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FUNCTION CLASSES AND RELATIONAL CONSTRAINTS STABLE UNDER COMPOSITIONS WITH CLONES

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Abstract

The general Galois theory for functions and relational constraints over arbitrary sets described in the authors' previous paper is refined by imposing algebraic conditions on relations.

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1. Introduction

In this paper we extend the results obtained in [3] by considering closure conditions of a more general form on classes of functions of several variables, and by restricting relational constraints to consist of invariant relations.

In fact, Theorems 2.1 and 3.2 in [3] correspond to Theorems 2 and 7 below, respectively, in the particular case $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{P}$, where $\mathcal{P}$ denotes the smallest clone containing only projections.

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2. Basic notions and preliminary results

Throughout the paper, let A, B, E and G be arbitrary nonempty sets. Given a nonnegative integer m, the elements of A^m are viewed as unary functions from the von Neumann ordinal $m = \{0, \ldots, m-1\}$ to A.

A *function of several variables on A to B* (or simply, *function on A to B*) is a map $f : A^n \to B$, for some positive integer n called the *arity* of f. A *class* of functions on A to B is a subset $\mathcal{F} \subseteq \bigcup_{n\geq 1} B^{A^n}$. For a fixed arity n, the n different *projection maps* $(a_t \mid t \in n) \mapsto a_i$, $i \in n$, are also called *variables.* For $A = B = \{0,1\}$, a function on A to B is called a *Boolean function.*

If f is an n-ary function on B to E and $g_1, \ldots, g_n$ are all m-ary functions on A to B then the *composition* $f(g_1, \ldots, g_n)$ is an m-ary function on A to E, and its value on $(a_1, \ldots, a_m) \in A^m$ is $f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m))$. If $\mathcal{I} \subseteq \bigcup_{n\geq 1} E^{B^n}$ and $\mathcal{J} \subseteq \bigcup_{n\geq 1} B^{A^n}$ we define the *composition of $\mathcal{I}$ with $\mathcal{J}$*, denoted $\mathcal{I}\mathcal{J}$, by

$$\mathcal{I}\mathcal{J} = \{f(g_1, \ldots, g_n) \mid n, m \geq 1, f \text{ } n\text{-ary in } \mathcal{I},\ g_1, \ldots, g_n \text{ } m\text{-ary in } \mathcal{J}\}.$$

If $\mathcal{I}$ is a singleton, $\mathcal{I} = \{f\}$, then we write $f\mathcal{J}$ for $\{f\}\mathcal{J}$. We say that a class $\mathcal{I}$ of functions of several variables is *stable under right* (*left*) *composition with* $\mathcal{J}$ if, whenever the composition is well defined, $\mathcal{I}\mathcal{J} \subseteq \mathcal{I}$ ($\mathcal{J}\mathcal{I} \subseteq \mathcal{I}$, respectively). A *clone* on A is a set $\mathcal{C} \subseteq \bigcup_{n\geq 1} A^{A^n}$ that contains all projections and satisfies $\mathcal{C}\mathcal{C} \subseteq \mathcal{C}$ (or equivalently, $\mathcal{C}\mathcal{C} = \mathcal{C}$). Note that if $\mathcal{J}$ is a clone on A (on B) and $\mathcal{I} \subseteq \bigcup_{n\geq 1} B^{A^n}$, then $\mathcal{I}\mathcal{J} \subseteq \mathcal{I}$ if and only if $\mathcal{I}\mathcal{J} = \mathcal{I}$ ($\mathcal{J}\mathcal{I} \subseteq \mathcal{I}$ if and only if $\mathcal{J}\mathcal{I} = \mathcal{I}$, respectively). Note that stability under right composition with the clone $\mathcal{P} \subseteq \bigcup_{n\geq 1} A^{A^n}$ of all projections on A subsumes closure under the operations of identification of variables, permutation of variables and addition of inessential variables.

Associativity Lemma. *Let A, B, E and G be arbitrary nonempty sets, and consider function classes $\mathcal{I} \subseteq \bigcup_{n\geq 1} G^{E^n}$, $\mathcal{J} \subseteq \bigcup_{n\geq 1} E^{B^n}$, and $\mathcal{K} \subseteq \bigcup_{n\geq 1} B^{A^n}$. The following hold:*

(i) $(\mathcal{I}\mathcal{J})\mathcal{K} \subseteq \mathcal{I}(\mathcal{J}\mathcal{K})$;

(ii) *If $\mathcal{J}$ is stable under right composition with the clone of projections on B, then $(\mathcal{I}\mathcal{J})\mathcal{K} = \mathcal{I}(\mathcal{J}\mathcal{K})$.*

