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SPECIAL M-HYPERIDENTITIES IN BIREGULAR LEFTMOST GRAPH VARIETIES OF TYPE (2,0)

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Abstract

Graph algebras establish a connection between directed graphs without multiple edges and special universal algebras of type (2,0). We say that a graph G satisfies a term equation $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$. A class of graph algebras V is called a graph variety if $V = Mod_g\Sigma$ where Σ is a subset of $T(X) \times T(X)$. A graph variety $V' = Mod_g\Sigma^{'}$ is called a biregular leftmost graph variety if $\Sigma^{'}$ is a set of biregular leftmost term equations. A term equation $s \approx t$ is called an identity in a variety V if $\underline{A(G)}$ satisfies $s \approx t$ for all $G \in V$. An identity $s \approx t$ of a variety V is called a hyperidentity of a graph algebra $\underline{A(G)}$, $G \in V$ whenever the operation symbols occuring in s and t are replaced by any term operations of $\underline{A(G)}$ of the appropriate arity, the resulting identities hold in $\underline{A(G)}$. An identity $s \approx t$ of a variety V is called an M-hyperidentity of a graph algebra $\underline{A(G)}$, $G \in V$ whenever the operation symbols occuring in s and t are replaced by any term operations in a subgroupoid M of term operations of $\underline{A(G)}$ of the appropriate arity, the resulting identities hold in $\underline{A(G)}$.

In this paper we characterize special M-hyperidentities in each biregular leftmost graph variety. For identities, varieties and other basic concepts of universal algebra see e.g. [3].

Keywords: varieties, biregular leftmost graph varieties, identities, term, hyperidentity, M-hyperidentity, binary algebra, graph algebra.

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1. Introduction

An identity $s \approx t$ of terms s, t of any type τ is called a *hyperidentity* (*M-hyperidentity*) of an algebra $\underline{A}$ if whenever the operation symbols occurring in s and t are replaced by any term operations (any term operations in a subgroupoid M of term operations) of $\underline{A}$ of the appropriate arity, the resulting identity holds in $\underline{A}$. Hyperidentities can be defined more precisely by using the concept of a hypersubstitution, which was introduced by K. Denecke, D. Lau, R. Pöschel and D. Schweigert in [5].

We fix a type $\tau = (n_i)_{i \in I},\ n_i > 0$ for all $i \in I$, and operation symbols $(f_i)_{i \in I}$, where f_i is $n_i - ary$. Let $W_\tau(X)$ be the set of all terms of type τ over some fixed alphabet X, and let $Alg(\tau)$ be the class of all algebras of type τ. Then a mapping

$$\sigma : \{f_i | i \in I\} \longrightarrow W_\tau(X)$$

which assigns to every $n_i - ary$ operation symbol f_i an $n_i - ary$ term will be called a *hypersubstitution* of type τ (for short, a hypersubstitution). By $\hat{\sigma}$ we denote the extension of the hypersubstitution σ to a mapping

$$\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X).$$

The term $\hat{\sigma}[t]$ is defined inductively by

(i) $\hat{\sigma}[x] = x$ for any variable x in the alphabet X, and

(ii) $\hat{\sigma}[f_i(t_1, ..., t_{n_i})] = \sigma(f_i)^{W_\tau(X)}(\hat{\sigma}[t_1], ..., \hat{\sigma}[t_{n_i}])$.

Here $\sigma(f_i)^{W_\tau(X)}$ on the right hand side of (ii) is the operation induced by $\sigma(f_i)$ on the term algebra with the universe $W_\tau(X)$.

