

HYPERSATISFACTION OF FORMULAS IN AGEBRAIC SYSTEMS

KLAUS DENECKE AND DARA PHUSANGA

*Universität Potsdam Institut für Mathematik
Am Neuen Palais 10, D-14469 Potsdam (Germany)*

e-mail: kdenecke@rz.uni-potsdam.de

e-mail: p.phu2514@hotmail.com

Abstract

In [2] the theory of hyperidentities and solid varieties was extended to algebraic systems and solid model classes of algebraic systems. The disadvantage of this approach is that it needs the concept of a formula system. In this paper we present a different approach which is based on the concept of a relational clone. The main result is a characterization of solid model classes of algebraic systems. The results will be applied to study the properties of the monoid of all hypersubstitutions of an ordered algebra.

Keywords: algebraic system, formula, relational clone, hyperformula.

2000 Mathematics Subject Classification: 03B50, 08A30.

1. INTRODUCTION

A first application of the theory of hyperidentities and solid model classes can be found in [2]. The main problem of this approach is that the application of a hypersubstitution to an algebraic system does not give an algebraic system, but a structure which is called a formula system. In this paper we want to avoid this problem. Here we define the application of a hypersubstitution to the fundamental relations of a given algebraic system to be elements of the relational clone generated by the set of fundamental relations.

This leads to a derived algebraic system. It is not necessary to consider the class of formula systems.

We notice that our approach is an example of the concept of an institution (see [7]). We describe how hypersubstitution can be used to come from first order logic to a restricted version of second order logic.

An *algebraic system* of type (τ, τ') is a triple $\mathcal{A} := (A; (f_i^A)_{i \in I}, (\gamma_j^A)_{j \in J})$ consisting of a set A , an indexed set $(f_i^A)_{i \in I}$ of operations defined on A where $f_i^A : A^{n_i} \rightarrow A$ is n_i -ary and an indexed set of relations $\gamma_j^A \subseteq A^{n_j}$ where γ_j^A is n_j -ary. The pair (τ, τ') with $\tau = (n_i)_{i \in I}$, $\tau' = (n_j)_{j \in J}$ of sequences of integers $n_i, n_j \in \mathbb{N}^+ := \mathbb{N} \setminus \{0\}$, is called the type of the algebraic system. For algebraic systems of type (τ, τ') subsystems, homomorphic images, congruences and direct products can be defined and most theorems known from Universal Algebra are valid also in this situation ([5]).

Terms and formulas are expressions in a first-order language which are used to describe properties of algebraic systems and to classify algebraic systems. Our definition of terms and formulas goes back to [5] (see also [2]). Let $X_n = \{x_1, \dots, x_n\}$ be a finite set of variables, let $X = \{x_1, \dots, x_n, \dots\}$ be countably infinite, let $(f_i)_{i \in I}$ be an indexed set of operation symbols and let $(\gamma_j)_{j \in J}$ be an indexed set of relation symbols. Then the set $W_\tau(X_n)$ of all n -ary terms of type τ and the set $\mathcal{F}_{(\tau, \tau')}(W_\tau(X_n))$ of all n -ary formulas of type (τ, τ') are defined in the usual way by the following conditions:

- (i) Every $x_k \in X_n$ is an n -ary term of type τ .
- (ii) If t_1, \dots, t_{n_k} are n -ary terms of type τ and if f_k is an n_k -ary operation symbol of type τ , then $f_k(t_1, \dots, t_{n_k})$ is an n -ary term of type τ .

Let $W_\tau(X) := \bigcup_{n \geq 1} W_\tau(X_n)$ be the set of all terms of type τ . To define formulas of type (τ, τ') we need the logical connectives \vee and \neg , the quantifier \exists and the equation symbol \approx .

Definition 1.1. Let $n \geq 1$. An n -ary formula of type (τ, τ') is defined in the following inductive way:

- (i) If t_1, t_2 are n -ary terms of type τ , then the equation $t_1 \approx t_2$ is an n -ary formula of type (τ, τ') . All variables in $t_1 \approx t_2$ are free.

- (ii) If t_1, \dots, t_{n_j} are n -ary terms of type τ and if γ_j is an n_j -ary relation symbol, then $\gamma_j(t_1, \dots, t_{n_j})$ is an n -ary formula of type (τ, τ') . All variables in such a formula are free.
- (iii) If F is an n -ary formula of type (τ, τ') , then $\neg F$ is an n -ary formula of type (τ, τ') . All free variables in F are also free in $\neg F$. All bound variables in F are also bound in $\neg F$.
- (iv) If F_1 and F_2 are n -ary formulas of type (τ, τ') such that variables occurring simultaneously in both formulas are free in each of them, then $F_1 \vee F_2$ is an n -ary formula of type (τ, τ') . If a variable occurs in F_1 and F_2 and is not free in both formulas, then $F_1 \vee F_2$ is not a formula. Variables that are free in at least one of the formulas F_1 or F_2 are also free in $F_1 \vee F_2$. Variables that are bound in either F_1 or F_2 are also bound in $F_1 \vee F_2$.
- (v) If F is an n -ary formula of type (τ, τ') and $x_i \in X_n$ occurs freely in F , then $\exists x_i(F)$ is an n -ary formula of type (τ, τ') . The variable x_i is bound in the formula $\exists x_i(F)$ and all other free or bound variables in F are of the same nature in $\exists x_i(F)$.

Let $\mathcal{F}_{(\tau, \tau')}(W_\tau(X_n))$ be the set of all n -ary formulas of type (τ, τ') and let $\mathcal{F}_{(\tau, \tau')}(W_\tau(X)) := \bigcup_{n \geq 1} \mathcal{F}_{(\tau, \tau')}(W_\tau(X_n))$ be the set of all formulas of type (τ, τ') .

All free or bound variables occur in X_n or X , resp.. Our definition of formulas follows [5], pp. 115–116.

2. CLONES OF TERM OPERATIONS AND RELATIONAL CLONES OF ALGEBRAIC SYSTEMS

We denote by $O^n(A)$ the set of all n -ary operations defined on A and let $Rel^n(A)$ be the set of all n -ary relations defined on A . Then $O(A) := \bigcup_{n \geq 1} O^n(A)$, $Rel(A) := \bigcup_{n \geq 1} Rel^n(A)$ are the sets of all operations and the set of all relations defined on A . Let $S_m^{n,A}$ be the usual superposition operation for operations, i.e. for m -ary operations t_1^A, \dots, t_n^A and for an n -ary operation s^A on A we define:

$$\begin{aligned}
& S_m^{n,A}(s^A, t_1^A, \dots, t_n^A)(a_1, \dots, a_m) \\
& := s^A(t_1^A(a_1, \dots, a_m), \dots, t_n^A(a_1, \dots, a_m))
\end{aligned}$$

for all $a_1, \dots, a_m \in A, m, n \in \mathbb{N}^+$. The projection operations $e_k^{n,A} \in O^n(A)$ are defined by $e_k^{n,A}(a_1, \dots, a_n) := a_k, 1 \leq k \leq n$. Then one obtains a many-sorted algebra $((O^n(A))_{n \geq 1}, (S_m^{n,A})_{m,n \geq 1}, (e_k^{n,A})_{1 \leq k \leq n, n \in \mathbb{N}^+})$ which is called *clone of all operations* defined on A . This algebra satisfies the superassociative identity ;

$$\begin{aligned}
& S_m^{p,A}(s^A, S_m^{n,A}(t_1^A, s_1^A, \dots, s_n^A), \dots, S_m^{n,A}(t_p^A, s_1^A, \dots, s_n^A)) \\
& = S_m^{n,A}(S_n^{p,A}(s^A, t_1^A, \dots, t_p^A), s_1^A, \dots, s_n^A), \quad (m, n, p = 1, 2, \dots).
\end{aligned}$$

Sometimes one speaks of a clone (of operations) as a set of operations defined on the same set A , closed under all superposition operations $S_m^{n,A}, m, n \geq 1$, and containing all projections.

The clone generated by the fundamental operations $\{f_i^A | i \in I\}$ of the algebraic system $\mathcal{A} = (A; (f_i^A)_{i \in I}, (\gamma_j^A)_{j \in J})$ is called the *clone of term operations* of \mathcal{A} and is denoted by $T(\mathcal{A})$. We note that the elements of $T(\mathcal{A})$ can be obtained as induced term operations of \mathcal{A} . If $(W_\tau(X))^A$ denotes the set of all induced term operations, then $T(\mathcal{A}) = (W_\tau(X))^A$. It is well-known that for algebraic constructions like the formation of subsystems and homomorphic images the term operations of \mathcal{A} play a similar role as the fundamental operations.

Now we are looking for a clone of relations which is generated by the fundamental relations of the algebraic system \mathcal{A} . We will use the concept of a relational algebra (see [6]). Usually one considers the following operations on sets of relations:

Definition 2.1. Assume that $a_1, a_2, \dots, a_n, a_{n+1}, \dots, a_{n+m}$ are elements of the set A .

- (1) The operation $\xi : Rel^n(A) \rightarrow Rel^n(A)$ defines the cyclic permutation of the inputs by $\rho \mapsto \xi\rho$ with
$$\xi\rho := \{(a_1, a_2, \dots, a_n) \mid (a_2, \dots, a_n, a_1) \in \rho\}.$$

- (2) The operation $\tau : Rel^n(A) \rightarrow Rel^n(A)$ exchanges the first and the second input of each n -tuple belonging to the relation ρ by $\rho \mapsto \tau\rho$ with
- $$\tau\rho := \{(a_1, a_2, \dots, a_n) \mid (a_2, a_1, a_3, \dots, a_n) \in \rho\}.$$
- (3) The operation $\triangle : Rel^n(A) \rightarrow Rel^{n-1}(A)$ identifies the first and the second input of each n -tuple in ρ by
- $$\triangle\rho := \{(a_1, a_2, \dots, a_{n-1}) \mid (a_1, a_1, a_2, \dots, a_{n-1}) \in \rho\}.$$
- (4) $\circ : Rel^m(A) \times Rel^n(A) \rightarrow Rel^{n+m-2}(A)$ is the relational product of the relations ρ_1 and ρ_2 and is defined by $(\rho_1, \rho_2) \mapsto \rho_1 \circ \rho_2$ with
- $$\rho_1 \circ \rho_2 := \{(a_1, a_2, \dots, a_{n-1}, a_n, a_{n+1}, \dots, a_{n+m-2}) \mid \exists b \in A, \\ (a_1, a_2, \dots, a_{n-1}, b) \in \rho_2 \text{ and } (b, a_n, a_{n+1}, \dots, a_{n+m-2}) \in \rho_1\}.$$
- (5) The operation $\nabla : Rel^n(A) \rightarrow Rel^{n+1}(A)$ allows the addition of a fictitious coordinate and is defined by $\rho \mapsto \nabla\rho$ with
- $$\nabla\rho := \{(a_1, a_2, \dots, a_{n+1}) \mid (a_2, a_3, \dots, a_{n+1}) \in \rho\}.$$
- (6) The projection operations $pr_{\alpha_1, \dots, \alpha_r} : Rel^n(A) \rightarrow Rel^r(A)$ are defined by $\rho \mapsto pr_{\alpha_1, \dots, \alpha_r}(\rho)$ with
- $$pr_{\alpha_1, \dots, \alpha_r}(\rho) := \{(a_{\alpha_1}, \dots, a_{\alpha_r}) \in A^r \mid \forall j \in \{1, \dots, n\} \setminus \{\alpha_1, \dots, \alpha_r\} \\ \exists a_j \in A, ((a_1, \dots, a_n) \in \rho)\}.$$
- (7) $\times : Rel^n(A) \times Rel^m(A) \rightarrow Rel^{n+m}(A)$ is the cartesian product of two relations ρ_1, ρ_2 and is defined by $(\rho_1, \rho_2) \mapsto \rho_1 \times \rho_2$ with
- $$\rho_1 \times \rho_2 := \{(a_1, a_2, \dots, a_n, a_{n+1}, \dots, a_{n+m}) \mid (a_1, \dots, a_n) \in \rho_1 \text{ and } \\ (a_{n+1}, \dots, a_{n+m}) \in \rho_2\}.$$
- (8) $\cap : Rel^n(A) \times Rel^n(A) \rightarrow Rel^n(A)$ is the intersection of two relations and is defined by $(\rho_1, \rho_2) \mapsto \rho_1 \cap \rho_2$ with
- $$\rho_1 \cap \rho_2 := \{(a_1, \dots, a_n) \mid (a_1, \dots, a_n) \in \rho_1 \text{ and } (a_1, \dots, a_n) \in \rho_2\}.$$
- (9) Let ε be an equivalence relation on $\{1, \dots, n\}$. Then δ_n^ε is defined by
- $$\delta_n^\varepsilon := \{(a_1, \dots, a_n) \in A^n \mid (i, j) \in \varepsilon \Rightarrow a_i = a_j\}.$$
- To select δ_n^ε from $Rel^n(A)$ can be regarded as a nullary operation.

