom } \beta$, it is obvious that

Lemma 1.

$$1. \quad L_B R_A = \begin{cases} J_r, & \text{if } A = B, \\ 0, & \text{if } A \neq B. \end{cases}$$

$$2. \quad \alpha_{A,B} \alpha_{C,D} = \begin{cases} \alpha_{A,D}, & \text{if } B = C, \\ 0, & \text{if } B \neq C. \end{cases}$$

Proposition 1 [7]. $\langle J_r \rangle = I(n, r)$, for $0 \leq r \leq n-1$.

Now we begin with the description of the maximal subsemigroups of the ideals of the semigroup IO_n .

Let us denote by $Dec(\Lambda_r)$ the set of all decompositions (N_1, N_2) of Λ_r , i.e. $N_1 \cup N_2 = \Lambda_r$ and $N_1 \cap N_2 = \emptyset$ where $N_1, N_2 \neq \emptyset$.

Definition 1. Let $(N_1, N_2) \in Dec(\Lambda_r)$ ($r = 1, \dots, n-1$). Then we put

$$S_{(N_1, N_2)} := I(n, r-1) \cup \{\alpha_{A,B} : A \in N_1 \text{ or } B \in N_2\}.$$

The maximal subsemigroups of the ideal $I(n, n) = IO_n$ were described by Ganyushkin and Mazorchuk:

Theorem 1 [12]. *A subsemigroup S of IO_n is maximal if and only if $S = I(n, n-1)$ or $S = \{\alpha_n\} \cup S_{(N_1, N_2)}$, where $(N_1, N_2) \in Dec(\Lambda_{n-1})$.*

In the following, we will consider the maximal subsemigroups of the ideals $I(n, r)$ for $r = 1, \dots, n-1$.

Lemma 2. *Every maximal subsemigroup in $I(n, r)$ contains the ideal $I(n, r-1)$.*

Proof. Let S be a maximal subsemigroup of $I(n, r)$. Assume that $J_r \subseteq S$, then according to Proposition 1 it follows that $I(n, r) = \langle J_r \rangle \subseteq S$, i.e. $S = I(n, r)$, a contradiction. Thus $J_r \not\subseteq S$. Then $S \cup I(n, r-1)$ is a proper subsemigroup of $I(n, r)$ since $I(n, r-1)$ is an ideal, and hence $S \cup I(n, r-1) = S$ by maximality of S . This implies $I(n, r-1) \subseteq S$. ■

Theorem 2. *Let $1 \leq r \leq n-1$. Then a subsemigroup S of $I(n, r)$ is maximal if and only if there is an element $(N_1, N_2) \in Dec(\Lambda_r)$ with $S = S_{(N_1, N_2)}$.*

Proof. Let $S = S_{(N_1, N_2)}$ for some $(N_1, N_2) \in Dec(\Lambda_r)$. Then

$$S = I(n, r-1) \cup \{\alpha_{A,B} : A \in N_1 \text{ or } B \in N_2\}.$$

Therefore, if $\alpha_{A,B} \notin S$ then $A \in N_2$ and $B \in N_1$, and thus $\alpha_{B,A} \in S$.

From Lemma 1 it follows that S is a semigroup. Really, let $\alpha_{A,B}, \alpha_{C,D} \in S$, i.e. $A, C \in N_1$ or $B, D \in N_2$ or $A \in N_1, D \in N_2$. Then we have $\alpha_{A,B}\alpha_{C,D} = \alpha_{A,D} \in S$ for $B = C$ and $\alpha_{A,B}\alpha_{C,D} = 0 \in I(n, r-1) \subseteq S$ for $B \neq C$.

Now we will show that S is maximal. Let $\alpha_{C,D} \in I(n, r) \setminus S$, i.e. $C \notin N_1$ and $D \notin N_2$. Then $D \in N_1$, since $N_1 \cup N_2 = \Lambda_r$ and so $\alpha_{D,P} \in S$ for all $P \in \Lambda_r$ and thus $R_D = \{\alpha_{D,P} : P \in \Lambda_r\} \subseteq S$. Moreover, we have $\alpha_{C,P} = \alpha_{C,D}\alpha_{D,P}$, for all $P \in \Lambda_r$, by Lemma 1. Thus we obtain the \mathcal{R} -class $R_C = \{\alpha_{C,P} : P \in \Lambda_r\} \subseteq \langle S \cup \{\alpha_{C,D}\} \rangle$. Moreover, $C \in N_2$ and so $L_C = \{\alpha_{P,C} : P \in \Lambda_r\} \subseteq S$. Using Lemma 1, we have $L_C R_C = J_r \subseteq \langle S \cup \{\alpha_{C,D}\} \rangle$. Thus, we obtain that $\langle S \cup \{\alpha_{C,D}\} \rangle = I(n, r)$. Therefore, S is a maximal subsemigroup of the ideal $I(n, r)$.