Graph algebras have been invented in [16] to obtain examples of non-finitely based finite algebras. To recall this concept, let $G = (V, E)$ be a (directed) graph with the vertex set V and the set of edges $E \subseteq V \times V$. Define the *graph algebra* $\underline{A(G)}$ corresponding to G with the underlying set $V \cup \{\infty\}$, where ∞ is a symbol outside V, and with two basic operations, namely a nullary operation pointing to ∞ and a binary one denoted by juxtaposition, given for $u, v \in V \cup \{\infty\}$ by

$$uv = \begin{cases} u, & \text{if} \quad (u, v) \in E, \\ \infty, & \text{otherwise.} \end{cases}$$

In [15] graph varieties had been investigated for finite undirected graphs in order to get graph theoretic results (structure theorems) from universal algebra via graph algebras. In [14] these investigations are extended to arbitrary (finite) directed graphs where the authors ask for a graph theoretic characterization of graph varieties, i.e., of classes of graphs which can be defined by identities for their corresponding graph algebras. The answer is a theorem of **Birkhoff-type**, which uses graph theoretic closure operations. *A class of finite directed graphs is equational* (i.e., a graph variety) *if and only if it is closed with respect to finite restricted pointed subproducts and isomorphic copies.*

In [6] M. Kapeedaeng and T. Poomsa-ard characterized all biregular leftmost graph varieties. In [1] Apinant Ananpinitwatna and Tiang Poomsa-ard characterized identities in all biregular leftmost graph varieties. In [7] J. Khampakdee and T. Poomsa-ard characterized hyperidentities in the class of $x(yx) \approx x(yy)$ graph algebras. In [10] T. Poomsa-ard characterized hyperidentities in the class of associative graph algebras. In [11, 12] T. Poomsa-ard, J. Wetweerapong and C. Samartkoon characterized hyperidentitis in the class of idempotent graph algebras and the class of transitive graph algebras respectively. In [2] Amporn Ananpinitwatna and Tiang Poomsa-ard characterized hyperidentities in all biregular leftmost graph varieties.

In this paper we characterize special M-hyperidentities in each biregular leftmost graph variety.

2. Terms, identities and graph varieties

Dealing with terms for graph algebras, the underlying formal language has to contain a binary operation symbol (juxtaposition) and a symbol for the constant ∞ (denoted by ∞, too).

Definition 2.1. The set $T(X)$ of all terms over the alphabet

$$X = \{x_1, x_2, x_3, ...\}$$

is defined inductively as follows:

(i) every variable $x_i, i = 1, 2, 3, ...$, and ∞ are terms;

(ii) if t_1 and t_2 are terms, then $t_1 t_2$ is a term;

(iii) $T(X)$ is the set of all terms which can be obtained from (i) and (ii) in finitely many steps.

Terms built up from the two-element set $X_2 = \{x_1, x_2\}$ of variables are thus binary terms. We denote the set of all binary terms by $T(X_2)$. The leftmost variable of a term t is denoted by $L(t)$ and rightmost variable of a term t is denoted by $R(t)$. A term, in which the symbol ∞ occurs is called a *trivial term.*

Definition 2.2. For each non-trivial term t of type $\tau = (2, 0)$ one can define a directed graph $G(t) = (V(t), E(t))$, where the vertex set $V(t)$ is the set of all variables occurring in t and the edge set $E(t)$ is defined inductively by

$$E(t) = \phi \text{ if } t \text{ is a variable and } E(t_1t_2) = E(t_1) \cup E(t_2) \cup \{(L(t_1), L(t_2))\}$$

where $t = t_1t_2$ is a compound term.

$L(t)$ is called the *root* of the graph $G(t)$, and the pair $(G(t), L(t))$ is the *rooted graph* corresponding to t. Formally, we assign the empty graph ϕ to every trivial term t.

Definition 2.3. We say that a graph $G = (V, E)$ *satisfies an identity* $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$ (i.e. we have $s = t$ for every assignment $V(s) \cup V(t) \rightarrow \overline{V} \cup \{\infty\}$), and in this case, we write $G \models s \approx t$. Given a class $\mathcal{G}$ of graphs and a set Σ of identities (i.e., $\Sigma \subset T(X) \times T(X)$) we introduce the following notation:

$G \models \Sigma$ if $G \models s \approx t$ for all $s \approx t \in \Sigma$,

$\mathcal{G} \models s \approx t$ if $G \models s \approx t$ for all $G \in \mathcal{G}$,

$\mathcal{G} \models \Sigma$ if $G \models \Sigma$ for all $G \in \mathcal{G}$,

$Id\mathcal{G} = \{s \approx t \mid s, t \in T(X),\ \mathcal{G} \models s \approx t,$