Let $\delta_3^{\{1; 2, 3\}}$ be the ternary relation of the form δ_n^ε where ε is given by the partition $\{\{1\}, \{2, 3\}\}$. The algebra $\mathcal{Rel}(A) := (Rel(A); \xi, \tau, \triangle, \circ, \delta_3^{\{1; 2, 3\}})$

of type $(1, 1, 1, 2, 0)$ is called the full relational algebra on A . Any subalgebra of $\mathcal{Rel}(A)$ is called a relational algebra on A . It turns out (see [6]) that relational algebras are also closed under the operations $\nabla, pr_{\alpha_1, \dots, \alpha_r}, \times$ and \cap and that they contain all relations δ_n^ε . Let Q be the universe of a relational algebra and let R be a non-empty subset of Q . Then we can form the relational subalgebra of the relational algebra \mathcal{Q} which is generated by R , i.e. the relational algebra with the universe $\langle R \rangle := \bigcap \{B \mid B \text{ is a relational subalgebra of } \mathcal{Q} \text{ and } R \subseteq B\}$.

Let $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I}, (\gamma_j^{\mathcal{A}})_{j \in J})$ be an algebraic system of type (τ, τ') . Then the relational subalgebra of $\mathcal{Rel}(A)$ which is generated by the set $\{\gamma_j^{\mathcal{A}} \mid j \in J\}$ is said to be the relational clone of the algebraic system \mathcal{A} and is denoted by $\mathcal{R}(\mathcal{A})$. More precisely, the set $\langle \{\gamma_j^{\mathcal{A}} \mid j \in J\} \rangle$ can be inductively defined by the following steps:

- (1) If $\rho^{\mathcal{A}} \in \{\gamma_j^{\mathcal{A}} \mid j \in J\}$, then $\rho^{\mathcal{A}} \in \langle \{\gamma_j^{\mathcal{A}} \mid j \in J\} \rangle$.
- (2) Assume that $\rho^{\mathcal{A}}, \rho_1^{\mathcal{A}}, \rho_2^{\mathcal{A}} \in \langle \{\gamma_j^{\mathcal{A}} \mid j \in J\} \rangle$, then $\xi \rho^{\mathcal{A}}, \tau \rho^{\mathcal{A}}, \triangle \rho^{\mathcal{A}}$ and $(\rho_1^{\mathcal{A}} \circ \rho_2^{\mathcal{A}}) \in \langle \{\gamma_j^{\mathcal{A}} \mid j \in J\} \rangle$.

Since $\delta_3^{\{1; 2, 3\}}$ belongs to the fundamental operations of every relational algebra, this relation is contained in $\langle \{\gamma_j^{\mathcal{A}} \mid j \in J\} \rangle$.

3. HYPERSUBSTITUTIONS FOR ALGEBRAIC SYSTEMS

Hypersubstitutions are originally defined as mappings which send operation symbols of a given type τ to terms and preserve arities. Every hypersubstitution σ can inductively be extended to a mapping $\widehat{\sigma} : W_\tau(X) \rightarrow W_\tau(X)$. In [2] hypersubstitutions were defined for algebraic systems of type (τ, τ') as mappings $\sigma : \{f_i \mid i \in I\}^2 \cup \{\gamma_j \mid j \in J\} \rightarrow \mathcal{F}_{(\tau, \tau')}(W_\tau(X))$. The disadvantage of this approach is that it needs the set $\mathcal{F}_{(\tau, \tau')}(W_\tau(X))$ of all formulas of type (τ, τ') . Our new approach maps fundamental operations to elements of $T(\mathcal{A})$ and fundamental relations to elements of $R(\mathcal{A})$.

Definition 3.1. Let $\sigma_F : \{f_i^{\mathcal{A}} \mid i \in I\} \rightarrow T(\mathcal{A})$ be a mapping assigning to every n_i -ary fundamental operation $f_i^{\mathcal{A}}$ of type τ an n_i -ary term operation $\sigma_F(f_i^{\mathcal{A}})$. Any such mapping σ_F will be called a concrete hypersubstitution.

Every concrete hypersubstitution induces a mapping $\widehat{\sigma}_F : T(\mathcal{A}) \rightarrow T(\mathcal{A})$ on the set of all term operations of type τ , as follows:

- (1) If $t^A = e_k^{n,A}$, then $\widehat{\sigma}_F[e_k^{n,A}] := e_k^{n,A}$.
- (2) If $t^A = f_i^A(t_1^A, \dots, t_{n_i}^A)$ and $t_1^A, \dots, t_{n_i}^A \in O^n(A)$, then

$$\widehat{\sigma}_F[f_i^A(t_1^A, \dots, t_{n_i}^A)] := S_n^{n_i, A}(\sigma_F(f_i^A), \widehat{\sigma}_F[t_1^A], \dots, \widehat{\sigma}_F[t_{n_i}^A]).$$

Definition 3.2. Let $\sigma_R : \{\gamma_j^A | j \in J\} \rightarrow R(\mathcal{A})$ be a mapping assigning to every n_j -ary fundamental relation γ_j^A of type τ' an n_j -ary relation $\sigma_R(\gamma_j^A)$ of the relational clone. Any such mapping σ_R will be called a relational hypersubstitution.

Every relational hypersubstitution of type τ' induces a mapping $\widehat{\sigma}_R : R(\mathcal{A}) \rightarrow R(\mathcal{A})$ on the relational clone, as follows:

- (1) If $\rho^A \in \{\gamma_j^A | j \in J\}$, then $\widehat{\sigma}_R[\rho^A] := \sigma_R(\rho^A)$ and

$$\widehat{\sigma}_R[\delta_3^{\{1; 2,3\}}] := \delta_3^{\{1; 2,3\}}.$$
- (2) If $\rho^A \in \langle \{\gamma_j^A | j \in J\} \rangle \setminus \{\gamma_j^A | j \in J\}$ and if we inductively assume that

$$\widehat{\sigma}_R[\rho^A], \widehat{\sigma}_R[\rho_1^A], \widehat{\sigma}_R[\rho_2^A]$$
 are already defined, then

$$\begin{aligned} \widehat{\sigma}_R[\xi \rho^A] &:= \xi(\widehat{\sigma}_R[\rho^A]), & \widehat{\sigma}_R[\tau \rho^A] &:= \tau(\widehat{\sigma}_R[\rho^A]), \\ \widehat{\sigma}_R[\triangle \rho^A] &:= \triangle(\widehat{\sigma}_R[\rho^A]), & \widehat{\sigma}_R[\rho_1^A \circ \rho_2^A] &:= \widehat{\sigma}_R[\rho_1^A] \circ \widehat{\sigma}_R[\rho_2^A]. \end{aligned}$$

Definition 3.3. Any mapping $\sigma : \{f_i^A | i \in I\} \cup \{\gamma_j^A | j \in J\} \rightarrow T(\mathcal{A}) \cup R(\mathcal{A})$ with $\sigma(f_i^A) := \sigma_F(f_i^A)$ for all $i \in I$ and $\sigma(\gamma_j^A) := \sigma_R(\gamma_j^A)$ for all $j \in J$ is called hypersubstitution for the algebraic system \mathcal{A} . Let $Relhyp_{\mathcal{A}}(\tau, \tau')$ be the set of all hypersubstitutions for the algebraic system \mathcal{A} .

Clearly any such hypersubstitution can be written as a pair $\sigma := (\sigma_F, \sigma_R)$. Using the extensions $\widehat{\sigma}_F$ and $\widehat{\sigma}_R$ we can define an extension $\widehat{\sigma} : T(\mathcal{A}) \cup R(\mathcal{A}) \rightarrow T(\mathcal{A}) \cup R(\mathcal{A})$ by $\widehat{\sigma} := (\widehat{\sigma}_F, \widehat{\sigma}_R)$. If $\sigma_1, \sigma_2 \in Relhyp_{\mathcal{A}}(\tau, \tau')$, then a product can be defined by $\sigma_1 \circ_{hr} \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$. For $\widehat{\sigma}_1 \circ \sigma_2$ we have $\widehat{\sigma}_1 \circ \sigma_2 = ((\widehat{\sigma}_1)_F, (\widehat{\sigma}_1)_R) \circ ((\sigma_2)_F, (\sigma_2)_R) = ((\widehat{\sigma}_1)_F \circ (\sigma_2)_F, (\widehat{\sigma}_1)_R \circ (\sigma_2)_R)$. Next we prove that the extension of this product is equal to the composition of the extensions.

Lemma 3.4. For any $\sigma_1, \sigma_2 \in Relhyp_{\mathcal{A}}(\tau, \tau')$ we have $(\sigma_1 \circ_{hr} \sigma_2)^{\widehat{}} = \widehat{\sigma}_1 \circ \widehat{\sigma}_2$.

Proof. Because of the last remark we have to show that $(\sigma_1 \circ_{hr} \sigma_2)^\wedge = ((\sigma_1 \circ_{hr} \sigma_2)^\wedge_F, (\sigma_1 \circ_{hr} \sigma_2)^\wedge_R) = ((\widehat{\sigma}_1)_F, (\widehat{\sigma}_1)_R) \circ ((\widehat{\sigma}_2)_F, (\widehat{\sigma}_2)_R) = ((\widehat{\sigma}_1)_F \circ (\widehat{\sigma}_2)_F, (\widehat{\sigma}_1)_R \circ (\widehat{\sigma}_2)_R)$, i.e. that $(\sigma_1 \circ_{hr} \sigma_2)^\wedge_F = (\widehat{\sigma}_1)_F \circ (\widehat{\sigma}_2)_F$ and $(\sigma_1 \circ_{hr} \sigma_2)^\wedge_R = (\widehat{\sigma}_1)_R \circ (\widehat{\sigma}_2)_R$. For the functional part we will give a proof by induction on the complexity of the definition of term operations.

(1) If $t^A = e_k^{n,A}$, then $(\sigma_1 \circ_{hr} \sigma_2)^\wedge_F [e_k^{n,A}]$

$$\begin{aligned} &= e_k^{n,A} \\ &= (\widehat{\sigma}_2)_F [e_k^{n,A}] \\ &= (\widehat{\sigma}_1)_F [(\widehat{\sigma}_2)_F [e_k^{n,A}]] \\ &= ((\widehat{\sigma}_1)_F \circ (\widehat{\sigma}_2)_F) [e_k^{n,A}]. \end{aligned}$$

(2) If $t^A = f_i^A(t_1^A, \dots, t_{n_i}^A)$ for $i \in I$ and if we assume that

$$(\sigma_1 \circ_{hr} \sigma_2)^\wedge_F [t_j^A] = ((\widehat{\sigma}_1)_F \circ (\widehat{\sigma}_2)_F) [t_j^A] \text{ for every } j \in \{1, \dots, n_i\}.$$

Then

$$\begin{aligned} &(\sigma_1 \circ_{hr} \sigma_2)^\wedge_F [f_i^A(t_1^A, \dots, t_{n_i}^A)] \\ &= S_n^{n_i, A}((\sigma_1 \circ_{hr} \sigma_2)_F(f_i^A), (\sigma_1 \circ_{hr} \sigma_2)^\wedge_F [t_1^A], \dots, (\sigma_1 \circ_{hr} \sigma_2)^\wedge_F [t_{n_i}^A]) \\ &= S_n^{n_i, A}(((\widehat{\sigma}_1)_F \circ (\widehat{\sigma}_2)_F)(f_i^A), ((\widehat{\sigma}_1)_F \circ (\widehat{\sigma}_2)_F)[t_1^A], \dots, ((\widehat{\sigma}_1)_F \circ (\widehat{\sigma}_2)_F)[t_{n_i}^A]) \\ &= S_n^{n_i, A}((\widehat{\sigma}_1)_F[(\sigma_2)_F(f_i^A)], (\widehat{\sigma}_1)_F[(\widehat{\sigma}_2)_F[t_1^A]], \dots, (\widehat{\sigma}_1)_F[(\widehat{\sigma}_2)_F[t_{n_i}^A]]) \\ &= (\widehat{\sigma}_1)_F[S_n^{n_i, A}((\sigma_2)_F(f_i^A), (\widehat{\sigma}_2)_F[t_1^A], \dots, (\widehat{\sigma}_2)_F[t_{n_i}^A])] \\ &\quad \text{(using that } (\widehat{\sigma}_1)_F \text{ is compatible with the operation } S_n^{n_i, A}, \text{ see e.g [1])} \\ &= (\widehat{\sigma}_1)_F[(\widehat{\sigma}_2)_F[f_i^A(t_1^A, \dots, t_{n_i}^A)]] \\ &= ((\widehat{\sigma}_1)_F \circ (\widehat{\sigma}_2)_F)[f_i^A(t_1^A, \dots, t_{n_i}^A)]. \end{aligned}$$

For the relational part we will give a proof by induction on the complexity of the definition of a relational clone.