For the converse part let S be a maximal subsemigroup of the ideal $I(n, r)$. From Lemma 2, we have that $I(n, r-1) \subseteq S$. Then $S = I(n, r-1) \cup T$, where $T \subseteq J_r$.

Let $\alpha_{A,B} \notin S$. Then $\langle S \cup \{\alpha_{A,B}\} \rangle = I(n, r)$. Let now $P, Q \in \Lambda_r$. Suppose that $\alpha_{P,Q} \notin S$. Then $\alpha_{P,Q} \in \langle S \cup \{\alpha_{A,B}\} \rangle$ and $\alpha_{P,Q} = \alpha_{P,A}\alpha_{A,B}\alpha_{B,Q}$. Moreover, $\alpha_{P,A} = \alpha_{P,A}\alpha_{A,B}\alpha_{B,A}$ and $\alpha_{B,Q} = \alpha_{B,A}\alpha_{A,B}\alpha_{B,Q}$. This shows that we need $\alpha_{P,A}$ and $\alpha_{B,Q}$ to generate $\alpha_{P,A}$ and $\alpha_{B,Q}$, respectively, with elements of $S \cup \{\alpha_{A,B}\}$. Hence $\alpha_{P,A}, \alpha_{B,Q} \in S$.

Assume that $\alpha_{Q,P} \notin S$. Then $\alpha_{Q,P} = \alpha_{Q,A}\alpha_{A,B}\alpha_{B,P}$ and by the same arguments, we obtain that $\alpha_{Q,A}, \alpha_{B,P} \in S$.

Further, from $\alpha_{Q,P} = \alpha_{Q,A}\alpha_{A,P}$ it follows that $\alpha_{A,P} \notin S$. But $\alpha_{P,Q} \notin \langle S \cup \{\alpha_{A,P}\} \rangle$ since $\alpha_{P,Q} = \alpha_{P,A}\alpha_{A,P}\alpha_{P,Q}$. This contradicts the maximality of S and thus $\alpha_{Q,P} \in S$. Hence if $\alpha_{P,Q} \notin S$ then $\alpha_{Q,P} \in S$ for any $P, Q \in \Lambda_r$. Therefore, for $N_1 = \{B : \alpha_{A,B} \notin S\}$ and $N_2 = \{A : \alpha_{A,B} \notin S\}$ we have that $S = S_{(N_1, N_2)}$. \blacksquare

There are exactly $2^{\binom{n}{r}} - 2$ maximal subsemigroups of the ideal $I(n, r)$, for $r = 1, \dots, n-1$ and $2^n - 1$ maximal subsemigroups of $I(n, n)$.

3. MAXIMAL SUBSEMIGROUPS OF THE IDEALS OF IM_n

The semigroup IM_n is the union of the \mathcal{J} -classes J_0, J_1, \dots, J_n , where

$$J_r := \{\alpha \in IM_n : \text{rank } \alpha = r\} \text{ for } r = 0, \dots, n.$$

It is well known that the ideal $I(n, r)$ ($r = 0, \dots, n$) of the semigroup IM_n is the union of \mathcal{J} -classes J_0, J_1, \dots, J_r , i.e.

$$I(n, r) = \{\alpha \in IM_n : \text{rank } \alpha \leq r\}.$$

Every principal factor on IM_n is a Rees quotient $I(n, r)/I(n, r-1)$ ($1 \leq r \leq n$) of which we think as $J_r \cup \{0\}$, where the product of two elements of J_r is taken to be zero if it falls into $I(n, r-1)$.

The \mathcal{R} -, \mathcal{L} - and \mathcal{H} - classes in J_r have the following form:

$$R_A := \{\alpha \in I(n, r) : \text{dom } \alpha = A\}, \quad A \in \Lambda_r;$$

$$L_B := \{\alpha \in I(n, r) : \text{im } \alpha = B\}, \quad B \in \Lambda_r;$$

$$H_{A,B} := \{\alpha_{A,B}, \beta_{A,B}\} = R_A \cap L_B, \quad A, B \in \Lambda_r.$$

The \mathcal{L} -class, \mathcal{R} -class and \mathcal{H} -class, respectively, containing the element $\alpha \in IM_n$ will be denoted by L_α , R_α , and H_α , respectively.