$Mod_g\Sigma = \{G \mid G \text{ is a graph and } G \models \Sigma\}, \qquad \mathcal{V}_g(\mathcal{G}) = Mod_gId\mathcal{G}.$

$\mathcal{V}_g(\mathcal{G})$ is called the *graph variety generated by* $\mathcal{G}$ and $\mathcal{G}$ is called *graph variety* if $\mathcal{V}_g(\mathcal{G}) = \mathcal{G}$. $\mathcal{G}$ is called *equational* if there exists a set Σ' of identities such that $\mathcal{G} = \mathrm{Mod}_g\Sigma'$. Obviously $\mathcal{V}_g(\mathcal{G}) = \mathcal{G}$ if and only if $\mathcal{G}$ is an equational class.

Let $var(t)$ be the set of all variables occurring in a term t. An equation $s \approx t$ is said to be *regular* if $var(s) = var(t)$, *leftmost* if $L(s) = L(t)$ and *biregular leftmost* if it is *regular, leftmost* and $|var(s)| = 2$. A graph variety $\mathcal{V} = Mod_g\Sigma''$ is said to be *biregular leftmost graph variety* if Σ'' is a set of biregular leftmost term equation.

3. Biregular leftmost graph varieties and identities

In [6] M. Kapeedaeng and T. Poomsa-ard characterized biregular leftmost graph varieties and found that $\mathcal{BRL} = \{\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_3, ..., \mathcal{K}_{28}\}$, where

$\mathcal{K}_0 = Mod\{xy \approx xy\}$, $\mathcal{K}_1 = Mod\{xy \approx x(yy)\}$,

$\mathcal{K}_2 = Mod\{xy \approx (xx)y\}$, $\mathcal{K}_3 = Mod\{xy \approx (xx)(yy)\}$,

$\mathcal{K}_4 = Mod\{xy \approx x(yx)\}$, $\mathcal{K}_5 = Mod\{xy \approx (xx)(yx)\}$,

$\mathcal{K}_6 = Mod\{x(yy) \approx (xx)y\}$, $\mathcal{K}_7 = Mod\{x(yy) \approx (xx)(yy)\}$,

$\mathcal{K}_8 = Mod\{x(yy) \approx x(yx)\}$, $\mathcal{K}_9 = Mod\{x(yy) \approx (xx)(yx)\}$,

$\mathcal{K}_{10} = Mod\{x(yy) \approx x((yy)x)\}$, $\mathcal{K}_{11} = Mod\{(xx)y \approx (xx)(yy)\}$,

$\mathcal{K}_{12} = Mod\{(xx)y \approx x(yx)\}$, $\mathcal{K}_{13} = Mod\{(xx)y \approx (xx)(yx)\}$,

$\mathcal{K}_{14} = Mod\{(xx)y \approx x((yy)x)\}$, $\mathcal{K}_{15} = Mod\{(xx)(yy) \approx x(yx)\}$,

$\mathcal{K}_{16} = Mod\{(xx)(yy) \approx (xx)(yx)\}$, $\mathcal{K}_{17} = Mod\{(xx)(yy) \approx (xx)((yy)x)\}$

$\mathcal{K}_{18} = Mod\{x(yx) \approx (xx)(yx)\}$, $\mathcal{K}_{19} = Mod\{(xx)(yx) \approx x((yy)x)\}$,

$\mathcal{K}_{20} = Mod_g\{xy \approx x(yy),\ (xx)y \approx x(yx)\} = \mathcal{K}_1 \cap \mathcal{K}_{12}$,

$\mathcal{K}_{21} = Mod_g\{xy \approx (xx)y,\ x(yy) \approx x(yx)\} = \mathcal{K}_2 \cap \mathcal{K}_8$,

$\mathcal{K}_{22} = Mod_g\{xy \approx x(yx),\ (xx)y \approx x(yy)\} = \mathcal{K}_4 \cap \mathcal{K}_6$,