(1) If $\rho^A \in \{\gamma_j^A \mid j \in J\}$, then

$$\begin{aligned} (\sigma_1 \circ_{hr} \sigma_2) \widehat{R}[\rho^A] &= ((\sigma_1)_R \circ (\sigma_2)_R)[\rho^A] \\ &= ((\widehat{\sigma}_1)_R \circ (\widehat{\sigma}_2)_R)(\rho^A) \\ &= (\widehat{\sigma}_1)_R[(\widehat{\sigma}_2)_R(\rho^A)] \\ &= (\widehat{\sigma}_1)_R[(\widehat{\sigma}_2)_R[\rho^A]] \\ &= ((\widehat{\sigma}_1)_R \circ (\widehat{\sigma}_2)_R)(\rho^A). \end{aligned}$$

(2) Assume that $\rho^A \in \langle \{\gamma_j^A \mid j \in J\} \setminus \{\gamma_j^A \mid j \in J\} \rangle$ and let us inductively assume that $(\sigma_1 \circ_{hr} \sigma_2) \widehat{R}[\rho^A] = ((\widehat{\sigma}_1)_R \circ (\widehat{\sigma}_2)_R)[\rho^A]$, $(\sigma_1 \circ_{hr} \sigma_2) \widehat{R}[\rho_1^A] = ((\widehat{\sigma}_1)_R \circ (\widehat{\sigma}_2)_R)[\rho_1^A]$, $(\sigma_1 \circ_{hr} \sigma_2) \widehat{R}[\rho_2^A] = ((\widehat{\sigma}_1)_R \circ (\widehat{\sigma}_2)_R)[\rho_2^A]$. Then

$$\begin{aligned} (\sigma_1 \circ_{hr} \sigma_2) \widehat{R}[\xi \rho^A] &= \xi((\sigma_1 \circ_{hr} \sigma_2) \widehat{R}[\rho^A]) \\ &= \xi(((\widehat{\sigma}_1)_R \circ (\widehat{\sigma}_2)_R)[\rho^A]) \\ &= \xi((\widehat{\sigma}_1)_R[(\widehat{\sigma}_2)_R[\rho^A]]) \\ &= (\widehat{\sigma}_1)_R[\xi((\widehat{\sigma}_2)_R[\rho^A])] \\ &= (\widehat{\sigma}_1)_R[(\widehat{\sigma}_2)_R[\xi \rho^A]] \\ &= ((\widehat{\sigma}_1)_R \circ (\widehat{\sigma}_2)_R)[\xi \rho^A]. \end{aligned}$$

The inductive steps for τ and Δ may be handled similarly.

$$\begin{aligned} (\sigma_1 \circ_{hr} \sigma_2) \widehat{R}[\rho_1^A \circ \rho_2^A] &= (\sigma_1 \circ_{hr} \sigma_2) \widehat{R}[\rho_1^A] \circ (\sigma_1 \circ_{hr} \sigma_2) \widehat{R}[\rho_2^A] \\ &= (((\widehat{\sigma}_1)_R \circ (\widehat{\sigma}_2)_R)(\rho_1^A)) \circ (((\widehat{\sigma}_1)_R \circ (\widehat{\sigma}_2)_R)(\rho_2^A)) \\ &= (\widehat{\sigma}_1)_R[(\widehat{\sigma}_2)_R[\rho_1^A]] \circ (\widehat{\sigma}_1)_R[(\widehat{\sigma}_2)_R[\rho_2^A]] \\ &= (\widehat{\sigma}_1)_R[(\widehat{\sigma}_2)_R[\rho_1^A] \circ (\widehat{\sigma}_2)_R[\rho_2^A]] \\ &= (\widehat{\sigma}_1)_R[(\widehat{\sigma}_2)_R[\rho_1^A \circ \rho_2^A]] \\ &= ((\widehat{\sigma}_1)_R \circ (\widehat{\sigma}_2)_R)[\rho_1^A \circ \rho_2^A]. \end{aligned}$$

Finally we have

$$(\sigma_1 \circ_{hr} \sigma_2) \widehat{R}[\delta_3^{\{1; 2,3\}}] = \delta_3^{\{1; 2,3\}} = (\widehat{\sigma}_2)_R[\delta_3^{\{1; 2,3\}}] = ((\widehat{\sigma}_1)_R \circ (\widehat{\sigma}_2)_R)[\delta_3^{\{1; 2,3\}}]. \blacksquare$$

An identity element with respect to the multiplication \circ_{hr} can be defined by $\sigma_{id} := ((\sigma_{id})_F, (\sigma_{id})_R)$ with $(\sigma_{id})_F(f_i^A) := f_i^A$ for all $i \in I$ and $(\sigma_{id})_R(\gamma_j^A) := \gamma_j^A$ for all $j \in J$.

By induction on the complexity of the definition of a term operation $t^A \in T(\mathcal{A})$ and of the definition of an element $\rho^A \in R(\mathcal{A})$ one can prove:

Lemma 3.5. *For any $t^A \in T(\mathcal{A})$ and for any $\rho^A \in R(\mathcal{A})$ the following equations are satisfied: $(\widehat{\sigma}_{id})_F[t^A] = t^A$ and $(\widehat{\sigma}_{id})_R[\rho^A] = \rho^A$.*

Then as a consequence of Lemma 3.4 and Lemma 3.5 we obtain.

Theorem 3.6. *For any algebraic system \mathcal{A} of type (τ, τ') , $(Relhyp_{\mathcal{A}}(\tau, \tau'), \circ_{hr}, \sigma_{id})$ is a monoid.*

Proof. Associativity of the operation \circ_{hr} follows from Lemma 3.4 by $(\sigma_1 \circ_{hr} \sigma_2) \circ_{hr} \sigma_3 = (\sigma_1 \circ_{hr} \sigma_2) \circ \sigma_3 = (\widehat{\sigma}_1 \circ \widehat{\sigma}_2) \circ \sigma_3 = \widehat{\sigma}_1 \circ (\widehat{\sigma}_2 \circ \sigma_3) = \widehat{\sigma}_1 \circ (\sigma_2 \circ_{hr} \sigma_3) = \sigma_1 \circ_{hr} (\sigma_2 \circ_{hr} \sigma_3)$. σ_{id} is an identity element since for any $\sigma \in Relhyp_{\mathcal{A}}(\tau, \tau')$ we get $\sigma_{id} \circ_{hr} \sigma = \widehat{\sigma}_{id} \circ \sigma = ((\widehat{\sigma}_{id})_F, (\widehat{\sigma}_{id})_R) \circ (\sigma_F, \sigma_R) = ((\widehat{\sigma}_{id})_F \circ \sigma_F, (\widehat{\sigma}_{id})_R \circ \sigma_R)$. From Lemma 3.5 we obtain for any $f_i^A, i \in I$, that $((\widehat{\sigma}_{id})_F \circ \sigma_F)(f_i^A) = (\widehat{\sigma}_{id})_F[\sigma_F(f_i^A)] = \sigma_F(f_i^A)$, i.e. $(\widehat{\sigma}_{id})_F \circ \sigma_F = \sigma_F$ and for any $\gamma_j^A, j \in J$ we have $((\widehat{\sigma}_{id})_R \circ \sigma_R)(\gamma_j^A) = (\widehat{\sigma}_{id})_R[\sigma_R(\gamma_j^A)] = \sigma_R(\gamma_j^A)$ i.e. $(\widehat{\sigma}_{id})_R \circ \sigma_R = \sigma_R$. This gives $\sigma_{id} \circ_{hr} \sigma = \sigma$. The equation $\sigma \circ_{hr} \sigma_{id} = \sigma$ is clear. ■

4. EXTENSION OF HYPERSUBSTITUTIONS TO MAPPINGS DEFINED ON THE REALIZATIONS OF FORMULAS

Classes of algebraic systems can be described as model classes of sets of formulas. For a given set of formulas the model class of this set consists precisely of those algebraic systems which satisfy all formulas of the given set. Since we want to use a stronger concept of satisfaction we have to apply hypersubstitutions to formulas. The first step is to define a mapping which maps a relation defined on a set A and an n_j -tuple of operations on A to a relation defined on $O^n(A)$. The operation $R_n^{n_j, A}$ with

$$R_n^{n_j, A} : Rel^{n_j}(A) \times (O^n(A))^{n_j} \rightarrow Rel^{n_j}(O^n(A))$$

by $(\gamma_j^A, f_1^A, \dots, f_{n_j}^A) \mapsto R_n^{n_j, A}(\gamma_j^A, f_1^A, \dots, f_{n_j}^A)$ where $R_n^{n_j, A}(\gamma_j^A, f_1^A, \dots, f_{n_j}^A)$ is true iff for all $a_1, \dots, a_n \in A$ we have $(f_1^A(a_1, \dots, a_n), \dots, f_{n_j}^A(a_1, \dots, a_n)) \in \gamma_j^A$ maps each n_j -ary relation γ_j^A and each n_j -tuple of n -ary operations on A to an n_j -ary relation $R_{\gamma_j^A}^A$ on $O^n(A)$ where $(f_1^A, \dots, f_{n_j}^A) \in R_{\gamma_j^A}^A$ iff for every n -tuple $\underline{a} = (a_1, \dots, a_n) \in A$ we have $(f_1^A(\underline{a}), \dots, f_{n_j}^A(\underline{a})) \in \gamma_j^A$. For the properties of the operations $R_n^{n_j, A}$, $n, n_j \in \mathbb{N}^+$ see [2].

The operations $R_n^{n_j, A}$ can especially be applied to the fundamental relations and term operations of the algebraic system $\mathcal{A} = (A; (f_i^A)_{i \in I}, (\gamma_j^A)_{j \in J})$ of type (τ, τ') . Then the extension to arbitrary relations of the relation algebra generated by $\{\gamma_j^A | j \in J\}$ can be defined as follows:

- (1) $R_n^{n_j, A}(\xi \rho^A, t_1^A, \dots, t_{n_j}^A) := \xi R_n^{n_j, A}(\rho^A, t_1^A, \dots, t_{n_j}^A),$
- (2) $R_n^{n_j, A}(\tau \rho^A, t_1^A, \dots, t_{n_j}^A) := \tau R_n^{n_j, A}(\rho^A, t_1^A, \dots, t_{n_j}^A),$
- (3) $R_n^{n_j, A}(\triangle \rho^A, t_1^A, \dots, t_{n_j}^A) := \triangle R_n^{n_j, A}(\rho^A, t_1^A, \dots, t_{n_j}^A),$
- (4) $R_n^{n_j, A}(\rho_1^A \circ \rho_2^A, t_1^A, \dots, t_{n_j}^A) := R_n^{n_j, A}(\rho_1^A, t_1^A, \dots, t_{n_j}^A) \circ R_n^{n_j, A}(\rho_2^A, t_1^A, \dots, t_{n_j}^A),$
- (5) $R_n^{3, A}(\delta_3^{\{1; 2, 3\}, A}, t_1^A, t_2^A, t_3^A) := \delta_3^{\{1; 2, 3\}, O^n(A)}.$

Besides formulas of type (τ, τ') we now define formulas of type (τ, τ'') introducing to each relation from $\langle \{\gamma_j^A | j \in J\} \rangle$ a new relation symbol R_l . This gives an indexed set $(R_l)_{l \in L}$ of relation symbols where R_l is n_l -ary. Let τ'' be the new type of all these relation symbols.

Let $\mathcal{F}_{(\tau, \tau'')}(W_\tau(X_n))$ be the set of all n -ary formulas of type (τ, τ'') and let $\mathcal{F}_{(\tau, \tau'')}(W_\tau(X)) := \bigcup_{n \geq 1} \mathcal{F}_{(\tau, \tau'')}(W_\tau(X_n))$ be the set of all formulas of type (τ, τ'') . Now we will define the realization of an n -ary formula of type (τ, τ'') as a relation defined on $O^n(A)$.

Definition 4.1. Let $\mathcal{A} = (A; (f_i^A)_{i \in I}, (\gamma_j^A)_{j \in J})$ be an algebraic system of type (τ, τ') . The realization of a formula F of type (τ, τ'') on \mathcal{A} is defined as follows :

- (i) If F has the form $s \approx t$, then $(s \approx t)^A := s^A = t^A$ where s^A, t^A are the term operations induced by the terms s, t on the algebra $(A; (f_i^A)_{i \in I})$.