Since the product $\alpha\beta$ for all $\alpha, \beta \in J_r$ belongs to the class J_r if and only if $\text{im } \alpha = \text{dom } \beta$, it is easy to show that

Lemma 3.

1. $L_B R_A = \begin{cases} J_r, & \text{if } A = B, \\ 0, & \text{if } A \neq B. \end{cases}$
2. $H_{A,B} H_{C,D} = \begin{cases} H_{A,D}, & \text{if } B = C, \\ 0, & \text{if } B \neq C. \end{cases}$

Let U be a subset of the semigroup IM_n . We denote by U^i (respectively U^a) the set of all isotone (respectively antitone) transformations in the set U . An immediate but important property is that the product of two isotone transformations or two antitone transformations is an isotone, and the product of an isotone transformation with an antitone transformation, or vice versa, is an antitone one.

Proposition 2. $J_r \subseteq \langle J_r^a \rangle$ and $J_r \subseteq \langle J_r^i \cup \{\beta_{A,B}\} \rangle$, for all $A, B \in \Lambda_r$.

Proof. Let $A, B \in \Lambda_r$. Then for all $C \in \Lambda_r$, we have $\alpha_{A,B} = \beta_{A,C}\beta_{C,B}$. Therefore, $J_r \subseteq \langle J_r^a \rangle$.

From $L_A^i \beta_{A,B} = L_B^a$ and $L_B^a R_B^i = J_r^a$, we have $J_r \subseteq \langle J_r^i \cup \{\beta_{A,B}\} \rangle$. ■

Proposition 3. $\langle J_r \rangle = I(n, r)$, for $0 \leq r \leq n - 1$.

Proof. Clearly $\langle J_0 \rangle = I(n, 0)$. In [5], it was shown that $J_{r-1}^i \subseteq J_r^i J_r^i$ and $J_{r-1}^a \subseteq J_{r-1}^i J_r^a J_{r-1}^i$ for $1 \leq r \leq n - 1$. Since $I(n, r) = J_0 \cup J_1 \cup \dots \cup J_r$, we have $\langle J_r \rangle = I(n, r)$. ■

From Proposition 2 and Proposition 3 we have

Corollary 1. Let $1 \leq r \leq n - 1$. Then $\langle J_r^a \rangle = \langle J_r^i \cup \{\beta_{A,B}\} \rangle = I(n, r)$, for all $A, B \in \Lambda_r$.

Now we begin with the description of the maximal subsemigroups of the ideals of the semigroup IM_n .

Clearly, the ideal $I(n, 1)$ of IM_n coincides with the ideal $I(n, 1)$ of IO_n . Thus the maximal subsemigroups of this ideal are characterized in Theorem 2 and there are exactly $2^n - 2$ such semigroups.

Now we will consider the maximal subsemigroups of the ideals $I(n, r)$ for $r = 2, \dots, n - 1$.

Lemma 4. Every maximal subsemigroup in $I(n, r)$ contains the ideal $I(n, r - 1)$.

The proof is similar as that in Lemma 2.

Theorem 3 Let $2 \leq r \leq n - 1$. Then a subsemigroup S of $I(n, r)$ is maximal if and only if it belongs to one of the following three types:

- (1) $S^{(1)} := I(n, r - 1) \cup J_r^i$;
- (2) $S_{(N_1, N_2)}^{(2)} := \bigcup \{H_\alpha : \alpha \in S_{(N_1, N_2)}\}$, for $(N_1, N_2) \in Dec(\Lambda_r)$;
- (3) $S_{(N_1, N_2)}^{(3)} := I(n, r - 1) \cup \{\alpha_{A,B} : A, B \in N_1 \text{ or } A, B \in N_2\} \cup$
 $\cup \{\beta_{A,B} : A \in N_1, B \in N_2 \text{ or } A \in N_2, B \in N_1\}$ for $(N_1, N_2) \in Dec(\Lambda_r)$.

Proof.