$\mathcal{K}_{23} = Mod_g\{x(yy) \approx (xx)y,\ x(yy) \approx x(yx)\} = \mathcal{K}_6 \cap \mathcal{K}_8$,

$\mathcal{K}_{24} = Mod_g\{x(yy) \approx (xx)y,\ x(yy) \approx (xx)(yx)\} = \mathcal{K}_6 \cap \mathcal{K}_9$,

$\mathcal{K}_{25} = Mod_g\{x(yy) \approx (xx)y,\ x(yx) \approx (xx)(yx)\} = \mathcal{K}_6 \cap \mathcal{K}_{18}$,

$\mathcal{K}_{26} = Mod_g\{x(yy) \approx (xx)(yy),\ x(yx) \approx (xx)(yx)\} = \mathcal{K}_7 \cap \mathcal{K}_{18}$,

$\mathcal{K}_{27} = Mod_g\{x(yy) \approx x((yy)x),\ (xx)y \approx (xx)(yx)\} = \mathcal{K}_{10} \cap \mathcal{K}_{13}$,

$\mathcal{K}_{28} = Mod_g\{(xx)y \approx (xx)(yy),\ x(yx) \approx (xx)(yx)\} = \mathcal{K}_{11} \cap \mathcal{K}_{18}$ is the set of all biregular leftmost graph varieties.

In [1] Apinant Ananpinitwatna and Tiang Poomsa-ard characterized identities in all biregular leftmost graph varieties. There are twentynine biregular leftmost graph varieties and we want to skip the proof about M-hyperidentity in some biregular leftmost graph variety.

So we will quote the theorems about identity in biregular leftmost graph variety only which will be needed as references. They are summarized in following table:

Table 1. The property of biregular graph varieties of terms s and t.

Variety	Property of s and t
$\mathcal{K}_1$	(i) $L(s) = L(t)$, $V(s) = V(t)$ (ii) for any $x \in V(s)$, there exists $y \in V(s)$ such that $(y, x) \in E(s)$ iff there exists $z \in V(t)$ such that $(z, x) \in E(t)$, (iii) for any $x, y \in V(s)$ with $x \neq y$, $(x, y) \in E(s)$ iff $(x, y) \in E(t)$.
$\mathcal{K}_2$	(i) $L(s) = L(t)$, $V(s) = V(t)$ (ii) for any $x \in V(s)$, there exists $y \in V(s)$ such that $(x, y) \in E(s)$ iff there exists $z \in V(t)$ such that $(x, z) \in E(t)$, (iii) for any $x, y \in V(s)$ with $x \neq y$, $(x, y) \in E(s)$ iff $(x, y) \in E(t)$.
$\mathcal{K}_4$	(i) $L(s) = L(t)$, $V(s) = V(t)$ (ii) for any $x \in V(s)$, $(x, x) \in E(s)$ iff $(x, x) \in E(t)$, (iii) for any $x, y \in V(s)$ with $x \neq y$, $(x, y) \in E(s)$ or $(y, x) \in E(s)$ iff $(x, y) \in E(t)$ or $(y, x) \in E(t)$.
$\mathcal{K}_9$	(i) $L(s) = L(t)$, $V(s) = V(t)$ (ii) for any $x \in V(s)$, there exists $y \in V_x(s)$ such that $(y, y) \in E(s)$ iff there exists $z \in V_x(t)$ such that $(z, z) \in E(t)$, (iii) for any $x, y \in V(s)$ with $x \neq y$, $(x, y) \in E(s)$ or $(y, x) \in E(s)$, and there exists $z \in V_x(s)$ such that $(z, z) \in E(s)$ iff $(x, y) \in E(t)$ or $(y, x) \in E(t)$, and there exists $z' \in V_x(t)$ such that $(z', z') \in E(t)$. Where for any non-trivial term t' and $x \in V(t)$, $V_x(t') = \{x' \in V(t') \mid \text{there exists a dipath from } x \text{ to } x' \text{ in } G(t')\}$.
$\mathcal{K}_{13}$	(i) $L(s) = L(t)$, $V(s) = V(t)$ (iii) for any $x \in V(s)$, $(x, x) \in E(s)$ iff $(x, x) \in E(t)$, (iii) for any $x, y \in V(s)$ with $x \neq y$, $(x, y) \in E(s)$ or $(y, x), (y, y) \in E(s)$ iff $(x, y) \in E(t)$ or $(y, x), (y, y) \in E(t)$.
$\mathcal{K}_{27}$	(i) $L(s) = L(t)$, $V(s) = V(t)$ (ii) for any $x \in V(s)$, $(x, x) \in E(s)$ iff $(x, x) \in E(t)$, (iii) for any $x, y \in V(s)$ with $x \neq y$, $(x, y) \in E(s)$ or $(y, x), (x, x) \in E(s)$ or $(y, x), (y, y) \in E(s)$ iff $(x, y) \in E(t)$ or $(y, x), (x, x) \in E(t)$ or $(y, x), (y, y) \in E(t)$.