Proof. The inclusion (i) is a direct consequence of the definition of function class composition. Property (ii) asserts that the converse inclusion also holds if $\mathcal{J}$ is stable under right composition with projections. This hypothesis means in particular that all functions obtained from members of $\mathcal{J}$ by permutation of variables and addition of inessential variables are also in $\mathcal{J}$. A typical function in $\mathcal{I}(\mathcal{J}\mathcal{K})$ is of the form

$$f(g_1(h_{11}, \ldots, h_{1m_1}), \ldots, g_n(h_{n1}, \ldots, h_{nm_n}))$$

where f is in $\mathcal{I}$, the g_i's are in $\mathcal{J}$, and the h_{ij}'s are in $\mathcal{K}$. By taking appropriate functions $g_1', \ldots, g_n'$ obtained from $g_1, \ldots, g_n$ by permutation of variables and addition of inessential variables, the function above can be expressed as

$$f(g_1'(h_{11}, \ldots, h_{1m_1}, \ldots, h_{n1}, \ldots, h_{nm_n}), \ldots, g_n'(h_{11}, \ldots, h_{1m_1}, \ldots, h_{n1}, \ldots, h_{nm_n}))$$

which is easily seen to be in $(\mathcal{I}\mathcal{J})\mathcal{K}$. ■

Note that statement (ii) of the Associativity Lemma applies, in particular, if $\mathcal{J}$ is any clone on $E = B$.

Let $\mathcal{F}$ be a class of functions on A to B. If $\mathcal{P}_A$ is the clone of all projections on A, then $\mathcal{F}\mathcal{P}_A = \mathcal{F}$ expresses closure under taking minors in [9], or closure under simple variable substitutions in the terminology of [3]. On the other hand, every such class $\mathcal{F}$ is stable under left composition with the clone $\mathcal{P}_B$ of all projections on B. If $A = B = \{0, 1\}$ and $\mathcal{L}_{01}$ is the clone (Post class) of constant preserving linear Boolean functions, then $\mathcal{F}\mathcal{L}_{01} = \mathcal{F}$ is equivalent to closure under substitution of triple sums $x+y+z$ for variables, while $\mathcal{L}_{01}\mathcal{F} = \mathcal{F}$ is equivalent to closure under taking triple sums of Boolean functions $f + g + h$ (see [1]).

Examples. (1) Recall that every Boolean function f is uniquely represented by a multilinear polynomial, called the *Zhegalkin* or the *Reed-Muller* polynomial representation of f. For each $k \geq 1$, let $\mathcal{D}^k$ be the class of all Boolean functions whose multilinear polynomial representation has degree less than k. Then each $\mathcal{D}^k$ is stable under both left and right composition with the Post class $\mathcal{L}_0$ of (0-preserving) linear Boolean functions (see [4]).

(2) Let $\mathcal{S}$ the class of all submodular functions on $\{0,1\}$ to the field $\mathbb{R}$ of real numbers, usually defined by the equation $\mathbf{f}(\mathbf{x})+\mathbf{f}(\mathbf{y}) \geq \mathbf{f}(\mathbf{x}\wedge\mathbf{y})+\mathbf{f}(\mathbf{x}\vee\mathbf{y})$ (see, e.g., [8]). The class $\mathcal{S}$ is stable under right composition with the Post class $\mathcal{I}$ consisting of all projections and constants, and under left composition with the clone $\mathcal{A}_{\mathbb{R}}$ of affine forms $a_1x_1 + \ldots + a_nx_n + b$ with nonnegative real coefficients a_i (see [6]).

An *m-ary relation on A* is a subset R of A^m. Thus the relation R is a class (set) of unary maps on m to A. A function f of several variables on A to A is said to *preserve* R if $fR \subseteq R$.

For a function class $\mathcal{F} \subseteq \bigcup_{n\geq 1} A^{A^n}$, an m-ary relation R on A is called an *$\mathcal{F}$-invariant relation* if $\mathcal{F}R \subseteq R$. In other words, R is an $\mathcal{F}$-invariant relation if every member of $\mathcal{F}$ preserves R. If two classes of functions $\mathcal{F}$ and $\mathcal{G}$ generate the same clone, then the $\mathcal{F}$-invariant relations are the same as the $\mathcal{G}$-invariant relations. (See Pöschel [10] and [11].)

Observe that we always have $R \subseteq \mathcal{F}R$ if $\mathcal{F}$ contains the projections, but we can have $R \subseteq \mathcal{F}R$ even if $\mathcal{F}$ contains no projections. (Take the Boolean triple sum $x_1 + x_2 + x_3$ as the only member of $\mathcal{F}$.)

For a clone $\mathcal{C}$ on a set A, the intersection of m-ary $\mathcal{C}$-invariant relations is always a $\mathcal{C}$-invariant relation. It is easy to see that, for an m-ary relation R, the smallest $\mathcal{C}$-invariant relation containing R in A^m is $\mathcal{C}R$, and it is said to be the $\mathcal{C}$-invariant relation *generated by* R. (See [10] and [11], where Pöschel denotes $\mathcal{C}R$ by $\Gamma_{\mathcal{C}}(R)$.)