(1) It is obvious that $S^{(1)} = I(n, r-1) \cup J_r^i$ is a semigroup, since $I(n, r-1)$ is an ideal and $(J_r^i)^2 \subseteq I^i(n, r) \subseteq I(n, r-1) \cup J_r^i$. From Proposition 2, we have that $J_r \subseteq \langle J_r^i \cup \{\beta_{A,B}\} \rangle$ for all $\beta_{A,B} \in J_r^a$. Since $I(n, r) \setminus S^{(1)} = J_r^a$, we obtain $\langle S^{(1)} \cup \{\beta_{A,B}\} \rangle = I(n, r)$ for all $\beta_{A,B} \in J_r^a$. Therefore, $S^{(1)}$ is maximal in $I(n, r)$.

(2) Let $S = S_{(N_1, N_2)}^{(2)}$ for some $(N_1, N_2) \in Dec(\Lambda_r)$. Then

$$S = I(n, r-1) \cup \{H_{A,B} : A \in N_1 \text{ or } B \in N_2\}.$$

From Lemma 3 it follows that S is a semigroup. Really, let $H_{A,B}, H_{C,D} \subseteq S$, i.e. $A, C \in N_1$ or $B, D \in N_2$ or $A \in N_1, D \in N_2$. Then we have $H_{A,B}H_{C,D} = H_{A,D} \subseteq S$ for $B = C$ and $H_{A,B}H_{C,D} \subseteq I(n, r-1) \subseteq S$ for $B \neq C$.

Now we will show that S is maximal. Let $H_{C,D} = \{\alpha_{C,D}, \beta_{C,D}\} \subseteq I(n, r) \setminus S$, i.e. $C \notin N_1$ and $D \notin N_2$. Then $D \in N_1$, since $N_1 \cup N_2 = \Lambda_r$ and so $H_{D,P} \in S$ for all $P \in \Lambda_r$ and thus $R_D = \bigcup_{P \in \Lambda_r} H_{D,P} \subseteq S$. Moreover, we have

$$H_{C,P} = H_{C,D}H_{D,P}, \text{ for } P \in \Lambda_r,$$

by Lemma 3. Thus we obtain the \mathcal{R} -class $R_C = \bigcup_{P \in \Lambda_r} H_{C,P} \subseteq \langle S \cup H_{C,D} \rangle$. Moreover, $C \in N_2$ and so $L_C = \bigcup_{P \in \Lambda_r} H_{P,C} \subseteq S$. Using Lemma 3, we have $L_C R_C = J_r \subseteq \langle S \cup H_{C,D} \rangle$. Since $\alpha_{C,D} = \beta_{C,D}\beta_{D,D}$ and $\beta_{C,D} = \alpha_{C,D}\beta_{D,D}$, where $\beta_{D,D} \in R_D \subseteq S$, we obtain that $\langle S \cup \{\alpha_{C,D}\} \rangle = I(n, r)$ and $\langle S \cup \{\beta_{C,D}\} \rangle = I(n, r)$. Therefore, S is a maximal subsemigroup of the ideal $I(n, r)$.

(3) Let $S = S_{(N_1, N_2)}^{(3)}$ for some $(N_1, N_2) \in Dec(\Lambda_r)$. From Lemma 3, it follows that S is a semigroup. We will show that S is maximal. Let

$$\begin{aligned} V := I(n, r) \setminus S = & \{\beta_{A,B} : A, B \in N_1 \text{ or } A, B \in N_2\} \cup \\ & \cup \{\alpha_{A,B} : A \in N_1, B \in N_2 \text{ or } A \in N_2, B \in N_1\} \end{aligned}$$

and let $\gamma \in V$. Then for the transformation γ we have four possibilities:

Let $\gamma \in \{\beta_{A,B} : A, B \in N_1\}$. Then $\alpha_{C,A} \in S$ (since $A \in N_1$) and so $\alpha_{C,A}\beta_{A,B} = \beta_{C,B} \in \langle S \cup \{\gamma\} \rangle$ for all $C \in N_1$. Also, we have $\beta_{C,A} \in S$ and thus $\beta_{C,A}\beta_{A,B} = \alpha_{C,B} \in \langle S \cup \{\gamma\} \rangle$ for all $C \in N_2$. Since $\alpha_{C,B} \in S$ for all $C \in N_1$ and $\beta_{C,B} \in S$ for all $C \in N_2$, we obtain $L_B = \bigcup_{C \in \Lambda_r} H_{C,B} \subseteq \langle S \cup \{\gamma\} \rangle$. Further, $\beta_{B,B} \in L_B$ and $\beta_{B,B}\beta_{B,D} = \alpha_{B,D}$ for all $D \in N_2$ as well as $\beta_{B,B}\alpha_{B,D} = \beta_{B,D}$ for all $D \in N_1$. Thus since $\alpha_{B,D} \in S$ for all $D \in N_1$ and $\beta_{B,D} \in S$ for all $D \in N_2$, we obtain $R_B = \bigcup_{D \in \Lambda_r} H_{B,D} \subseteq \langle S \cup \{\gamma\} \rangle$. From Lemma 3, we have $L_B R_B = J_r$ and therefore $\langle S \cup \{\gamma\} \rangle = I(n, r)$.