4. Hypersubstitution and proper hypersubstitution

Let $\mathcal{K}$ be a graph variety. Now we want to formulate precisely the concept of a graph hypersubstitution for graph algebras.

Definition 4.1. A mapping $\sigma : \{f, \infty\} \to T(X_2)$, where $X_2 = \{x_1, x_2\}$ and f is the operation symbol corresponding to the binary operation of a graph algebra is called *graph hypersubstitution* if $\sigma(\infty) = \infty$ and $\sigma(f) = s \in T(X_2)$. The graph hypersubstitution with $\sigma(f) = s$ is denoted by σ_s.

Definition 4.2. An identity $s \approx t$ is a $\mathcal{K}$ *graph hyperidentity* iff for all graph hypersubstitutions σ, the equations $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are identities in $\mathcal{K}$.

If we want to check that an identity $s \approx t$ is a hyperidentity in $\mathcal{K}$ we can restrict our consideration to a (small) subset of $Hyp\mathcal{G}$ - the set of all graph hypersubstitutions.

In [8] the following relation between hypersubstitutions was defined:

Definition 4.3. Two graph hypersubstitutions σ_1, σ_2 are called $\mathcal{K}$-*equivalent* iff $\sigma_1(f) \approx \sigma_2(f)$ is an idetity in $\mathcal{K}$. In this case we write $\sigma_1 \sim_{\mathcal{K}} \sigma_2$.

The following lemma was proved in [9].

Lemma 4.1. *If $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in Id\mathcal{K}$ and $\sigma_1 \sim_{\mathcal{K}} \sigma_2$ then, $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{K}$.*

Therefore, it is enough to consider the quotient set $Hyp\mathcal{G}/\sim_{\mathcal{K}}$.

In [13] it was shown that any non-trivial term t over the class of graph algebras has a uniquely determined normal form term $NF(t)$ and there is an algorithm to construct the normal form term to a given term t. Now, we want to describe how to construct the normal form term. Let t be a non-trivial term. The *normal form term* of t is the term $NF(t)$ constructed by the following algorithm:

(i) Construct $G(t) = (V(t), E(t))$.

(ii) Construct for every $x \in V(t)$ the list $l_x = (x_{i_1}, ..., x_{i_{k(x)}})$ of all out-neighbors (i.e. $(x, x_{i_j}) \in E(t), 1 \leq j \leq k(x)$) ordered by increasing indices $i_1 \leq ... \leq i_{k(x)}$ and let s_x be the term $(...((x x_{i_1}) x_{i_2}) ... x_{i_{k(x)}})$.

(iii) Starting with $x := L(t), Z := V(t), s := L(t)$, choose the variable $x_i \in Z \cap V(s)$ with the least index i, substitute the first occurrence of x_i by the term s_{x_i}, denote the resulting term again by s and put $Z := Z \setminus \{x_i\}$. While $Z \neq \phi$ continue this procedure. The resulting term is the normal form $NF(t)$.

The algorithm stops after a finite number of steps, since $G(t)$ is a rooted graph. Without difficulties one shows $G(NF(t)) = G(t), L(NF(t)) = L(t)$.

The following definition was given in [4].

Definition 4.4. The graph hypersubstitution $\sigma_{NF(t)}$, is called *normal form graph hypersubstitution.* Here $NF(t)$ is the normal form of the binary term t.