- (ii) If F has the form $R_l(t_1, \dots, t_{n_l})$, then

$$R_l(t_1, \dots, t_{n_l})^{\mathcal{A}} := R_n^{n_l, \mathcal{A}}(R_l^{\mathcal{A}}, t_1^{\mathcal{A}}, \dots, t_{n_l}^{\mathcal{A}}).$$

Here $R_l^{\mathcal{A}}$ is an element of the relational algebra generated by $\{\gamma_j^{\mathcal{A}} | j \in J\}$

- (iii) If the formula has the form $\neg F$, then $(\neg F)^{\mathcal{A}} := \neg F^{\mathcal{A}}$
(where $\neg F^{\mathcal{A}}$ is true iff $F^{\mathcal{A}}$ is not true).
- (iv) If the formula has the form $F_1 \vee F_2$, then $(F_1 \vee F_2)^{\mathcal{A}} := F_1^{\mathcal{A}} \vee F_2^{\mathcal{A}}$
(where $F_1^{\mathcal{A}} \vee F_2^{\mathcal{A}}$ is true iff $F_1^{\mathcal{A}}$ is true or $F_2^{\mathcal{A}}$ is true).
- (v) If the formula has the form $\exists x_i(F)$, then $(\exists x_i(F))^{\mathcal{A}} := \exists x_i(F)^{\mathcal{A}}$ (where $\exists x_i(F)^{\mathcal{A}}$ is true iff there exists an a_i such that after substituting a_i for x_i the realization $F^{\mathcal{A}}$ becomes true).

Let $(\mathcal{F}_{(\tau, \tau'')}(W_{\tau}(X_n)))^{\mathcal{A}}$ be the set of all realizations of n -ary formulas of type (τ, τ'') and let $(W_{\tau}(X_m))^{\mathcal{A}}$ be the set of all m -ary term operations induced by terms of type τ on the algebra $(A; (f_i^{\mathcal{A}})_{i \in I})$ of type τ . Then for all $m, n \in N^+$ an operation

$$\dot{R}_m^{n, \mathcal{A}} : (\mathcal{F}_{(\tau, \tau'')}(W_{\tau}(X_n)))^{\mathcal{A}} \times ((W_{\tau}(X_m))^{\mathcal{A}} \rightarrow \text{Rel}^{n_j}(O^m(A)))$$

is inductively defined in the following way:

Definition 4.2. Let \mathcal{A} be an algebraic system of type (τ, τ') . For any $F^{\mathcal{A}} \in (\mathcal{F}_{(\tau, \tau'')}(W_{\tau}(X_n)))^{\mathcal{A}}$ and any n -tuple of m -ary term operations $\dot{R}_m^{n, \mathcal{A}}(F^{\mathcal{A}}, t_1^{\mathcal{A}}, \dots, t_n^{\mathcal{A}})$ is defined inductively by the following steps :

- (i) If $F^{\mathcal{A}}$ has the form $s^{\mathcal{A}} = t^{\mathcal{A}}$, then $\dot{R}_m^{n, \mathcal{A}}(s^{\mathcal{A}} = t^{\mathcal{A}}, t_1^{\mathcal{A}}, \dots, t_n^{\mathcal{A}})$
 $:= S_m^{n, \mathcal{A}}(s^{\mathcal{A}}, t_1^{\mathcal{A}}, \dots, t_n^{\mathcal{A}}) = S_m^{n, \mathcal{A}}(t^{\mathcal{A}}, t_1^{\mathcal{A}}, \dots, t_n^{\mathcal{A}}).$
- (ii) If F has the form $R_l^{\mathcal{A}}(s_1, \dots, s_{n_l})$, then $\dot{R}_m^{n, \mathcal{A}}(R_l^{\mathcal{A}}(s_1, \dots, s_{n_l})^{\mathcal{A}}, t_1^{\mathcal{A}}, \dots, t_n^{\mathcal{A}})$
 $= R_l^{\mathcal{A}}(S_m^{n, \mathcal{A}}(s_1^{\mathcal{A}}, t_1^{\mathcal{A}}, \dots, t_n^{\mathcal{A}}), \dots, S_m^{n, \mathcal{A}}(s_{n_l}^{\mathcal{A}}, t_1^{\mathcal{A}}, \dots, t_n^{\mathcal{A}})).$
- (iii) If $F^{\mathcal{A}}$ has the form $(\neg F)^{\mathcal{A}}$, then $\dot{R}_m^{n, \mathcal{A}}((\neg F)^{\mathcal{A}}, t_1^{\mathcal{A}}, \dots, t_n^{\mathcal{A}})$
 $:= \neg \dot{R}_m^{n, \mathcal{A}}(F^{\mathcal{A}}, t_1^{\mathcal{A}}, \dots, t_n^{\mathcal{A}}).$

- (iv) If F^A has the form $(F_1 \vee F_2)^A$, then $\dot{R}_m^{n,A}((F_1 \vee F_2)^A, t_1^A, \dots, t_n^A)$
 $:= \dot{R}_m^{n,A}(F_1^A, t_1^A, \dots, t_n^A) \vee \dot{R}_m^{n,A}(F_2^A, t_1^A, \dots, t_n^A).$
- (v) If F^A has the form $(\exists x_i(F))^A$, then $\dot{R}_m^{n,A}((\exists x_i(F))^A, t_1^A, \dots, t_n^A)$
 $:= \exists x_i(\dot{R}_m^{n,A}(F^A, t_1^A, \dots, t_n^A)).$

The operation $\dot{R}_m^{n,A}$, $m, n \in \mathbb{N}^+$ satisfies an equation similar to the supparasociative identity mentioned in Section 2.

In Section 3 we defined extensions $\hat{\sigma}$ of hypersubstitutions for the algebraic system \mathcal{A} as mappings $\hat{\sigma} : T(\mathcal{A}) \cup R(\mathcal{A}) \rightarrow T(\mathcal{A}) \cup R(\mathcal{A})$. Now we define another kind of extension of hypersubstitution σ^A which is also based on a pair (σ_F, σ_R) , $\sigma_F : \{f_i^A | i \in I\} \rightarrow T(\mathcal{A})$, $\sigma_R : \{\gamma_j^A | j \in J\} \rightarrow R(\mathcal{A})$. We need the fact that for any relation symbol R_l corresponding to a relation in $R(\mathcal{A})$ there exists a formula F such that the realization of F gives R_l^A (see [6]).

Definition 4.3. Let $\sigma \in Relhyp_A(\tau, \tau')$. Then we define a mapping

$$\dot{\sigma}^A : (\mathcal{F}_{(\tau, \tau'')} (W_\tau(X)))^A \rightarrow (\mathcal{F}_{(\tau, \tau'')} (W_\tau(X)))^A$$

inductively as follows :

- (i) $\dot{\sigma}^A[s^A = t^A] := \hat{\sigma}_F[s^A] = \hat{\sigma}_F[t^A].$
- (ii) $\dot{\sigma}^A[R_l^A(t_1^A, \dots, t_{n_l}^A)] := R_n^{n_l, A}(\hat{\sigma}_R[R_l^A], \hat{\sigma}_F[t_1^A], \dots, \hat{\sigma}_F[t_{n_l}^A]).$
 Here $\hat{\sigma}_R[R_l^A]$ is an element of the relational clone generated by $\{\gamma_j^A | j \in J\}.$
- (iii) $\dot{\sigma}^A[\neg F^A] := \neg(\dot{\sigma}^A[F^A]).$
- (iv) $\dot{\sigma}^A[F_1^A \vee F_2^A] := \dot{\sigma}^A[F_1^A] \vee \dot{\sigma}^A[F_2^A].$
- (v) $\dot{\sigma}^A[\exists x_i(F^A)] := \exists x_i(\dot{\sigma}^A[F^A]).$

Theorem 4.4. Let $\sigma \in Relhyp_A(\tau, \tau')$, let $m, n \in \mathbb{N}^+$. Then $\dot{\sigma}^A$ is a mapping from $((\mathcal{F}_{(\tau, \tau'')} (W_\tau(X_n)))^A)_{n \geq 1}$ to $((\mathcal{F}_{(\tau, \tau'')} (W_\tau(X_n)))^A)_{n \geq 1}$ with $\dot{\sigma}^A[\dot{R}_m^{n,A}(F^A, s_1^A, \dots, s_n^A)] = \dot{R}_m^{n,A}(\dot{\sigma}^A[F^A], \hat{\sigma}_F[s_1^A], \dots, \hat{\sigma}_F[s_n^A])$ for all $F \in \mathcal{F}_{(\tau, \tau'')} (W_\tau(X_n))$ and $s_1, \dots, s_n \in W_\tau(X_n).$

Proof. We will give a proof by induction on the definition of the realization of formula F^A .

(i) If F^A has the form $t_1^A = t_2^A$, then

$$\begin{aligned}
& \dot{\sigma}^A[\dot{R}_m^{n,A}(t_1^A = t_2^A, s_1^A, \dots, s_n^A)] \\
&= \dot{\sigma}^A[S_m^{n,A}(t_1^A, s_1^A, \dots, s_n^A) = S_m^{n,A}(t_2^A, s_1^A, \dots, s_n^A)] \\
&= \widehat{\sigma}_F[(S_m^{n,A}(t_1^A, s_1^A, \dots, s_n^A))] = \widehat{\sigma}_F[S_m^{n,A}(t_2^A, s_1^A, \dots, s_n^A)] \\
&= S_m^{n,A}(\widehat{\sigma}_F[t_1^A], \widehat{\sigma}_F[s_1^A], \dots, \widehat{\sigma}_F[s_n^A]) = S_m^{n,A}(\widehat{\sigma}_F[t_2^A], \widehat{\sigma}_F[s_1^A], \dots, \widehat{\sigma}_F[s_n^A]) \\
&= \dot{R}_m^{n,A}(\widehat{\sigma}_F[t_1^A] = \widehat{\sigma}_F[t_2^A], \widehat{\sigma}_F[s_1^A], \dots, \widehat{\sigma}_F[s_n^A]) \\
&= \dot{R}_m^{n,A}(\dot{\sigma}^A[t_1^A = t_2^A], \widehat{\sigma}_F[s_1^A], \dots, \widehat{\sigma}_F[s_n^A]).
\end{aligned}$$

(ii) If F^A has the form $R_l^A(t_1^A, \dots, t_{n_l}^A)$, then

$$\begin{aligned}
& \dot{\sigma}^A[\dot{R}_m^{n,A}(R_l^A(t_1^A, \dots, t_{n_l}^A), s_1^A, \dots, s_n^A)] \\
&= \dot{\sigma}^A[R_l^A(S_m^{n,A}(t_1^A, s_1^A, \dots, s_n^A), \dots, S_m^{n,A}(t_{n_l}^A, s_1^A, \dots, s_n^A))] \\
&= R_m^{n_l,A}(\widehat{\sigma}_R[R_l^A], \widehat{\sigma}_F[S_m^{n,A}(t_1^A, s_1^A, \dots, s_n^A)], \dots, \widehat{\sigma}_F[S_m^{n,A}(t_{n_l}^A, s_1^A, \dots, s_n^A)]) \\
&= R_m^{n_l,A}(\widehat{\sigma}_R[R_l^A], S_m^{n,A}(\widehat{\sigma}_F[t_1^A], \widehat{\sigma}_F[s_1^A], \dots, \widehat{\sigma}_F[s_n^A]), \dots, \\
&\quad S_m^{n,A}(\widehat{\sigma}_F[t_{n_l}^A], \widehat{\sigma}_F[s_1^A], \dots, \widehat{\sigma}_F[s_n^A])) \\
&= \dot{R}_m^{n,A}(R_m^{n_l,A}(\widehat{\sigma}_R([R_l^A], \widehat{\sigma}_F[t_1^A], \dots, \widehat{\sigma}_F[t_{n_l}^A]), \widehat{\sigma}_F[s_1^A], \dots, \widehat{\sigma}_F[s_n^A])) \\
&\quad \text{by Lemma 4.3} \\
&= \dot{R}_m^{n,A}(\dot{\sigma}^A[R_l^A(t_1^A, \dots, t_{n_l}^A)], \widehat{\sigma}_F[s_1^A], \dots, \widehat{\sigma}_F[s_n^A]).
\end{aligned}$$

(iii) If F^A has the form $\neg F^A$ and assume that

$\dot{\sigma}^A[\dot{R}_m^{n,A}(F^A, s_1^A, \dots, s_n^A)] = \dot{R}_m^{n,A}(\dot{\sigma}^A[F^A], \widehat{\sigma}_F[s_1^A], \dots, \widehat{\sigma}_F[s_n^A])$, then

$$\begin{aligned}
& \dot{\sigma}^A[\dot{R}_m^{n,A}(\neg F^A, s_1^A, \dots, s_n^A)] \\
&= \neg(\dot{\sigma}^A[\dot{R}_m^{n,A}(F^A, s_1^A, \dots, s_n^A)]) \\
&= \neg(\dot{R}_m^{n,A}(\dot{\sigma}^A[F^A], \widehat{\sigma}_F[s_1^A], \dots, \widehat{\sigma}_F[s_n^A])) \\
&= \dot{R}_m^{n,A}(\dot{\sigma}^A[\neg F^A], \widehat{\sigma}_F[s_1^A], \dots, \widehat{\sigma}_F[s_n^A]).
\end{aligned}$$

The inductive steps for $F^A = F_1^A \vee F_2^A$ and $F^A = \exists x_i(F^A)$ may be handled similarly. ■

The next lemma shows that the \cdot^A -extension of a hypersubstitution is compatible with the product.