– For $\gamma \in \{\beta_{A,B} : A, B \in N_2\}$, the proof is similar.

– Let $\gamma \in \{\alpha_{A,B} : A \in N_1, B \in N_2\}$. Then $\alpha_{C,A} \in S$ (since $A \in N_1$) and so $\alpha_{C,A}\alpha_{A,B} = \alpha_{C,B} \in \langle S \cup \{\gamma\} \rangle$ for all $C \in N_1$. Also, we have $\beta_{C,A} \in S$ and thus $\beta_{C,A}\alpha_{A,B} = \beta_{C,B} \in \langle S \cup \{\gamma\} \rangle$ for all $C \in N_2$. Since $\alpha_{C,B} \in S$ for all $C \in N_2$ and $\beta_{C,B} \in S$ for all $C \in N_1$, we obtain $L_B = \bigcup_{C \in \Lambda_r} H_{C,B} \subseteq \langle S \cup \{\gamma\} \rangle$. Further, $\beta_{B,B} \in L_B$ and $\beta_{B,B}\alpha_{B,D} = \beta_{B,D}$ for all $D \in N_2$ as well as $\beta_{B,B}\beta_{B,D} = \alpha_{B,D}$ for all $D \in N_1$. Thus since $\alpha_{B,D} \in S$ for all $D \in N_2$ and $\beta_{B,D} \in S$ for all $D \in N_1$, we obtain $R_B = \bigcup_{D \in \Lambda_r} H_{B,D} \subseteq \langle S \cup \{\gamma\} \rangle$. From Lemma 3, we have $L_B R_B = J_r$ and therefore $\langle S \cup \{\gamma\} \rangle = I(n, r)$.

– For $\gamma \in \{\alpha_{A,B} : A \in N_2, B \in N_1\}$, the proof is similar.

Altogether, this shows that S is maximal.

For the converse part let S be a maximal subsemigroup of the ideal $I(n, r)$. From Lemma 4, we have that $I(n, r-1) \subseteq S$. Then $S = I(n, r-1) \cup T$, where $T \subseteq J_r$. We consider two cases for the set T .

1. Let $T = J_r^i$. Then $S = I(n, r-1) \cup J_r^i = S^{(1)}$.
2. Let now $T \neq J_r^i$. Assume that $J_r^i \subseteq T$. Then $T = J_r^i \cup T'$ where $\emptyset \neq T' \subseteq J_r^a$. From Corollary 1, we have $S = I(n, r)$, a contradiction. Thus $J_r^i \not\subseteq T$. We also have that $J_r^a \not\subseteq T$ since $\langle J_r^a \rangle = I(n, r)$.

Admit that $H_{A,B} \subseteq S$ or $H_{A,B} \cap S = \emptyset$, for all $A, B \in \Lambda_r$. Assume that $S^i = S \cap I^i(n, r)$ is not a maximal subsemigroup of $I^i(n, r)$. Then there is an isotone transformation $\alpha_{A,B} \in I(n, r) \setminus S$ such that $\langle S^i \cup \{\alpha_{A,B}\} \rangle$ is a proper subset of $I^i(n, r)$. Therefore, there exists an $\alpha_{C,D} \in I(n, r) \setminus S$ such that $\alpha_{C,D} \notin \langle S^i \cup \{\alpha_{A,B}\} \rangle$. But $\langle S \cup \{\alpha_{A,B}\} \rangle = I(n, r)$ since S is maximal and $\alpha_{C,D} = \beta_{C,A}\alpha_{A,B}\beta_{B,D}$. Moreover, $\beta_{C,A} = \beta_{C,A}\alpha_{A,B}\alpha_{B,A} = \alpha_{C,A}\alpha_{A,B}\beta_{B,A}$ and $\beta_{B,D} = \beta_{B,A}\alpha_{A,B}\alpha_{B,D} = \alpha_{B,A}\alpha_{A,B}\beta_{B,D}$. This shows that we need