Since for any binary term t the rooted graphs of t and $NF(t)$ are the same, we have $t \approx NF(t) \in Id\mathcal{K}$. Then for any graph hypersubstitution σ_t with $\sigma_t(f) = t \in T(X_2)$, one obtains $\sigma_t \sim_{\mathcal{K}} \sigma_{NF(t)}$.

In [4] all rooted graphs with at most two vertices were considered. Then we formed the corresponding binary terms and used the algorithm to construct normal form terms. The result is given in the Table 2.

Table 2. Normal form terms.

normal form term	graph hypers	normal form term	graph hypers
x_1x_2	σ_0	x_1	σ_1
x_2	σ_2	x_1x_1	σ_3
x_2x_2	σ_4	x_2x_1	σ_5
$(x_1x_1)x_2$	σ_6	$(x_2x_1)x_2$	σ_7
$x_1(x_2x_2)$	σ_8	$x_2(x_1x_1)$	σ_9
$(x_1x_1)(x_2x_2)$	σ_{10}	$(x_2(x_1x_1))x_2$	σ_{11}
$x_1(x_2x_1)$	σ_{12}	$x_2(x_1x_2)$	σ_{13}
$(x_1x_1)(x_2x_1)$	σ_{14}	$(x_2(x_1x_2))x_2$	σ_{15}
$x_1((x_2x_1)x_2)$	σ_{16}	$x_2((x_1x_1)x_2)$	σ_{17}
$(x_1x_1)((x_2x_1)x_2)$	σ_{18}	$(x_2((x_1x_1)x_2))x_2$	σ_{19}

Let $M_{\mathcal{G}}$ be the set of all normal form graph hypersubstitutions. Then we get,

$M_{\mathcal{G}} =$
$\{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$.

The concept of a proper hypersubstitution of a class of algebras was introduced in [9].

Definition 4.5. A hypersubstitution σ is called *proper with respect to a class* $\mathcal{K}$ *of algebras* if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}$ for all $s \approx t \in Id\mathcal{K}$.

The following lemma was proved in [4].

Lemma 4.2. *For each non-trivial term* $s, (s \neq x \in X)$ *and for all* $u, v \in X$, *we have*

$$E(\hat{\sigma}_6[s]) = E(s) \cup \{(u,u) | (u,v) \in E(s)\},$$

$$E(\hat{\sigma}_8[s]) = E(s) \cup \{(v,v) | (u,v) \in E(s)\},$$

and

$$E(\hat{\sigma}_{12}[s]) = E(s) \cup \{(v,u) | (u,v) \in E(s)\}.$$

By the similar way we prove that,

$$E(\hat{\sigma}_{10}[s]) = E(s) \cup \{(u,u),(v,v) | (u,v) \in E(s)\}.$$

For any non-trivial term t the dual term t^d is defined in the following way:

If $t = x \in X$, then $x^d = x$, if $t = t_1 t_2$, then $t^d = {t_2}^d {t_1}^d$. The dual term t^d can be obtained by application of the graph hypersubstitution $\sigma_5, \hat{\sigma}_5[t] = t^d$. If $s \approx t$ is a graph identity, then $s^d \approx t^d$ is called dual identity and the rooted graph corresponding to t^d is dual to $(G(t), L(t))$.

Let $PM_{\mathcal{K}}$ be the set of all proper graph hypersubstitutions with respect to the class $\mathcal{K}$. In [2] it was found out that,

$PM_{\mathcal{K}_1} = \{\sigma_0, \sigma_6, \sigma_{12}\}$ $\qquad$ $PM_{\mathcal{K}_2} = \{\sigma_0, \sigma_8, \sigma_{12}\}$

$PM_{\mathcal{K}_4} = \{\sigma_0, \sigma_{10}\}$ $\qquad$ $PM_{\mathcal{K}_9} = \{\sigma_0, \sigma_6, \sigma_8, \sigma_{12}\}$

$PM_{\mathcal{K}_{13}} = \{\sigma_0, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{16}\}$ $\qquad$ $PM_{\mathcal{K}_{27}} = \{\sigma_0, \sigma_{10}, \sigma_{12}\}$.