Lemma 4.5. *Let $\sigma_1, \sigma_2 \in Relhyp_A(\tau, \tau')$. Then we have $(\sigma_1 \circ_{hr} \sigma_2)^A = \dot{\sigma}_1^A \circ \dot{\sigma}_2^A$.*

Proof. We will give a proof by induction on the definition of the realization of the formula F^A .

(i) If F^A has the form $s^A = t^A$, then

$$\begin{aligned}
 & (\sigma_1 \circ_{hr} \sigma_2)^A [s^A = t^A] \\
 &= ((\sigma_1 \circ_{hr} \sigma_2)_F \widehat{[s^A]} = (\sigma_1 \circ_{hr} \sigma_2)_F \widehat{[t^A]}) \\
 &= (((\widehat{\sigma}_1)_F \circ (\widehat{\sigma}_2)_F)[s^A] = (((\widehat{\sigma}_1)_F \circ (\widehat{\sigma}_2)_F))[t^A]) \\
 &= ((\widehat{\sigma}_1)_F[(\widehat{\sigma}_2)_F[s^A]] = (\widehat{\sigma}_1)_F[(\widehat{\sigma}_2)_F[t^A]]) \\
 &= (\dot{\sigma}_1^A[(\widehat{\sigma}_2)_F[s^A]] = (\widehat{\sigma}_2)_F[t^A]) \\
 &= (\dot{\sigma}_1^A[\dot{\sigma}_2^A[s^A = t^A]]) \\
 &= (\dot{\sigma}_1^A \circ \dot{\sigma}_2^A)[s^A = t^A].
 \end{aligned}$$

(ii) If F^A has the form $R_l^A(t_1^A, \dots, t_{n_l}^A)$, then

$$\begin{aligned}
 & (\sigma_1 \circ_{hr} \sigma_2)^A [R_l^A(t_1^A, \dots, t_{n_l}^A)] \\
 &= R_m^{n_l, A}((\sigma_1 \circ_{hr} \sigma_2)_R \widehat{[R_l^A]}, (\sigma_1 \circ_{hr} \sigma_2)_F \widehat{[t_1^A]}, \dots, (\sigma_1 \circ_{hr} \sigma_2)_F \widehat{[t_{n_l}^A]}) \\
 &= R_m^{n_l, A}((\widehat{\sigma}_1)_R[(\widehat{\sigma}_2)_R(R_l^A)], (\widehat{\sigma}_1)_F[(\widehat{\sigma}_2)_F[t_1^A]], \dots, (\widehat{\sigma}_1)_F[(\widehat{\sigma}_2)_F[t_{n_l}^A]]) \\
 &= \dot{\sigma}_1^A[R_m^{n_l, A}((\widehat{\sigma}_2)_R(R_l^A), (\widehat{\sigma}_2)_F[t_1^A], \dots, (\widehat{\sigma}_2)_F[t_{n_l}^A])] \\
 &= \dot{\sigma}_1^A[\dot{\sigma}_2^A[R_l^A(t_1^A, \dots, t_{n_l}^A)]] \\
 &= (\dot{\sigma}_1^A \circ \dot{\sigma}_2^A)[R_l^A(t_1^A, \dots, t_{n_l}^A)].
 \end{aligned}$$

(iii) If F^A has the form $\neg F^A$ and if we assume that

$$(\sigma_1 \circ_{hr} \sigma_2)^A [F^A] = (\dot{\sigma}_1^A \circ \dot{\sigma}_2^A)[F^A], \text{ then}$$

$$\begin{aligned}
& (\sigma_1 \circ_{hr} \sigma_2)^A[\neg F^A] \\
&= \neg((\sigma_1 \circ_{hr} \sigma_2)^A[F^A]) \\
&= \neg((\dot{\sigma}_1^A \circ \dot{\sigma}_2^A)[F^A]) \\
&= \neg(\dot{\sigma}_1^A[\dot{\sigma}_2^A[F^A]]) \\
&= \dot{\sigma}_1^A[\neg(\dot{\sigma}_2^A[F^A])] \\
&= \dot{\sigma}_1^A[\dot{\sigma}_2^A[\neg F^A]] \\
&= (\dot{\sigma}_1^A \circ \dot{\sigma}_2^A)[\neg F^A].
\end{aligned}$$

Cases (iv) and (v) may be handled similarly. ■

We can also show that the extension of the identity hypersubstitution maps all realizations of formulas to themselves.

Lemma 4.6. *For $\sigma_{id} \in Relhyp_A(\tau, \tau')$ we have $\dot{\sigma}_{id}^A[F^A] = F^A$ for all $F^A \in (\mathcal{F}_{(\tau, \tau'')}(W_\tau(X)))^A$.*

Proof. The proof is by a straightforward induction on the realization of a formula F^A .

- (i) If F^A has the form $s^A = t^A$, then $\dot{\sigma}_{id}^A[s^A = t^A] = ((\widehat{\sigma}_{id})_F[s^A] = (\widehat{\sigma}_{id})_F[t^A]) = (s^A = t^A)$.
- (ii) If F^A has the form $R_l^A(t_1^A, \dots, t_{n_l}^A)$, then
$$\begin{aligned}
\dot{\sigma}_{id}^A[R_l^A(t_1^A, \dots, t_{n_l}^A)] &= R_m^{n_l, A}((\widehat{\sigma}_{id})_R(R_l^A), (\widehat{\sigma}_{id})_F([t_1^A]), \dots, (\widehat{\sigma}_{id})_F([t_{n_l}^A])) \\
&= R_m^{n_l, A}(R_l^A, t_1^A, \dots, t_{n_l}^A) = R_l^A(t_1^A, \dots, t_{n_l}^A).
\end{aligned}$$
- (iii) If F^A has the form $\neg F^A$ and assume that $\dot{\sigma}_{id}^A[F^A] = F^A$, then
$$\dot{\sigma}_{id}^A[\neg F^A] = \neg(\dot{\sigma}_{id}^A[F^A]) = \neg F^A.$$
- (iv) If F^A has the form $F_1^A \vee F_2^A$ and assume that $\dot{\sigma}_{id}^A[F_j^A] = F_j^A; j \in \{1, 2\}$, then
$$\dot{\sigma}_{id}^A[F_1^A \vee F_2^A] = \dot{\sigma}_{id}^A[F_1^A] \vee \dot{\sigma}_{id}^A[F_2^A] = F_1^A \vee F_2^A.$$
- (v) If F^A has the form $\exists x_i(F^A)$ and assume that $\dot{\sigma}_{id}^A[F^A] = F^A$, then
$$\dot{\sigma}_{id}^A[\exists x_i(F^A)] = \exists x_i(\dot{\sigma}_{id}^A[F^A]) = \exists x_i(F^A).$$

■

5. APPLICATION OF HYPERSUBSTITUTIONS TO ALGEBRAIC SYSTEMS

The main result of this section is Theorem 5.5, which is the basis of the so-called “conjugate property” and connects the application of hypersubstitution to an algebraic system with that to the realization of a formula. It should be mentioned that this parallels the satisfaction condition in the theory of institutions (see [7] and [8]).

If $\mathcal{F}^A \subseteq (\mathcal{F}_{(\tau, \tau'')}(W_\tau(X)))^A$ is a set of realizations of formulas of type (τ, τ'') on the algebraic system \mathcal{A} , then we define

$$\chi_h^F : \mathcal{P}((\mathcal{F}_{(\tau, \tau'')}(W_\tau(X)))^A) \rightarrow \mathcal{P}((\mathcal{F}_{(\tau, \tau'')}(W_\tau(X)))^A) \text{ by}$$

$$\chi_h^F(\mathcal{F}^A) := \{\dot{\sigma}^A[F^A] \mid \sigma \in Relhyp_{\mathcal{A}}(\tau, \tau'), F^A \in \mathcal{F}^A\}.$$

Then

$$\chi_h^F(\mathcal{F}^A) := \bigcup_{\sigma \in Relhyp_{\mathcal{A}}(\tau, \tau')} \bigcup_{F^A \in \mathcal{F}^A} \dot{\sigma}^A[F^A].$$

Lemma 5.1. χ_h^F has the properties of a completely additive closure operator.

Proof. By Lemma 4.6 $\dot{\sigma}_{id}^A[F^A] = F^A$ and this shows that χ_h^F is extensive. By definition

$$\chi_h^F(\mathcal{F}^A) := \bigcup_{\sigma \in Relhyp_{\mathcal{A}}(\tau, \tau')} \bigcup_{F^A \in \mathcal{F}^A} \dot{\sigma}^A[F^A]$$

and therefore χ_h^F is completely additive and thus monotone, i.e., if $\mathcal{F}_1^A \subseteq \mathcal{F}_2^A \subseteq \mathcal{F}_{(\tau, \tau'')}(W_\tau(X))^A$, then $\chi_h^F(\mathcal{F}_1^A) \subseteq \chi_h^F(\mathcal{F}_2^A)$. By extensivity $\chi_h^F(\mathcal{F}^A) \subseteq \chi_h^F(\chi_h^F(\mathcal{F}^A))$. If conversely $F_1^A \in \chi_h^F(\chi_h^F(\mathcal{F}^A))$ then there are $\sigma_1, \sigma_2 \in Relhyp_{\mathcal{A}}(\tau, \tau')$ and $F_2^A \in \mathcal{F}^A$ such that $F_1^A = \dot{\sigma}_1^A[\dot{\sigma}_2^A[F_2^A]]$. By Lemma 4.5 this means $F_1^A = (\sigma_1 \circ_{hr} \sigma_2)^A[F_2^A]$ for $(\sigma_1 \circ_{hr} \sigma_2) \in Relhyp_{\mathcal{A}}(\tau, \tau')$ and therefore $(\sigma_1 \circ_{hr} \sigma_2)^A[F_2^A] \in \chi_h^F(\mathcal{F}^A)$. Then $\chi_h^F(\chi_h^F(\mathcal{F}^A)) \subseteq \chi_h^F(\mathcal{F}^A)$. ■

Every hypersubstitution for algebraic systems can be applied to algebraic systems $\mathcal{A} = (A; (f_i^A)_{i \in I}, (\gamma_j^A)_{j \in J})$. For $\sigma \in Relhyp_{\mathcal{A}}(\tau, \tau')$ we define the derived algebraic system $\sigma(\mathcal{A}) := (A; (\sigma_F(f_i^A))_{i \in I}, (\sigma_R(\gamma_j^A))_{j \in J})$. For each algebraic system of type (τ, τ') and each $\sigma \in Relhyp_{\mathcal{A}}(\tau, \tau')$ we define a closure operator

$$\chi_h^A : \mathcal{P}((Algsys(\tau, \tau'))) \rightarrow \mathcal{P}((Algsys(\tau, \tau'))) \text{ by}$$

$$\chi_h^A(\mathcal{K}) := \{\sigma(\mathcal{A}) \mid \sigma \in Relhyp_{\mathcal{A}}(\tau, \tau')\}$$

and for $\mathcal{K} \subseteq Algsys(\tau, \tau')$ we have $\chi_h^A(\mathcal{K}) := \bigcup_{\mathcal{A} \in \mathcal{K}} \chi_h^A(\mathcal{A})$. Then we have

Lemma 5.2. χ_h^A has the properties of a completely additive closure operator.

Although the proof is very similar to that of Lemma 5.1, we will not omit it since there are a few differences.