$\beta_{C,A}$ or $\alpha_{C,A}$ and $\beta_{B,D}$ or $\alpha_{B,D}$ to generate $\beta_{C,A}$ and $\beta_{B,D}$, respectively, with elements of $S \cup \{\alpha_{A,B}\}$. This implies that $\beta_{C,A}, \alpha_{C,A}, \beta_{B,D}, \alpha_{B,D} \in S$, since we assume that $H_{A,B} \subseteq S$ or $H_{A,B} \cap S = \emptyset$, for all $A, B \in \Lambda_r$. Hence $\alpha_{C,D} = \alpha_{C,A}\alpha_{A,B}\alpha_{B,D} \in \langle S^i \cup \{\alpha_{A,B}\} \rangle$, a contradiction. Therefore, we obtain that S^i is maximal in $I^i(n, r)$. Since all maximal subsemigroups of the ideal $I^i(n, r)$ are of type $S_{(N_1, N_2)}$ we have $S = \cup\{H_\alpha : \alpha \in S^i\} = S_{(N_1, N_2)}^{(2)}$, for some $(N_1, N_2) \in Dec(\Lambda_r)$.

Now, admit that $|H_{A,B} \cap S| = 1$, for some $A, B \in \Lambda_r$. Suppose that $\alpha_{A,B} \notin S$ and $\beta_{A,B} \in S$. Then from $\alpha_{A,B} = \beta_{A,B}\beta_{B,B}$ and $\alpha_{A,B} = \beta_{A,A}\beta_{A,B}$, it follows that $\beta_{A,A}, \beta_{B,B} \notin S$. Moreover, from $\beta_{A,B}\alpha_{B,A} = \beta_{A,A} \notin S$, we get $\alpha_{B,A} \notin S$. Assume that $\beta_{B,A} \notin S$. Then $\beta_{B,A} \in \langle S \cup \{\alpha_{B,A}\} \rangle$, because of the maximality of S , and since $\beta_{B,A} = \beta_{B,B}\alpha_{B,A}\alpha_{A,A} = \alpha_{B,B}\alpha_{B,A}\beta_{A,A}$, we obtain $\beta_{A,A} \in S$ or $\beta_{B,B} \in S$, a contradiction, and thus $\beta_{B,A} \in S$.

Further, let $P, Q \in \Lambda_r$. Suppose that $\alpha_{P,Q} \notin S$. Then from $\alpha_{P,Q} = \alpha_{P,A}\beta_{A,B}\beta_{B,Q}$, it follows that if $\alpha_{P,A} \in S$ then $\beta_{B,Q} \notin S$ and vice versa. Also from $\alpha_{P,Q} = \beta_{P,A}\beta_{A,B}\alpha_{B,Q}$, it follows that if $\beta_{P,A} \in S$ then $\alpha_{B,Q} \notin S$ and vice versa. Moreover, $\alpha_{P,Q} \in \langle S \cup \{\alpha_{A,B}\} \rangle$ since S is maximal. Hence $\alpha_{P,Q} = \alpha_{P,A}\alpha_{A,B}\alpha_{B,Q} = \beta_{P,A}\alpha_{A,B}\beta_{B,Q}$. Therefore, we have $\alpha_{P,A}, \alpha_{B,Q} \in S$ and $\beta_{P,A}, \beta_{B,Q} \notin S$ or vice versa.

Assume that $\beta_{P,Q} \notin S$. Then $\beta_{P,Q} \in \langle S \cup \{\alpha_{A,B}\} \rangle$ and so $\beta_{P,Q} = \alpha_{P,A}\alpha_{A,B}\beta_{B,Q} = \beta_{P,A}\alpha_{A,B}\alpha_{B,Q}$. But we obtain already that if $\alpha_{P,A}, \alpha_{B,Q} \in S$ then $\beta_{P,A}, \beta_{B,Q} \notin S$ or vice versa. Therefore, $\beta_{P,Q} \notin \langle S \cup \{\alpha_{A,B}\} \rangle$. This contradicts the maximality of S and thus $\beta_{P,Q} \in S$.

Further, from $\alpha_{P,Q} = \beta_{P,Q}\beta_{Q,Q}$ and $\alpha_{P,Q} = \beta_{P,P}\beta_{P,Q}$, it follows that $\beta_{P,P}, \beta_{Q,Q} \notin S$. Moreover, from $\beta_{P,Q}\alpha_{Q,P} = \beta_{P,P} \notin S$, we get $\alpha_{Q,P} \notin S$. Assume that $\beta_{Q,P} \notin S$. Then $\beta_{Q,P} \in \langle S \cup \{\alpha_{Q,P}\} \rangle$, because of the maximality of S , and since $\beta_{Q,P} = \beta_{Q,Q}\alpha_{Q,P}\alpha_{P,P} = \alpha_{Q,Q}\alpha_{Q,P}\beta_{P,P}$, we obtain $\beta_{P,P} \in S$ or $\beta_{Q,Q} \in S$, a contradiction, and thus $\beta_{Q,P} \in S$.