3. Classes of functions definable by constraints consisting of invariant relations

Consider arbitrary nonempty sets A and B. An m-ary *A-to-B constraint* (or simply, m-ary *constraint*, when the underlying sets are understood from the context) is a couple (R,S) where $R \subseteq A^m$ and $S \subseteq B^m$. The relations R and S are called the *antecedent* and *consequent*, respectively, of the relational constraint (Pippenger [9]). Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be clones on A and B, respectively. If R is a $\mathcal{C}_1$-invariant relation and S is a $\mathcal{C}_2$-invariant relation, we say that (R,S) is a *$(\mathcal{C}_1,\mathcal{C}_2)$-constraint*. A function $f : A^n \longrightarrow B$, $n \geq 1$, is said to *satisfy* an m-ary A-to-B constraint (R,S) if $fR \subseteq S$. The following result generalizes Lemma 1 in [1]:

Lemma 1. *Consider arbitrary nonempty sets A and B. Let f be a function on A to B and let $\mathcal{C}$ be a clone on A. If every function in $f\mathcal{C}$ satisfies an A-to-B constraint (R,S), then f satisfies $(\mathcal{C}R,S)$.*

Proof. The assumption means that $(f\mathcal{C})R \subseteq S$. By the Associativity Lemma, $(f\mathcal{C})R = f(\mathcal{C}R)$, and thus $f(\mathcal{C}R) \subseteq S$. ∎

A class $\mathcal{K} \subseteq \bigcup_{n\geq 1} B^{A^n}$ of functions on A to B is said to be *locally closed* if for every function f on A to B the following holds: if every finite restriction of f (i.e restriction to a finite subset) coincides with a finite restriction of some member of $\mathcal{K}$, then f belongs to $\mathcal{K}$.

A class $\mathcal{K} \subseteq \bigcup_{n\geq 1} B^{A^n}$ of functions on A to B is said to be *definable* (or *defined*) by a set $\mathcal{T}$ of A-to-B constraints, if $\mathcal{K}$ is the class of all those functions which satisfy every constraint in $\mathcal{T}$.

Examples. (Using the notation of the Examples in the previous section.)

(1) For each $k \geq 1$, let M_k be a $2^k \times k$ matrix whose set of rows is exactly the set of characteristic vectors of subsets of $\{1, \ldots, k\}$. Consider the following relations $R_k = \{\mathbf{x} \in \{0,1\}^{2^k} : M_k\mathbf{y} = \mathbf{x},\ \text{for some } \mathbf{y} \in \{0,1\}^k\}$ and $S_k = \{(x_1, \ldots, x_{2^k}) \in B^{2^k} : x_1 + \ldots + x_{2^k} = 0\}$. Then for each $k \geq 1$, (R_k, S_k) is a $(\mathcal{L}_0, \mathcal{L}_0)$-constraint which defines the class $\mathcal{D}^k$ of Boolean functions of degree less than k (see [4]).

(2) Consider the relations R and S on $\{0,1\}$ and $\mathbb{R}$, respectively, given by $R = \{(0,0,0,0), (1,1,1,1), (0,1,0,1), (1,0,0,1)\} \subseteq \{0,1\}^4$ and $S = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 \geq x_3 + x_4\}$. Then (R, S) is a $(\mathcal{I}, \mathcal{A}_{\mathbb{R}})$-constraint and it is not difficult to verify that (R, S) defines the class $\mathcal{S}$ of submodular functions.

Theorem 2. *Consider arbitrary nonempty sets A and B and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be clones on A and B, respectively. For any function class $\mathcal{K} \subseteq \bigcup_{n\geq 1} B^{A^n}$ the following conditions are equivalent:*

(i) *$\mathcal{K}$ is locally closed and it is stable both under right composition with $\mathcal{C}_1$ and under left composition with $\mathcal{C}_2$;*

(ii) *$\mathcal{K}$ is definable by some set of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints.*

Proof. To show that (ii)⇒ (i), assume that $\mathcal{K}$ is definable by some set $\mathcal{T}$ of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints. For every (R, S) in $\mathcal{T}$, we have $\mathcal{K}R \subseteq S$. Since R is a $\mathcal{C}_1$-invariant relation, $\mathcal{K}R = \mathcal{K}(\mathcal{C}_1R)$. By the Associativity Lemma, $\mathcal{K}(\mathcal{C}_1R) = (\mathcal{K}\mathcal{C}_1)R$, and therefore $(\mathcal{K}\mathcal{C}_1)R = \mathcal{K}R \subseteq S$. Since this is true for every (R, S) in $\mathcal{T}$ we must have $\mathcal{K}\mathcal{C}_1 \subseteq \mathcal{K}$.