Proof. Additivity follows from the definition and monotonicity follows from additivity. Let σ_{id} be the identity hypersubstitution for algebraic systems. Then $\sigma_{id}(\mathcal{A}) = (A; ((\sigma_{id})_F(f_i^A))_{i \in I}, ((\sigma_{id})_R(\gamma_j^A))_{j \in J}) = (A; (f_i^A)_{i \in I}, (\gamma_j^A)_{j \in J})$. As a consequence, χ_h^A is extensive. The inclusion $\chi_h^A(\mathcal{K}) \subseteq \chi_h^A(\chi_h^A(\mathcal{K}))$ follows from extensivity. If $\mathcal{A} \in \chi_h^A(\chi_h^A(\mathcal{K}))$, then there are $\sigma, \sigma' \in Relhyp_{\mathcal{A}}(\tau, \tau')$ and an algebraic system $\mathcal{A}' \in \mathcal{K}$ such that

$$\begin{aligned} \mathcal{A} &= \sigma(\sigma'(\mathcal{A}')) \\ &= (A; (\widehat{\sigma}_F[\sigma'_F(f_i^A)])_{i \in I}, (\widehat{\sigma}_R[\sigma'_R(\gamma_j^A)])_{j \in J}) \\ &= (A; ((\sigma \circ_{hr} \sigma')_F(f_i^A))_{i \in I}, ((\sigma \circ_{hr} \sigma')_R(\gamma_j^A))_{j \in J}). \end{aligned}$$

■

Lemma 5.3. For any term $t \in W_\tau(X)$ and for any $R^A \in \langle \{\gamma_j^A \mid j \in J\} \rangle$ we have

$$\widehat{\sigma}_F[t^A] = t^{\sigma(A)}, \quad \widehat{\sigma}_R[R^A] = R^{\sigma(A)}.$$

Proof. $\widehat{\sigma}_F[t^A] = t^{\sigma(A)}$ can be obtained from [4], Lemma 3.2.1. Next we have to show by induction on the definition of the relational clone that $\widehat{\sigma}_R[R^A] = (R)^{\sigma(A)}$.

If $\rho^A \in \{\gamma_j^A \mid j \in J\}$, then $\widehat{\sigma}_R[\rho^A] = \sigma_R(\rho^A) = \rho^{\sigma(A)}$ by definition of the derived algebraic system.

Assume that for $\rho^A \in \langle \{\gamma_j^A \mid j \in J\} \rangle \setminus \{\gamma_j^A \mid j \in J\}$ we have already $\widehat{\sigma}_R[\rho^A] = \rho^{\sigma(A)}$, $\widehat{\sigma}_R[\rho_1^A] = \rho_1^{\sigma(A)}$, $\widehat{\sigma}_R[\rho_2^A] = \rho_2^{\sigma(A)}$. Then we get

$$\widehat{\sigma}_R[\xi \rho^A] = \xi(\widehat{\sigma}_R[\rho^A]) = \xi(\rho^{\sigma(A)}).$$

$$\widehat{\sigma}_R[\tau \rho^A] = \tau(\widehat{\sigma}_R[\rho^A]) = \tau(\rho^{\sigma(A)}).$$

$$\widehat{\sigma}_R[\triangle \rho^A] = \triangle(\widehat{\sigma}_R[\rho^A]) = \triangle(\rho^{\sigma(A)}).$$

$$\widehat{\sigma}_R[\rho_1^A \circ \rho_2^A] = \widehat{\sigma}_R[\rho_1^A] \circ \widehat{\sigma}_R[\rho_2^A] = \rho_1^{\sigma(A)} \circ \rho_2^{\sigma(A)}.$$

$$\widehat{\sigma}_R[\delta_3^{\{1; 2,3\}, A}] = \delta_3^{\{1; 2,3\}, \sigma(A)}.$$

■

Lemma 5.4. *Let $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I}, (\gamma_j^{\mathcal{A}})_{j \in J})$ be an algebraic system of type (τ, τ') . Then for each $F^{\mathcal{A}} \in (\mathcal{F}_{(\tau, \tau')}(W_{\tau}(X)))^{\mathcal{A}}$ we have*

$$F^{\sigma(\mathcal{A})} = \dot{\sigma}^{\mathcal{A}}[F^{\mathcal{A}}].$$

Proof. We will give a proof by induction on the definition of the realization of a formula.

(i) Let $F^{\mathcal{A}}$ be the formula $t_1^{\mathcal{A}} = t_2^{\mathcal{A}}$, then

$$\begin{aligned} & \dot{\sigma}^{\mathcal{A}}[(t_1^{\mathcal{A}} = t_2^{\mathcal{A}})] \\ &= (\widehat{\sigma}_F[t_1^{\mathcal{A}}] = \widehat{\sigma}_F[t_2^{\mathcal{A}}]) \\ &= (t_1^{\sigma(\mathcal{A})} = t_2^{\sigma(\mathcal{A})}) \\ &= (t_1 \approx t_2)^{\sigma(\mathcal{A})} \end{aligned}$$

where $t_1^{\sigma(\mathcal{A})}, t_2^{\sigma(\mathcal{A})}$ are the term operations induced by terms t_1, t_2 , respectively, on the algebra $(A; (\sigma_F(f_i)^{\mathcal{A}})_{i \in I})$.

(ii) If $F^{\mathcal{A}}$ has the form $R_l^{\mathcal{A}}(t_1^{\mathcal{A}}, \dots, t_{n_l}^{\mathcal{A}})$ then

$$\begin{aligned} & \dot{\sigma}^{\mathcal{A}}[R_l^{\mathcal{A}}(t_1^{\mathcal{A}}, \dots, t_{n_l}^{\mathcal{A}})] \\ &= R_n^{n_l, \mathcal{A}}((\widehat{\sigma}_R(R_l^{\mathcal{A}}), \widehat{\sigma}_F[t_1^{\mathcal{A}}]^{\mathcal{A}}, \dots, \widehat{\sigma}_F[t_{n_l}^{\mathcal{A}}]^{\mathcal{A}})) \\ &= R_n^{n_l, \mathcal{A}}((R_l^{\mathcal{A}})^{\sigma(\mathcal{A})}, t_1^{\sigma(\mathcal{A})}, \dots, t_{n_l}^{\sigma(\mathcal{A})}) \\ &= (R_l^{\mathcal{A}}(t_1, \dots, t_{n_l}))^{\sigma(\mathcal{A})}. \end{aligned}$$

(iii) If $F^{\mathcal{A}}$ has the form $\neg F^{\mathcal{A}}$ and assume that $F^{\sigma(\mathcal{A})} = \dot{\sigma}^{\mathcal{A}}[F^{\mathcal{A}}]$, then

$$\begin{aligned} & \dot{\sigma}^{\mathcal{A}}[\neg F^{\mathcal{A}}] \\ &= \neg \dot{\sigma}^{\mathcal{A}}[F^{\mathcal{A}}] \\ &= \neg F^{\sigma(\mathcal{A})} \\ &= (\neg F)^{\sigma(\mathcal{A})}. \end{aligned}$$

The inductive steps for $F^{\mathcal{A}} = F_1^{\mathcal{A}} \vee F_2^{\mathcal{A}}$ and $F^{\mathcal{A}} = \exists x_i(F^{\mathcal{A}})$ may be handled similarly. ■

Now we prove:

Theorem 5.5. *Let \mathcal{A} be an algebraic system of type (τ, τ') and let $F^{\mathcal{A}}$ be the realization of a formula of type (τ, τ') . Let $\sigma \in Relhyp_{\mathcal{A}}(\tau, \tau')$. Then*

$$\dot{\sigma}^{\mathcal{A}}[F^{\mathcal{A}}] \text{ is true in } \mathcal{A} \text{ iff } F^{\sigma(\mathcal{A})} \text{ is true in } \sigma(\mathcal{A}).$$

Proof. In fact, using Lemma 5.4 we have that $\dot{\sigma}^{\mathcal{A}}[F^{\mathcal{A}}]$ is true in \mathcal{A} iff $F^{\sigma(\mathcal{A})}$ is true in $\sigma(\mathcal{A})$. ■

Because of its properties we say that (χ_h^A, χ_h^F) forms a conjugate pair of completely additive closure operators.

6. SOLID MODEL CLASSES

An algebraic system $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I}, (\gamma_j^{\mathcal{A}})_{j \in J})$ of type (τ, τ') satisfies a formula $F \in \mathcal{F}_{(\tau, \tau')}(W_{\tau}(X))$ if for any replacements of the free variables in F by elements from A and for any replacements of the operation symbols and relation symbols in F by the corresponding fundamental operations and relations of \mathcal{A} the arising realization $F^{\mathcal{A}}$ is satisfied. In this case we write $\mathcal{A} \models F$.

Let $Algsys(\tau, \tau')$ be the class of all algebraic systems of type (τ, τ') . Then the satisfaction relation \models defines a Galois connection between $Algsys(\tau, \tau')$ and $\mathcal{F}_{(\tau, \tau')}(W_{\tau}(X))$ by

$$Mod\mathcal{F} := \{\mathcal{A} \in Algsys(\tau, \tau') \mid \forall F \in \mathcal{F} (\mathcal{A} \models F)\} \text{ for subsets}$$

$$\mathcal{F} \subseteq \mathcal{F}_{(\tau, \tau')}(W_{\tau}(X))$$

and

$$Th\mathcal{K} := \{F \in \mathcal{F}_{(\tau, \tau')}(W_{\tau}(X)) \mid \forall \mathcal{A} \in \mathcal{K} (\mathcal{A} \models F)\} \text{ for subclasses}$$

$$\mathcal{K} \subseteq Algsys(\tau, \tau').$$

Then (Th, Mod) satisfies the conditions of a Galois-connection. The collection of all fixed points with respect to $ThMod$ and with respect to $ModTh$, respectively form complete lattices.

Definition 6.1. Let \mathcal{A} be an algebraic system of type (τ, τ') and let $F^{\mathcal{A}}$ be the realization of a formula of type (τ, τ') in \mathcal{A} . Then we say that the formula F hypersatisfies \mathcal{A} if $\dot{\sigma}^{\mathcal{A}}[F^{\mathcal{A}}]$ is true in \mathcal{A} , for all $\sigma \in Relhyp_{\mathcal{A}}(\tau, \tau')$. In this case we write

$$\mathcal{A} \models_H F.$$

We notice that F is a formula of type (τ, τ'') , but that $\dot{\sigma}^{\mathcal{A}}[F^{\mathcal{A}}]$ is the realization of a formula of type (τ, τ') .

The relation \models_H defines a second Galois connection $(HTh, HMod)$ by

$$HMod\mathcal{F} := \{\mathcal{A} \in Algsys(\tau, \tau') \mid \forall F \in \mathcal{F} (\mathcal{A} \models_H F)\}$$

for subsets $\mathcal{F} \subseteq \mathcal{F}_{(\tau, \tau')}(W_{\tau}(X))$ and

$$HTh\mathcal{K} := \{F \in \mathcal{F}_{(\tau, \tau')}(W_{\tau}(X)) \mid \forall \mathcal{A} \in \mathcal{K} (\mathcal{A} \models_H F)\}$$

for subclasses $\mathcal{K} \subseteq Algsys(\tau, \tau')$.

Moreover, the fixed points of the closure operators $HThHMod$ and $HModHTh$ form two more complete lattices. The fixed points of the operator $HModHTh$ are called solid model classes.

The relationships between these two Galois connections and the conjugate pair (χ_h^A, χ_h^F) of completely additive closure operators can be described by using the general theory of conjugate pairs of additive closure operators ([4]). One of the results which can be obtained using this theory is the characterization of solid model classes and the fixed points with respect to the closure operator $HThHMod$.

Theorem 6.2. Let \mathcal{K} be a class of algebraic systems of type (τ, τ') of the form $\mathcal{K} = Mod\mathcal{F}$ for some set \mathcal{F} of formulas of type (τ, τ') . Then the following four propositions are equivalent:

(i) $\mathcal{K} = HModHTh\mathcal{K}$.

(ii) $\chi_h^A[\mathcal{K}] = \mathcal{K}$.

$$(iii) \quad Th\mathcal{K} = HTh\mathcal{K}.$$

$$(iv) \quad \chi_h^F[Th\mathcal{K}] = Th\mathcal{K}.$$

Dually, the four propositions (i'), (ii'), (iii') and (iv') are also pairwise equivalent :

$$(i') \quad \mathcal{F} = HThHMod\mathcal{F}.$$

$$(ii') \quad \chi_h^F[\mathcal{F}] = \mathcal{F}.$$

$$(iii') \quad Mod\mathcal{F} = HMod\mathcal{F}.$$

$$(iv') \quad \chi_h^A[Mod\mathcal{F}] = Mod\mathcal{F}.$$

This means that \mathcal{K} is a solid model class iff it is a fixed point with respect to the closure operator $HModHTh\mathcal{K}$ iff all derived algebraic systems belong to \mathcal{K} iff it is closed under the application of the operator χ_h^A . From the general theory of conjugate pairs of additive closure operators follows also that the collection of all solid model classes of type (τ, τ') forms a complete sublattice of the lattice of all model classes of this type.