Analogously, if $\beta_{P,Q} \notin S$ we have that $\beta_{Q,P} \notin S$ and $\alpha_{P,Q}, \alpha_{Q,P} \in S$.

Suppose that $\alpha_{P,Q} \in S$ for some $P, Q \in \Lambda_r$. Then $\beta_{P,Q} \notin S$. Otherwise, from $\alpha_{A,B} = \alpha_{A,P}\beta_{P,Q}\beta_{Q,B} \notin S$ it follows

- i) $\alpha_{A,P} \notin S$ and $\beta_{Q,B} \in S$, i.e. $\beta_{A,P} \in S$ and $\beta_{Q,B} \in S$;
- ii) $\alpha_{A,P} \in S$ and $\beta_{Q,B} \notin S$, i.e. $\alpha_{A,P} \in S$ and $\alpha_{Q,B} \in S$;
- iii) $\alpha_{A,P} \notin S$ and $\beta_{Q,B} \notin S$, i.e. $\beta_{A,P} \in S$ and $\alpha_{Q,B} \in S$.

Then $\alpha_{A,B} = \beta_{A,P}\alpha_{P,Q}\beta_{Q,B} = \alpha_{A,P}\alpha_{P,Q}\alpha_{Q,B} = \beta_{A,P}\beta_{P,Q}\alpha_{Q,B}$, which contradicts that $\alpha_{A,B} \notin S$.

The proof when $\alpha_{A,B} \in S$ and $\beta_{A,B} \notin S$ is similar.

Finally, we obtain that

$$(1) \quad \alpha_{P,Q} \in S \iff \beta_{P,Q} \notin S$$

for $P, Q \in \Lambda_r$.

Let $\rho_r := \{(P, Q) : \alpha_{P,Q} \in S\}$. Obviously, ρ_r is an equivalence relation on Λ_r with $\Lambda_r/\rho_r = \{N_1, N_2, \dots, N_m\}$ ($m \geq 2$). Indeed, ρ_r is reflexive since $E_r \subseteq S$, symmetric because of the previous considerations and transitive since $\alpha_{P,Q}\alpha_{Q,R} = \alpha_{P,R} \in S$ for $\alpha_{P,Q}, \alpha_{Q,R} \in S$. Moreover, $m \geq 2$ becomes clear by $J_r^i \not\subseteq T$. Assume that the decomposition contains more than two elements, i.e. $m > 2$. Then there are N_1, N_2, N_3 in our decomposition such that $A \in N_1$, $B \in N_2$ and $C \in N_3$. Thus $\alpha_{A,B} = \beta_{A,C}\beta_{C,B} \in S$, a contradiction. Therefore, $\Lambda_r/\rho_r = \{N_1, N_2\}$ and $S = S_{(N_1, N_2)}^{(3)}$, because of (1). \blacksquare

There are exactly $2^{\binom{n}{r}} - 2$ maximal subsemigroups of the ideal $I^i(n, r)$ and exactly $2^{\binom{n}{r}} - 2$ maximal subsemigroups of type (3). Taking $I(n, r-1) \cup J_r^i$ into account, we get $2^{\binom{n}{r}+1} - 3$ maximal subsemigroups of the ideal $I(n, r)$, for $r = 2, \dots, n-1$.

Finally, we characterize the maximal subsemigroups of the ideal $I(n, n) = IM_n$.