For every (R, S) in $\mathcal{T}$, we have $\mathcal{K}R \subseteq S$, and therefore $\mathcal{C}_2(\mathcal{K}R) \subseteq \mathcal{C}_2 S$. By the Associativity Lemma, $(\mathcal{C}_2\mathcal{K})R \subseteq \mathcal{C}_2(\mathcal{K}R) \subseteq \mathcal{C}_2 S$, and $\mathcal{C}_2 S = S$ because S is a $\mathcal{C}_2$-invariant relation. Thus $(\mathcal{C}_2\mathcal{K})R \subseteq S$ for every (R, S) in $\mathcal{T}$, and we must have $\mathcal{C}_2\mathcal{K} \subseteq \mathcal{K}$.

To see that $\mathcal{K}$ is locally closed, consider $f \notin \mathcal{K}$, say of arity $n \geq 1$, and let (R, S) be an m-ary $(\mathcal{C}_1, \mathcal{C}_2)$-constraint that is satisfied by every function g in $\mathcal{K}$ but not satisfied by f. Hence for some $\mathbf{a}^1, \ldots, \mathbf{a}^n$ in R, $f(\mathbf{a}^1, \ldots, \mathbf{a}^n) \notin S$ but $g(\mathbf{a}^1, \ldots, \mathbf{a}^n) \in S$, for every n-ary function g in $\mathcal{K}$. Thus the restriction of f to the finite set $\{(\mathbf{a}^1(i), \ldots, \mathbf{a}^n(i)) : i \in m\}$ does not coincide with that of any member of $\mathcal{K}$.

To prove (i)$\Rightarrow$(ii), we show that for every function g not in $\mathcal{K}$, there is a $(\mathcal{C}_1, \mathcal{C}_2)$-constraint (R, S) which is satisfied by every member of $\mathcal{K}$ but not satisfied by g. The class $\mathcal{K}$ will then be definable by the set $\mathcal{T}$ of those $(\mathcal{C}_1, \mathcal{C}_2)$-constraints that are satisfied by all members of $\mathcal{K}$.

Note that $\mathcal{K}$ is a fortiori stable under right composition with the clone containing all projections, that is, $\mathcal{K}$ is closed under simple variable substitutions. We may assume that $\mathcal{K}$ is nonempty. Suppose that g is an n-ary function on A to B not in $\mathcal{K}$. Since $\mathcal{K}$ is locally closed, there is a finite restriction g_F of g to a finite subset $F \subseteq A^n$ such that g_F disagrees with every function in $\mathcal{K}$ restricted to F. Suppose that F has size m, and let $\mathbf{a}^1, \ldots, \mathbf{a}^n$ be m-tuples in A^m, such that $F = \{(\mathbf{a}^1(i), \ldots, \mathbf{a}^n(i)) : i \in m\}$. Define R_0 to be the set containing $\mathbf{a}^1, \ldots, \mathbf{a}^n$, and let $S = \{f(\mathbf{a}^1, \ldots, \mathbf{a}^n) : f \in \mathcal{K}, f$ n-ary$\}$. Clearly, (R_0, S) is not satisfied by g, and it is not difficult to see that every member of $\mathcal{K}$ satisfies (R_0, S). As $\mathcal{K}$ is stable under left composition with $\mathcal{C}_2$, it follows that S is a $\mathcal{C}_2$-invariant relation. Let R be the $\mathcal{C}_1$-invariant relation generated by R_0, i.e. $R = \mathcal{C}_1 R_0$. By Lemma 1, the constraint (R, S) constitutes indeed the desired separating $(\mathcal{C}_1, \mathcal{C}_2)$-constraint. ∎

This generalizes the characterizations of closed classes of functions given in Pippenger [9] as well as in [1] and [3] by considering arbitrary underlying sets, possible infinite, and closure conditions of a more general form. We obtain as special cases of Theorem 2 the characterizations given in Theorem 2.1 of [3] and, in the finite case, in Theorem 3.2 of [9], by considering $\mathcal{C}_1 = \mathcal{P}_A$ and $\mathcal{C}_2 = \mathcal{P}_B$, and $\mathcal{C}_1 = \mathcal{U}_A$ and $\mathcal{C}_2 = \mathcal{P}_B$, respectively, where $\mathcal{U}_A$ is a clone containing only functions on A having at most one essential variable, and $\mathcal{P}_A$ and $\mathcal{P}_B$ are the clones of all projections on A and B, respectively. Taking $A = B = \{0, 1\}$ and $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{L}_{01}$, we get the characterization of classes of Boolean functions definable by sets of affine constraints given in [1].

4. Sets of invariant constraints characterized by functions of several variables

In order to discuss sets of constraints determined by functions of several variables, we need to recall the following concepts and constructions introduced in [9] and [3].