7. EXAMPLE

As an example we consider the algebraic system $\mathcal{A} = (A; max, min, \leq)$ of type $((2, 2), (2))$ where \leq is a partial order relation on A which is invariant under the binary operations min and max with respect to \leq . For the operations min, max we introduce the operation symbols f_1 and f_2 , respectively and for \leq we use the relation symbol γ . Clearly, the algebra $\mathcal{A}' = (A; max, min)$ is a distributive lattice. Then the two-generated free algebra of the variety $V(\mathcal{A}')$ generated by \mathcal{A}' contains only 4 elements, namely

$$\begin{aligned} & W_{(2,2)}(X_2)/Id\mathcal{V}(\mathcal{A}') \\ &= \{[x_1]_{Id\mathcal{V}(\mathcal{A}')}, [x_2]_{Id\mathcal{V}(\mathcal{A}')}, [f_1(x_1, x_2)]_{Id\mathcal{V}(\mathcal{A}')}, [f_2(x_1, x_2)]_{Id\mathcal{V}(\mathcal{A}')}\}. \end{aligned}$$

The set $\{e_1^{2,A}, e_2^{2,A}, f_1^A = min, f_2^A = max\}$ is the set of all binary term operations of \mathcal{A}' . The relational clone of \mathcal{A} is $\{\leq, \leq^{-1}, A^2, \Delta_A\}$. Then all concrete hypersubstitutions are given by mappings from $\{f_1^A, f_2^A\}$ to $W_{(2,2)}(X_2)/Id\mathcal{V}(\mathcal{A}')$. This means that we have the following 16 concrete hypersubstitutions:

$$\begin{array}{ll}
 (\sigma_1)_F : & f_1^A \mapsto e_1^{2,A} \quad (\sigma_2)_F : f_1^A \mapsto e_1^{2,A} \\
 & f_2^A \mapsto e_1^{2,A}, \quad f_2^A \mapsto e_2^{2,A}, \\
 (\sigma_3)_F : & f_1^A \mapsto e_1^{2,A} \quad (\sigma_4)_F : f_1^A \mapsto e_1^{2,A} \\
 & f_2^A \mapsto f_1^A, \quad f_2^A \mapsto f_2^A, \\
 (\sigma_5)_F : & f_1^A \mapsto e_2^{2,A} \quad (\sigma_6)_F : f_1^A \mapsto e_2^{2,A} \\
 & f_2^A \mapsto e_1^{2,A}, \quad f_2^A \mapsto e_2^{2,A}, \\
 (\sigma_7)_F : & f_1^A \mapsto e_2^{2,A} \quad (\sigma_8)_F : f_1^A \mapsto e_2^{2,A} \\
 & f_2^A \mapsto f_1^A, \quad f_2^A \mapsto f_2^A, \\
 (\sigma_9)_F : & f_1^A \mapsto f_1^A \quad (\sigma_{10})_F : f_1^A \mapsto f_1^A \\
 & f_2^A \mapsto e_1^{2,A}, \quad f_2^A \mapsto e_2^{2,A}, \\
 (\sigma_{11})_F : & f_1^A \mapsto f_1^A \quad (\sigma_{12})_F : f_1^A \mapsto f_1^A \\
 & f_2^A \mapsto f_1^A, \quad f_2^A \mapsto f_2^A, \\
 (\sigma_{13})_F : & f_1^A \mapsto f_2^A \quad (\sigma_{14})_F : f_1^A \mapsto f_2^A \\
 & f_2^A \mapsto e_1^{2,A}, \quad f_2^A \mapsto e_2^{2,A}, \\
 (\sigma_{15})_F : & f_1^A \mapsto f_2^A \quad (\sigma_{16})_F : f_1^A \mapsto f_2^A \\
 & f_2^A \mapsto f_1^A, \quad f_2^A \mapsto f_2^A.
 \end{array}$$

The 4 relational hypersubstitutions are the 4 mappings $\{\gamma_j^A\} \rightarrow \{\leq, \leq^{-1}, A^2, \Delta_A\}$ given by

$$\begin{array}{ll}
 (\sigma_1)_R : & \gamma^A \mapsto \leq, \quad (\sigma_2)_R : \gamma^A \mapsto \leq^{-1}, \\
 (\sigma_3)_R : & \gamma^A \mapsto A^2, \quad (\sigma_4)_R : \gamma^A \mapsto \Delta_A.
 \end{array}$$

Then the set $Relhyp_{\mathcal{A}}((2, 2), (2))$ consists of 64 elements which are given by the following table.

	$(\sigma_1)_R$	$(\sigma_2)_R$	$(\sigma_3)_R$	$(\sigma_4)_R$
$(\sigma_1)_F$	σ_1	σ_2	σ_3	σ_4
$(\sigma_2)_F$	σ_5	σ_6	σ_7	σ_8
$(\sigma_3)_F$	σ_9	σ_{10}	σ_{11}	σ_{12}
$(\sigma_4)_F$	σ_{13}	σ_{14}	σ_{15}	σ_{16}
$(\sigma_5)_F$	σ_{17}	σ_{18}	σ_{19}	σ_{20}
$(\sigma_6)_F$	σ_{21}	σ_{22}	σ_{23}	σ_{24}
$(\sigma_7)_F$	σ_{25}	σ_{26}	σ_{27}	σ_{28}
$(\sigma_8)_F$	σ_{29}	σ_{30}	σ_{31}	σ_{32}
$(\sigma_9)_F$	σ_{33}	σ_{34}	σ_{35}	σ_{36}
$(\sigma_{10})_F$	σ_{37}	σ_{38}	σ_{39}	σ_{40}
$(\sigma_{11})_F$	σ_{41}	σ_{42}	σ_{43}	σ_{44}
$(\sigma_{12})_F$	σ_{45}	σ_{46}	σ_{47}	σ_{48}
$(\sigma_{13})_F$	σ_{49}	σ_{50}	σ_{51}	σ_{52}
$(\sigma_{14})_F$	σ_{53}	σ_{54}	σ_{55}	σ_{56}
$(\sigma_{15})_F$	σ_{57}	σ_{58}	σ_{59}	σ_{60}
$(\sigma_{16})_F$	σ_{61}	σ_{62}	σ_{63}	σ_{64}

Let us give two examples for the application of an extension of a hypersubstitution to a formula which is satisfied in \mathcal{A} . Consider $\leq (e_1^{2,A}, e_1^{2,A})$. Then

$$\begin{aligned} & \dot{\sigma}_1^A[\leq (e_1^{2,A}, e_1^{2,A})] \\ &= R_2^{2,A}((\sigma_1)_R(\leq), (\hat{\sigma}_1)_F[e_1^{2,A}], (\hat{\sigma}_1)_F[e_1^{2,A}]) = \leq (e_1^{2,A}, e_1^{2,A}). \end{aligned}$$

In a similar way, we proceed for all other hypersubstitutions and see that the resulting formulas are always satisfied in \mathcal{A} . This shows that $\gamma(x_1, x_1)$ is a hyperformula.

As a second example consider $f_1^A(e_1^{2,A}, e_1^{2,A}) = e_1^{2,A}$. Then

$$\begin{aligned} \dot{\sigma}_1^A[f_1^A(e_1^{2,A}, e_1^{2,A})] &= e_1^{2,A} \\ &= (\sigma_1)_F(f_1^A)((\hat{\sigma}_1)_F[e_1^{2,A}], (\hat{\sigma}_1)_F[e_1^{2,A}]) = (\hat{\sigma}_1)_F[e_1^{2,A}] \\ &= e_1^{2,A}(e_1^{2,A}, e_1^{2,A}) = e_1^{2,A} \\ &= e_1^{2,A} = e_1^{2,A}. \end{aligned}$$

In a similar way, we proceed in all other cases. This shows that $f_1(x_1, x_1) \approx x_1$ is a hyperformula.

By definition of the multiplication in $\mathcal{Relhyp}_{\mathcal{A}}((2, 2), (2))$ the multiplication tables of the concrete hypersubstitutions and of the relational hypersubstitutions determine completely the multiplication in $\mathcal{Relhyp}_{\mathcal{A}}((2, 2), (2))$. These tables are given by

\circ_h	$(\sigma_1)_F$	$(\sigma_2)_F$	$(\sigma_3)_F$	$(\sigma_4)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_7)_F$	$(\sigma_8)_F$
$(\sigma_1)_F$	$(\sigma_1)_F$	$(\sigma_2)_F$	$(\sigma_3)_F$	$(\sigma_4)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_7)_F$	$(\sigma_8)_F$
$(\sigma_2)_F$	$(\sigma_2)_F$	$(\sigma_2)_F$	$(\sigma_1)_F$	$(\sigma_2)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$
$(\sigma_3)_F$	$(\sigma_3)_F$	$(\sigma_2)_F$	$(\sigma_1)_F$	$(\sigma_3)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_5)_F$	$(\sigma_7)_F$
$(\sigma_4)_F$	$(\sigma_4)_F$	$(\sigma_2)_F$	$(\sigma_1)_F$	$(\sigma_4)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_5)_F$	$(\sigma_8)_F$
$(\sigma_5)_F$	$(\sigma_5)_F$	$(\sigma_2)_F$	$(\sigma_2)_F$	$(\sigma_1)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_5)_F$	$(\sigma_5)_F$
$(\sigma_6)_F$	$(\sigma_6)_F$	$(\sigma_2)_F$	$(\sigma_2)_F$	$(\sigma_6)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_6)_F$	$(\sigma_6)_F$
$(\sigma_7)_F$	$(\sigma_7)_F$	$(\sigma_2)_F$	$(\sigma_2)_F$	$(\sigma_3)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_6)_F$	$(\sigma_6)_F$
$(\sigma_8)_F$	$(\sigma_8)_F$	$(\sigma_2)_F$	$(\sigma_2)_F$	$(\sigma_4)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_6)_F$	$(\sigma_8)_F$
$(\sigma_9)_F$	$(\sigma_9)_F$	$(\sigma_2)_F$	$(\sigma_3)_F$	$(\sigma_1)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_7)_F$	$(\sigma_5)_F$
$(\sigma_{10})_F$	$(\sigma_{10})_F$	$(\sigma_2)_F$	$(\sigma_3)_F$	$(\sigma_2)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_7)_F$	$(\sigma_6)_F$
$(\sigma_{11})_F$	$(\sigma_{11})_F$	$(\sigma_2)_F$	$(\sigma_3)_F$	$(\sigma_3)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_7)_F$	$(\sigma_7)_F$
$(\sigma_{12})_F$	$(\sigma_{12})_F$	$(\sigma_2)_F$	$(\sigma_3)_F$	$(\sigma_4)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_7)_F$	$(\sigma_8)_F$
$(\sigma_{13})_F$	$(\sigma_{13})_F$	$(\sigma_2)_F$	$(\sigma_4)_F$	$(\sigma_1)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_7)_F$	$(\sigma_5)_F$
$(\sigma_{14})_F$	$(\sigma_{14})_F$	$(\sigma_2)_F$	$(\sigma_4)_F$	$(\sigma_6)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_8)_F$	$(\sigma_6)_F$
$(\sigma_{15})_F$	$(\sigma_{15})_F$	$(\sigma_2)_F$	$(\sigma_8)_F$	$(\sigma_3)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_8)_F$	$(\sigma_7)_F$
$(\sigma_{16})_F$	$(\sigma_{16})_F$	$(\sigma_2)_F$	$(\sigma_4)_F$	$(\sigma_4)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_8)_F$	$(\sigma_8)_F$