For $A \in \Lambda_{n-1}$ we put $\bar{A} := \{n+1-i : i \in A\}$ and for $N \subseteq \mathcal{P}(X_n)$ we set $\bar{N} := \{\bar{A} : A \in N\}$. Then we have

$$(2) \quad \begin{aligned} \beta_{A, \bar{A}} \alpha_{\bar{A}, B} &= \beta_n \alpha_{\bar{A}, B} = \beta_{A, B}, \\ \beta_{A, \bar{A}} \beta_{\bar{A}, B} &= \beta_n \beta_{\bar{A}, B} = \alpha_{A, B}, \\ \alpha_{B, A} \beta_{A, \bar{A}} &= \alpha_{B, A} \beta_n = \beta_{B, \bar{A}}, \\ \beta_{B, A} \beta_{A, \bar{A}} &= \beta_{B, A} \beta_n = \alpha_{B, \bar{A}}. \end{aligned}$$

Theorem 4. *A subsemigroup S of IM_n is maximal if and only if it belongs to one of the following three types:*

- (1) $T := I(n, n-1) \cup \{\alpha_n\}$;
- (2) $T_{(N_1, N_2)} := J_n \cup \{H_\alpha : \alpha \in S_{(N_1, N_2)}\}$, for $(N_1, N_2) \in Dec(\Lambda_{n-1})$
with $\overline{N}_1 = N_1$ and $\overline{N}_2 = N_2$;
- (3) $T_{(N, \overline{N})} := J_n \cup I(n, n-2) \cup \{\alpha_{A, B} : A, B \in N \text{ or } A, B \in \overline{N}\}$
 $\cup \{\beta_{A, B} : A \in N, B \in \overline{N} \text{ or } A \in \overline{N}, B \in N\}$ for $(N, \overline{N}) \in Dec(\Lambda_{n-1})$.

Proof. It is clear that T is a maximal subsemigroup of IM_n . Further, we put

$$Inv := \{\beta_{A, \overline{A}} : A \in \Lambda_{n-1}\}.$$

Let $(N_1, N_2) \in Dec(\Lambda_{n-1})$ be a decomposition with the required properties. Since $Inv \subseteq T_{(N_1, N_2)}$ and by (2) it is easy to verify that $T_{(N_1, N_2)}$ is a subsemigroup of IM_n . Since $T_{(N_1, N_2)} \setminus J_n$ is a maximal subsemigroup of $I(n, n-1)$ by Theorem 3 and $J_n \subseteq T_{(N_1, N_2)}$, it follows that $T_{(N_1, N_2)}$ is a maximal subsemigroup of IM_n . Analogously, one can show that $T_{(N, \overline{N})}$ is a maximal subsemigroup of IM_n .

For the converse part, let S be maximal in IM_n . Admit that $J_n \not\subseteq S$. Then it is easy to see that $S = T$. Now suppose that $J_n \subseteq S$. Assume that $Inv \not\subseteq S$. Then there is an $A \in \Lambda_{n-1}$ with $\beta_{A, \overline{A}} \notin S$. Since S is maximal, we have $IM_n = \langle S \cup \{\beta_{A, \overline{A}}\} \rangle = S \cup \{\beta_{A, \overline{A}}\}$ by (2). Thus $S = IM_n \setminus \{\beta_{A, \overline{A}}\}$. But $\beta_{A, \overline{A}} = \alpha_{A, B} \beta_{B, \overline{A}}$ for some $B \in \Lambda_{n-1}$ with $B \neq A$. Since $\alpha_{A, B}, \beta_{B, \overline{A}} \in S$, we have $S = IM_n$, a contradiction. Hence $Inv \subseteq S$. Let $S_{n-1} := S \cap I(n, n-1)$. Assume that S_{n-1} is not a maximal subsemigroup of $I(n, n-1)$. Clearly, $S_{n-1} \neq I(n, n-1)$. Let $\gamma \in I(n, n-1) \setminus S_{n-1}$. Then for all $\delta \in I(n, n-1)$, we have $\delta \in \langle S \cup \{\gamma\} \rangle = \langle S_{n-1} \cup \{\gamma\} \rangle \cup J_n$ by (2) and since $Inv \subseteq S$. This shows that $\delta \in \langle S_{n-1} \cup \{\gamma\} \rangle$ and thus $\langle S_{n-1} \cup \{\gamma\} \rangle = I(n, n-1)$. Consequently, S_{n-1} is a maximal subsemigroup of $I(n, n-1)$. Using Theorem 3 we choose all decompositions $(N_1, N_2) \in Dec(\Lambda_{n-1})$ such that $Inv \subseteq S_{(N_1, N_2)}^{(2)}$ and $Inv \subseteq S_{(N_1, N_2)}^{(3)}$, respectively. In this way we obtain the semigroups $T_{(N_1, N_2)}$ and $T_{(N, \overline{N})}$. ■

It is straightforward to calculate that there are exactly $2^{\frac{n+1}{2}} - 1$ maximal subsemigroups of IM_n if n is odd and exactly $2^{\frac{n}{2}} - 1$ maximal subsemigroups of IM_n if n is even.

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