Given maps $f : A \to B$ and $g : C \to D$, their composition $g \circ f$ is defined only if $B = C$. Removing this restriction, the *concatenation* of f and g, denoted simply gf, is defined as the map with domain $f^{-1}[B \cap C]$ and codomain D given by $(gf)(a) = g(f(a))$ for all $a \in f^{-1}[B \cap C]$. Clearly, if $B = C$ then $gf = g \circ f$, thus concatenation subsumes and extends functional composition.

Let $(g_i)_{i \in I}$ be a family of maps, $g_i : A_i \to B_i$ such that $A_i \cap A_j = \emptyset$ whenever $i \neq j$. The *(piecewise) sum of the family* $(g_i)_{i \in I}$, denoted $\Sigma_{i \in I} g_i$, is the map from $\bigcup_{i \in I} A_i$ to $\bigcup_{i \in I} B_i$ whose restriction to each A_i agrees with g_i. If I is finite, we may use the infix $+$ notation.

For $B \subseteq A$, ι_{AB} denotes the canonical injection (inclusion map) from B to A. Note that the restriction $f|_B$ of any map $f : A \to C$ to the subset B is given by the concatenation $f\iota_{AB}$.

Let $=_A$ be the equality relation on a set A. The *binary A-to-B equality constraint* is simply $(=_A, =_B)$. A constraint (R, S) is called the *empty constraint* if both antecedent and consequent are empty. For every $m \geq 1$, the constraints (A^m, B^m) are said to be *trivial.* Note that every function on A to B satisfies each of these constraints.

A constraint (R, S) is said to be a *relaxation* of a constraint (R_0, S_0) if $R \subseteq R_0$ and $S \supseteq S_0$. Given a nonempty family of constraints $(R, S_j)_{j \in J}$ of the same arity (and antecedent), the constraint $(R, \cap_{j \in J} S_j)$ is said to be obtained from $(R, S_j)_{j \in J}$ by *intersecting consequents.*

Let m and n be positive integers (viewed as ordinals, i.e., $m = \{0, \ldots, m-1\}$). Let $h : n \to m \cup V$, where V is an arbitrary set of symbols containing no ordinals called *existentially quantified indeterminate indices*, or simply *indeterminates*, and $\sigma : V \to A$ any map called a *Skolem map.* Then each m-tuple $\mathbf{a} \in A^m$, being a map $\mathbf{a} : m \to A$, gives rise to an n-tuple $(\mathbf{a} + \sigma)h \in A^n$.

Let $H = (h_j)_{j \in J}$ be a nonempty family of maps $h_j : n_j \to m \cup V$, where each n_j is a positive integer (recall $n_j = \{0, \ldots, n_j - 1\}$).

Then H is called a *minor formation scheme* with *target* m, *indeterminate set* V and *source family* $(n_j)_{j\in J}$. Let $(R_j)_{j\in J}$ be a family of relations (of various arities) on the same set A, each R_j of arity n_j, and let R be an m-ary relation on A. We say that R is a *restrictive conjunctive minor* of the family $(R_j)_{j\in J}$ *via* H, or simply a *restrictive conjunctive minor* of the family $(R_j)_{j\in J}$, if for every m-tuple $\mathbf{a}$ in A^m, the condition $R(\mathbf{a})$ implies that there is a Skolem map $\sigma : V \to A$ such that, for all j in J, we have $R_j[(\mathbf{a}+\sigma)h_j]$. On the other hand, if for every m-tuple $\mathbf{a}$ in A^m, the condition $R(\mathbf{a})$ holds whenever there is a Skolem map $\sigma : V \to A$ such that, for all j in J, we have $R_j[(\mathbf{a}+\sigma)h_j]$, then we say that R is an *extensive conjunctive minor* of the family $(R_j)_{j\in J}$ *via* H, or simply an *extensive conjunctive minor* of the family $(R_j)_{j\in J}$. If R is both a restrictive conjunctive minor and an extensive conjunctive minor of the family $(R_j)_{j\in J}$ via H, then R is said to be a *tight conjunctive minor* of the family $(R_j)_{j\in J}$ *via* H, or *tight conjunctive minor* of the family. Note that given a scheme H and a family $(R_j)_{j\in J}$, there is a unique tight conjunctive minor of the family $(R_j)_{j\in J}$ via H.

If $(R_j, S_j)_{j\in J}$ is a family of A-to-B constraints (of various arities) and (R, S) is an A-to-B constraint such that for a scheme H

(i) R is a restrictive conjunctive minor of $(R_j)_{j\in J}$ via H,

(ii) S is an extensive conjunctive minor of $(S_j)_{j\in J}$ via H,

then (R, S) is said to be a *conjunctive minor* of the family $(R_j, S_j)_{j\in J}$ *via* H, or simply a *conjunctive minor* of the family of constraints.