\circ_h	$(\sigma_9)_F$	$(\sigma_{10})_F$	$(\sigma_{11})_F$	$(\sigma_{12})_F$	$(\sigma_{13})_F$	$(\sigma_{14})_F$	$(\sigma_{15})_F$	$(\sigma_{16})_F$
$(\sigma_1)_F$	$(\sigma_9)_F$	$(\sigma_{10})_F$	$(\sigma_{11})_F$	$(\sigma_{12})_F$	$(\sigma_{13})_F$	$(\sigma_{14})_F$	$(\sigma_{15})_F$	$(\sigma_{16})_F$
$(\sigma_2)_F$	$(\sigma_1)_F$	$(\sigma_2)_F$	$(\sigma_1)_F$	$(\sigma_2)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$
$(\sigma_3)_F$	$(\sigma_1)_F$	$(\sigma_2)_F$	$(\sigma_1)_F$	$(\sigma_3)_F$	$(\sigma_3)_F$	$(\sigma_{10})_F$	$(\sigma_9)_F$	$(\sigma_{11})_F$
$(\sigma_4)_F$	$(\sigma_1)_F$	$(\sigma_2)_F$	$(\sigma_1)_F$	$(\sigma_4)_F$	$(\sigma_{13})_F$	$(\sigma_{14})_F$	$(\sigma_{13})_F$	$(\sigma_{16})_F$
$(\sigma_5)_F$	$(\sigma_5)_F$	$(\sigma_1)_F$	$(\sigma_6)_F$	$(\sigma_5)_F$	$(\sigma_1)_F$	$(\sigma_2)_F$	$(\sigma_2)_F$	$(\sigma_1)_F$
$(\sigma_6)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_6)_F$	$(\sigma_6)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_6)_F$	$(\sigma_6)_F$
$(\sigma_7)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_6)_F$	$(\sigma_7)_F$	$(\sigma_9)_F$	$(\sigma_6)_F$	$(\sigma_{10})_F$	$(\sigma_{11})_F$
$(\sigma_8)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_6)_F$	$(\sigma_8)_F$	$(\sigma_{13})_F$	$(\sigma_{14})_F$	$(\sigma_{14})_F$	$(\sigma_{16})_F$
$(\sigma_9)_F$	$(\sigma_9)_F$	$(\sigma_{10})_F$	$(\sigma_{11})_F$	$(\sigma_{10})_F$	$(\sigma_1)_F$	$(\sigma_2)_F$	$(\sigma_3)_F$	$(\sigma_1)_F$
$(\sigma_{10})_F$	$(\sigma_9)_F$	$(\sigma_{10})_F$	$(\sigma_{11})_F$	$(\sigma_{10})_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_7)_F$	$(\sigma_6)_F$
$(\sigma_{11})_F$	$(\sigma_9)_F$	$(\sigma_{10})_F$	$(\sigma_{11})_F$	$(\sigma_{11})_F$	$(\sigma_9)_F$	$(\sigma_{14})_F$	$(\sigma_{11})_F$	$(\sigma_{11})_F$
$(\sigma_{12})_F$	$(\sigma_9)_F$	$(\sigma_{10})_F$	$(\sigma_{11})_F$	$(\sigma_{12})_F$	$(\sigma_{13})_F$	$(\sigma_{14})_F$	$(\sigma_{15})_F$	$(\sigma_{16})_F$
$(\sigma_{13})_F$	$(\sigma_{13})_F$	$(\sigma_{14})_F$	$(\sigma_{16})_F$	$(\sigma_{13})_F$	$(\sigma_1)_F$	$(\sigma_2)_F$	$(\sigma_4)_F$	$(\sigma_1)_F$
$(\sigma_{14})_F$	$(\sigma_{13})_F$	$(\sigma_{14})_F$	$(\sigma_{16})_F$	$(\sigma_{10})_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_8)_F$	$(\sigma_6)_F$
$(\sigma_{15})_F$	$(\sigma_{13})_F$	$(\sigma_{14})_F$	$(\sigma_{16})_F$	$(\sigma_{15})_F$	$(\sigma_9)_F$	$(\sigma_{10})_F$	$(\sigma_{12})_F$	$(\sigma_{11})_F$
$(\sigma_{16})_F$	$(\sigma_{13})_F$	$(\sigma_{14})_F$	$(\sigma_{16})_F$	$(\sigma_{16})_F$	$(\sigma_{13})_F$	$(\sigma_{14})_F$	$(\sigma_{16})_F$	$(\sigma_{16})_F$

\circ_r	$(\sigma_1)_R$	$(\sigma_2)_R$	$(\sigma_3)_R$	$(\sigma_4)_R$
$(\sigma_1)_R$	$(\sigma_1)_R$	$(\sigma_2)_R$	$(\sigma_3)_R$	$(\sigma_4)_R$
$(\sigma_2)_R$	$(\sigma_2)_R$	$(\sigma_1)_R$	$(\sigma_3)_R$	$(\sigma_1)_R$
$(\sigma_3)_R$	$(\sigma_3)_R$	$(\sigma_3)_R$	$(\sigma_3)_R$	$(\sigma_3)_R$
$(\sigma_4)_R$	$(\sigma_4)_R$	$(\sigma_4)_R$	$(\sigma_4)_R$	$(\sigma_4)_R$

Idempotent elements in $\mathcal{Relhyp}_{\mathcal{A}}((2,2),(2))$ are pairs (σ_F, σ_R) , where σ_F and σ_R are idempotent.

We get that

$\sigma_1, \sigma_3, \sigma_4, \sigma_5, \sigma_7, \sigma_8, \sigma_{13}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{19}, \sigma_{20}, \sigma_{21}, \sigma_{23}, \sigma_{24}, \sigma_{29}, \sigma_{31}, \sigma_{32}, \sigma_{33}, \sigma_{35}, \sigma_{36}, \sigma_{37}, \sigma_{39}, \sigma_{40}, \sigma_{41}, \sigma_{43}, \sigma_{44}, \sigma_{45}, \sigma_{47}, \sigma_{48}, \sigma_{61}, \sigma_{63}, \sigma_{64}$ are all idempotent elements.

The following multiplication tables show that the idempotent relational hypersubstitutions and the idempotent concrete hypersubstitutions form

subsemigroups. Consequently, the set of all idempotent elements of $\mathcal{Relhyp}_{\mathcal{A}}((2, 2), (2))$ forms a subsemigroup of $\mathcal{Relhyp}_{\mathcal{A}}((2, 2), (2))$.

\circ_r	$(\sigma_1)_R$	$(\sigma_3)_R$	$(\sigma_4)_R$
$(\sigma_1)_R$	$(\sigma_1)_R$	$(\sigma_3)_R$	$(\sigma_4)_R$
$(\sigma_3)_R$	$(\sigma_3)_R$	$(\sigma_3)_R$	$(\sigma_3)_R$
$(\sigma_4)_R$	$(\sigma_4)_R$	$(\sigma_4)_R$	$(\sigma_4)_R$

\circ_h	$(\sigma_1)_F$	$(\sigma_2)_F$	$(\sigma_4)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_8)_F$	$(\sigma_9)_F$	$(\sigma_{10})_F$	$(\sigma_{11})_F$	$(\sigma_{12})_F$	$(\sigma_{16})_F$
$(\sigma_1)_F$	$(\sigma_1)_F$	$(\sigma_2)_F$	$(\sigma_4)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_8)_F$	$(\sigma_9)_F$	$(\sigma_{10})_F$	$(\sigma_{11})_F$	$(\sigma_{12})_F$	$(\sigma_{16})_F$
$(\sigma_2)_F$	$(\sigma_2)_F$	$(\sigma_2)_F$	$(\sigma_2)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_6)_F$	$(\sigma_1)_F$	$(\sigma_2)_F$	$(\sigma_1)_F$	$(\sigma_2)_F$	$(\sigma_6)_F$
$(\sigma_4)_F$	$(\sigma_4)_F$	$(\sigma_2)_F$	$(\sigma_4)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_8)_F$	$(\sigma_1)_F$	$(\sigma_2)_F$	$(\sigma_1)_F$	$(\sigma_4)_F$	$(\sigma_{16})_F$
$(\sigma_5)_F$	$(\sigma_5)_F$	$(\sigma_2)_F$	$(\sigma_1)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_5)_F$	$(\sigma_5)_F$	$(\sigma_1)_F$	$(\sigma_6)_F$	$(\sigma_5)_F$	$(\sigma_1)_F$
$(\sigma_6)_F$	$(\sigma_6)_F$	$(\sigma_2)_F$	$(\sigma_6)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_6)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_6)_F$	$(\sigma_6)_F$	$(\sigma_6)_F$
$(\sigma_8)_F$	$(\sigma_8)_F$	$(\sigma_2)_F$	$(\sigma_4)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_8)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_6)_F$	$(\sigma_8)_F$	$(\sigma_{16})_F$
$(\sigma_9)_F$	$(\sigma_9)_F$	$(\sigma_2)_F$	$(\sigma_1)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_5)_F$	$(\sigma_9)_F$	$(\sigma_{10})_F$	$(\sigma_{11})_F$	$(\sigma_{10})_F$	$(\sigma_1)_F$
$(\sigma_{10})_F$	$(\sigma_{10})_F$	$(\sigma_2)_F$	$(\sigma_2)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_6)_F$	$(\sigma_9)_F$	$(\sigma_{10})_F$	$(\sigma_{11})_F$	$(\sigma_{10})_F$	$(\sigma_6)_F$
$(\sigma_{11})_F$	$(\sigma_{11})_F$	$(\sigma_2)_F$	$(\sigma_3)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_7)_F$	$(\sigma_9)_F$	$(\sigma_{10})_F$	$(\sigma_{11})_F$	$(\sigma_{11})_F$	$(\sigma_{11})_F$
$(\sigma_{12})_F$	$(\sigma_{12})_F$	$(\sigma_2)_F$	$(\sigma_4)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_7)_F$	$(\sigma_8)_F$	$(\sigma_{10})_F$	$(\sigma_{11})_F$	$(\sigma_{12})_F$	$(\sigma_{16})_F$
$(\sigma_{16})_F$	$(\sigma_{16})_F$	$(\sigma_2)_F$	$(\sigma_4)_F$	$(\sigma_5)_F$	$(\sigma_6)_F$	$(\sigma_8)_F$	$(\sigma_{13})_F$	$(\sigma_{14})_F$	$(\sigma_{16})_F$	$(\sigma_{16})_F$	$(\sigma_{16})_F$

Every idempotent element of a semigroup $(S; \cdot)$ is regular, i.e. satisfies $a \cdot b \cdot a = a$ for some $b \in S$. We want to determine all regular elements of $\mathcal{Relhyp}_{\mathcal{A}}((2, 2), (2))$. Again we have only to check the concrete hypersubstitutions and the relational hypersubstitutions. All relational hypersubstitutions except $(\sigma_2)_R$ are idempotent. But $(\sigma_2)_R$ satisfies $(\sigma_2)_R^3 = (\sigma_2)_R \circ_r (\sigma_1)_R = (\sigma_2)_R$. Thus, every relational hypersubstitution satisfies $\sigma_R^3 = \sigma_R$ and is regular.

Now we consider the non-idempotent concrete hypersubstitutions

$$(\sigma_3)_F, (\sigma_7)_F, (\sigma_{13})_F, (\sigma_{14})_F, (\sigma_{15})_F.$$

Then we have

$$((\sigma_3)_F)^3 = (\sigma_3)_F \circ_h (\sigma_1)_F = (\sigma_3)_F$$

$$((\sigma_{13})_F)^3 = (\sigma_{13})_F \circ_h (\sigma_1)_F = (\sigma_{13})_F$$

$$((\sigma_{15})_F)^3 = (\sigma_{15})_F \circ_h (\sigma_{12})_F = (\sigma_{15})_F$$

This shows that $(\sigma_3)_F, (\sigma_{13})_F, (\sigma_{15})_F$ are regular. For $(\sigma_7)_F$ and $(\sigma_{14})_F$ we get

$$(\sigma_7)_F = (\sigma_7)_F \circ_h (\sigma_{15})_F \circ_h (\sigma_7)_F$$

$$(\sigma_{14})_F = (\sigma_{14})_F \circ_h (\sigma_3)_F \circ_h (\sigma_{14})_F.$$

This shows that $\mathcal{Relhyp}_{\mathcal{A}}((2, 2), (2))$ is a regular semigroup which contains the set of all idempotent elements as a subsemigroup, i.e. $\mathcal{Relhyp}_{\mathcal{A}}((2, 2), (2))$ is an orthodox semigroup. Further, one can check that $((\sigma_7)_F)^3 = ((\sigma_7)_F)^2$ and $((\sigma_{14})_F)^3 = ((\sigma_{14})_F)^2$. Therefore the elements of the semigroup $\mathcal{Relhyp}_{\mathcal{A}}((2, 2), (2))$ have order 1 or 2.

REFERENCES

- [1] K. Denecke and S.L. Wismath, *Hyperidentities and Clones*, Gordon and Breach Science Publishers 2000.
- [2] K. Denecke and D. Phusanga, *Hyperformulas and Solid Algebraic Systems*, *Studia Logica* **90** (2) (2008), 263–286.
- [3] E. Graczyńska and D. Schweigert, *Hyperidentities of a given type*, *Algebra Universalis* **27** (1990), 305–318.
- [4] J. Koppitz and K. Denecke, *M-solid Varieties*, Springer 2006.
- [5] A.I. Mal'cev, *Algebraic Systems*, Akademie-Verlag, Berlin 1973.

- [6] R. Pöschel and L.A. Kalužnin, *Funktionen-und Relationenalgebren*, VEB Deutscher Verlag der Wissenschaften, Berlin 1979.
- [7] J.A. Goguen and R.M. Burstall, *Introducing Institution*, in Proceeding of the Logic of Programming Workshop (1984), 221–256.
- [8] J.A. Goguen and R.M. Burstall, *Institutions: Abstract Model Theory for Specification and Programming*, Journal of the ACM **39** (1) (1992), 95–146.

Received 15 May 2009

Revised 3 November 2009