If both R and S are tight conjunctive minors of the respective families via H, the constraint (R, S) is said to be a *tight conjunctive minor* of the family $(R_j, S_j)_{j\in J}$ *via* H, or simply a *tight conjunctive minor* of the family of constraints. Note that given a scheme H and a family $(R_j, S_j)_{j\in J}$, there is a unique tight conjunctive minor of the family via the scheme H.

We say that a class $\mathcal{T}$ of relational constraints is *closed under formation of conjunctive minors* if whenever every member of the nonempty family $(R_j, S_j)_{j\in J}$ of constraints is in $\mathcal{T}$, all conjunctive minors of the family $(R_j, S_j)_{j\in J}$ are also in $\mathcal{T}$.

The following lemma from [3] shows that closure under formation of conjunctive minors is a necessary condition for a set of constraints to be determined by functions of several variables.

Lemma 3. *Let (R, S) be a conjunctive minor of a nonempty family $(R_j, S_j)_{j \in J}$ of A-to-B constraints. If $f : A^n \to B$ satisfies every (R_j, S_j) then f satisfies (R, S).*

A set $\mathcal{T}$ of relational constraints is said to be *locally closed* if for every A-to-B constraint (R, S) the following holds: if every relaxation of (R, S) with finite antecedent coincides with some member of $\mathcal{T}$, then (R, S) belongs to $\mathcal{T}$. The following result from [3] (see Theorem 3.2) provides necessary and sufficient conditions for a set of constraints to be determined by functions of several variables.

Theorem 4. *Consider arbitrary nonempty sets A and B. Let $\mathcal{T}$ be a set of A-to-B relational constraints. Then the following are equivalent:*

(i) *$\mathcal{T}$ is locally closed and contains the binary equality constraint, the empty constraint, and it is closed under formation of conjunctive minors;*

(ii) *There is a class $\mathcal{F}$ of functions on A to B such that the constraints satisfied by all functions in $\mathcal{F}$ are exactly the members of $\mathcal{T}$.*

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be clones on arbitrary nonempty sets A and B, respectively. Among all A-to-B constraints, observe that the empty constraint and the equality constraint are $(\mathcal{C}_1, \mathcal{C}_2)$-constraints.

The following Lemma is essentially a restatement, in a variant form, of the closure condition given by Szabó in [13] on the set of relations preserved by a clone of functions. We indicate a proof via Lemma 3 above.

Lemma 5 (Szabó)**.** *Let $\mathcal{C}$ be a clone on an arbitrary nonempty set A. If R is a tight conjunctive minor of a nonempty family $(R_j)_{j \in J}$ of $\mathcal{C}$-invariant relations, then R is a $\mathcal{C}$-invariant relation.*

Proof. Let R be a tight conjunctive minor of a nonempty family $(R_j)_{j \in J}$ of $\mathcal{C}$-invariant relations. We have to prove that every function in $\mathcal{C}$ preserves R or, equivalently, that every function in $\mathcal{C}$ satisfies the A-to-A constraint (R, R). Since $(R_j)_{j \in J}$ is a nonempty family of $\mathcal{C}$-invariant relations, every function in $\mathcal{C}$ preserves every member of the family $(R_j)_{j \in J}$, that is, every function in $\mathcal{C}$ satisfies every member of the family $(R_j, R_j)_{j \in J}$ of A-to-A constraints. From Lemma 3 above, it follows that every member of $\mathcal{C}$ satisfies (R, R), that is, R is a $\mathcal{C}$-invariant relation. ∎

Thus every tight conjunctive minor (R,S) of a nonempty family $(R_j,S_j)_{j\in J}$ of $(\mathcal{C}_1,\mathcal{C}_2)$-constraints is a $(\mathcal{C}_1,\mathcal{C}_2)$-constraint. However, not all relaxations of $(\mathcal{C}_1,\mathcal{C}_2)$-constraints are $(\mathcal{C}_1,\mathcal{C}_2)$-constraints and so not all conjunctive minors of a nonempty family $(R_j,S_j)_{j\in J}$ of $(\mathcal{C}_1,\mathcal{C}_2)$-constraints are $(\mathcal{C}_1,\mathcal{C}_2)$-constraints. A relaxation (R,S) of an A-to-B constraint (R_0,S_0) is called a $(\mathcal{C}_1,\mathcal{C}_2)$-*relaxation* of (R_0,S_0) if (R,S) is a $(\mathcal{C}_1,\mathcal{C}_2)$-constraint. Similarly, a conjunctive minor (R,S) of a nonempty family $(R_j,S_j)_{j\in J}$ of A-to-B constraints is called a $(\mathcal{C}_1,\mathcal{C}_2)$-*conjunctive minor* of the family $(R_j,S_j)_{j\in J}$, if (R,S) is a $(\mathcal{C}_1,\mathcal{C}_2)$-constraint.

A set $\mathcal{T}$ of $(\mathcal{C}_1,\mathcal{C}_2)$-constraints is said to be *closed under formation of* $(\mathcal{C}_1,\mathcal{C}_2)$-*conjunctive minors* if whenever every member of the nonempty family $(R_j,S_j)_{j\in J}$ of constraints is in $\mathcal{T}$, all $(\mathcal{C}_1,\mathcal{C}_2)$-conjunctive minors of the family $(R_j,S_j)_{j\in J}$ are also in $\mathcal{T}$. The following result extends Lemma 1 in [2].

Lemma 6. *Let* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *be clones on arbitrary nonempty sets* A *and* B*, respectively. Let* $\mathcal{T}_0$ *be a set of* $(\mathcal{C}_1,\mathcal{C}_2)$*-constraints, closed under* $(\mathcal{C}_1,\mathcal{C}_2)$*-relaxations. Define* $\mathcal{T}$ *to be the set of all relaxations of the various constraints in* $\mathcal{T}_0$*. Then* $\mathcal{T}_0$ *is the set of* $(\mathcal{C}_1,\mathcal{C}_2)$*-constraints which are in* $\mathcal{T}$*, and the following are equivalent:*

(a) $\mathcal{T}_0$ *is closed under formation of* $(\mathcal{C}_1,\mathcal{C}_2)$*-conjunctive minors;*

(b) $\mathcal{T}$ *is closed under taking conjunctive minors.*

Proof. Clearly, the first claim holds, and it is easy to see that (b)$\Rightarrow$ (a). To prove implication (a)$\Rightarrow$(b), assume (a). Let (R,S) be a conjunctive minor of a nonempty family $(R_j,S_j)_{j\in J}$ of A-to-B constraints in $\mathcal{T}$ via a scheme $H=(h_j)_{j\in J}$, $h_j : n_j \to m\cup V$. We have to prove that $(R,S)\in\mathcal{T}$.

Since for every $j\in J$, $(R_j,S_j)\in\mathcal{T}$, there is a nonempty family $(R_j^0,S_j^0)_{j\in J}$ of $(\mathcal{C}_1,\mathcal{C}_2)$-constraints in $\mathcal{T}_0$ such that, for each j in J, (R_j,S_j) is a relaxation of (R_j^0,S_j^0). So let (R_0,S_0) be the tight conjunctive minor of the family $(R_j^0,S_j^0))_{j\in J}$ via the scheme H. From Lemma 5, it follows that R_0 is a $\mathcal{C}_1$-invariant relation and S_0 a $\mathcal{C}_2$-invariant relation, and since $\mathcal{T}_0$ is closed under formation of $(\mathcal{C}_1,\mathcal{C}_2)$-conjunctive minors, we have $(R_0,S_0)\in\mathcal{T}_0$.

Let us prove that (R,S) is a relaxation of (R_0,S_0) and, thus, that $(R,S)\in\mathcal{T}$. Since R is a restrictive conjunctive minor of the family $(R_j)_{j\in J}$ via the scheme $H=(h_j)_{j\in J}$, we have that for every m-tuple $\mathbf{a}$ in R there is a Skolem map $\sigma : V\to A$ such that, for all j in J, the n_j-tuple $(\mathbf{a}+\sigma)h_j$ is in R_j. Since $R_j \subseteq R_j^0$ for every j in J, it follows that $(\mathbf{a} + \sigma)h_j$ is in R_j^0 for every j in J. Thus $\mathbf{a}$ is in R_0 and we conclude $R \subseteq R_0$.

By analogous reasoning one can easily verify that $\mathbf{b}$ is in S whenever $\mathbf{b}$ is in S_0, i.e that $S \supseteq S_0$. Thus (R, S) is a relaxation of (R_0, S_0) and so $(R, S) \in \mathcal{T}$, and the proof of *(a)* is complete. ∎

Let $\mathcal{T}_0$ be a set of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints. We say that $\mathcal{T}_0$ is $(\mathcal{C}_1, \mathcal{C}_2)$-*locally closed* if the set $\mathcal{T}$ of all relaxations of the various constraints in $\mathcal{T}_0$ is locally closed.

We can now extend Theorem 4 above to sets of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints.

Theorem 7. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be clones on arbitrary nonempty sets A and B, respectively, and let $\mathcal{T}_0$ be a set of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints. Then the following are equivalent:*

(i) *$\mathcal{T}_0$ is $(\mathcal{C}_1, \mathcal{C}_2)$-locally closed, contains the binary equality constraint, the empty constraint, and it is closed under formation of $(\mathcal{C}_1, \mathcal{C}_2)$-conjunctive minors;*

(ii) *There is a class $\mathcal{F}$ of functions on A to B such that the $(\mathcal{C}_1, \mathcal{C}_2)$-constraints satisfied by all functions in $\mathcal{F}$ are exactly the members of $\mathcal{T}_0$.*

Proof. To prove implication (ii) ⇒(i), assume (ii). Let $\mathcal{K}$ be the set of all functions satisfying every constraint in $\mathcal{T}_0$. Note that $\mathcal{T}_0$ is closed under $(\mathcal{C}_1, \mathcal{C}_2)$-relaxations. By Theorem 2, we have $\mathcal{C}_2\mathcal{K} = \mathcal{K}$, and $\mathcal{K}\mathcal{C}_1 = \mathcal{K}$. We may assume that $\mathcal{K} \neq \emptyset$. Let $\mathcal{T}$ be the set of all those constraints (not necessarily $(\mathcal{C}_1, \mathcal{C}_2)$-constraints) satisfied by every function in $\mathcal{K}$. Observe that $\mathcal{T}_0$ is the set of all $(\mathcal{C}_1, \mathcal{C}_2)$-constraints which are in $\mathcal{T}$. We show that $\mathcal{T}$ is the set of all relaxations of members of $\mathcal{T}_0$.

Let (R, S) be a constraint in $\mathcal{T}$. From the definition of $\mathcal{T}$, it follows that $\mathcal{K}R \subseteq S$. Note that $\mathcal{K}$ is stable under right composition with the clone of projections on A, because $\mathcal{K}\mathcal{C}_1 = \mathcal{K}$. Thus by the Associativity Lemma it follows that $\mathcal{C}_2(\mathcal{K}R) = (\mathcal{C}_2\mathcal{K})R$. Since $\mathcal{C}_2\mathcal{K} = \mathcal{K}$, we have that $\mathcal{C}_2(\mathcal{K}R) = \mathcal{K}R$, i.e. $\mathcal{K}R$ is a $\mathcal{C}_2$-invariant relation. Also, again because $\mathcal{K}\mathcal{C}_1 = \mathcal{K}$, by Lemma 1 we conclude that every function in $\mathcal{K}$ satisfies $(\mathcal{C}_1 R, \mathcal{K}R)$. Clearly, $(\mathcal{C}_1 R, \mathcal{K}R)$ is a $(\mathcal{C}_1, \mathcal{C}_2)$-constraint, therefore it belongs to $\mathcal{T}_0$. Thus every constraint (R, S) in $\mathcal{T}$ is a relaxation of a member of $\mathcal{T}_0$, namely, a relaxation of $(\mathcal{C}_1 R, \mathcal{K}R)$.

By Theorem 4 above, we have that $\mathcal{T}$ is locally closed and contains the binary equality constraint, the empty constraint, and it is closed under formation of conjunctive minors. Since the binary equality constraint and the empty constraint are $(\mathcal{C}_1, \mathcal{C}_2)$-constraints, it follows from Lemma 6 that (i) holds.

To prove implication (i) $\Rightarrow$(ii), it is enough to show that for every $(\mathcal{C}_1, \mathcal{C}_2)$-constraint (R, S) not in $\mathcal{T}_0$, there is a function g which satisfies every constraint in $\mathcal{T}_0$, but does not satisfy (R, S).

Let $\mathcal{T}$ be the set of relaxations of the various $(\mathcal{C}_1, \mathcal{C}_2)$-constraints in $\mathcal{T}_0$. Observe that $(R, S) \notin \mathcal{T}$, otherwise (R, S) would be a $(\mathcal{C}_1, \mathcal{C}_2)$-relaxation of some $(\mathcal{C}_1, \mathcal{C}_2)$-constraint in $\mathcal{T}_0$, contradicting the fact implied by (i) that $\mathcal{T}_0$ is closed under taking $(\mathcal{C}_1, \mathcal{C}_2)$-relaxations. Clearly, $\mathcal{T}$ is locally closed, contains the binary equality constraint, and the empty constraint. From Lemma 6, it follows that $\mathcal{T}$ is closed under taking conjunctive minors. By Theorem 4, there is a function g which does not satisfy (R, S) but satisfies every constraint in $\mathcal{T}$ and so, in particular, g satisfies every constraint in $\mathcal{T}_0$. Thus we have (i)$\Rightarrow$ (ii). ∎

Theorem 7 generalizes the characterizations of closed classes of constraints given in Pippenger [9] and also in [2] as well as [3] by considering both arbitrary, possibly infinite, underlying sets, and closure conditions of a more general form on classes of relational constraints.

Theorems 2 and 7 may also be viewed as analogues, with constraints instead of relations, of the characterization given by Pöschel, as part of Theorem 3.2 in [12], of the closed sets in a class of Galois connections between operations and relations of a prescribed type on a set A.

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