5. Special M-hyperidentities

We know that a graph identity $s \approx t$ is a graph hyperidentity, if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is a graph identity for all $\sigma \in M_{\mathcal{G}}$. Let M be a subgroupoid of $M_{\mathcal{G}}$. Then, a graph identity $s \approx t$ is an M-graph hyperidentity (M-hyperidentity), if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is a graph identity for all $\sigma \in M$. In [3] K. Denecke and S.L. Wismath defined special subgroupoid of M_g as the following.

Definition 5.1.

(i) A hypersubstitution $\sigma \in Hyp(\tau)$ is said to be leftmost if for every $i \in I$, the first variable in $\hat{\sigma}[f_i(x_1, ..., x_{n_i})]$ is x_1. Let $Left(\tau)$ be the set of all leftmost hypersubstitutions of type τ.

(ii) A hypersubstitution $\sigma \in Hyp(\tau)$ is said to be outermost if for every $i \in I$, the first variable in $\hat{\sigma}[f_i(x_1, ..., x_{n_i})]$ is x_1 and the last variable is x_{n_i}. Let $Out(\tau)$ be the set of all outermost hypersubstitutions of type τ.

(iii) A hypersubstitution $\sigma \in Hyp(\tau)$ is said to be rightmost if for every $i \in I$, the last variable in $\hat{\sigma}[f_i(x_1, ..., x_{n_i})]$ is x_{n_i}. Let $Right(\tau)$ be the set of all rightmost hypersubstitutions of type τ. Note that $Out(\tau) = Right(\tau) \cap Left(\tau)$.

(iv) A hypersubstitution $\sigma \in Hyp(\tau)$ is called regular if for every $i \in I$, each of the variables $x_1, ..., x_{n_i}$ occurs in $\hat{\sigma}[f_i(x_1, ..., x_{n_i})]$. Let $Reg(\tau)$ be the set of all regular hypersubstitutions of type τ.

(v) A hypersubstitution $\sigma \in Hyp(\tau)$ is called symmetrical if for every $i \in I$, there is a permutation s_i on the set $\{1, ..., n_i\}$ such that $\hat{\sigma}[f_i(x_1, ..., x_{n_i})] = f_i(x_{s_i(1)}, ..., x_{s_i(n_i)})$. Let $D(\tau)$ be the set of all symmetrical hypersubstitutions of type τ.

(vi) We will call a hypersubstitution σ of type τ a pre-hypersubstitution if for every $i \in I$, the term $\sigma(f_i)$ is not a variable. Let $Pre(\tau)$ be the set of all pre-hypersubstitutions of type τ.

From Definition 5.1, we have:

$M_{Left} = \{\sigma_0, \sigma_1, \sigma_3, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{14}, \sigma_{16}, \sigma_{18}\}$.

$M_{Right} = \{\sigma_0, \sigma_2, \sigma_4, \sigma_6, \sigma_7, \sigma_8, \sigma_{10}, \sigma_{11}, \sigma_{13}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$.

$M_{Out} = \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{16}, \sigma_{18}\}$.

$M_{Reg} = \{\sigma_0, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$.

$M_D = \{\sigma_0, \sigma_5\}$.

$M_{Pre} = \{\sigma_0, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$.

Definition 5.2. Let V be a graph variety of type τ, and let $s \approx t$ be an identity of V. Let M be a subgroupoid of $Hyp(\tau)$. Then $s \approx t$ is called an M-hyperidentity with respect to V, if for every $\sigma \in M$, $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity of V.

For any biregular leftmost graph variety $\mathcal{K}$ and for any $s \approx t \in Id\mathcal{K}$. We want to characterize the property of s and t such that $s \approx t$ is an M_{Left}-hyperidentity, M_{Right}-hyperidentity, M_{Out}-hyperidentity, M_{Reg}-hyperidentity, M_D-hyperidentity and M_{Pre}-hyperidentity with respect to $\mathcal{K}$ for all biregular leftmost graph varieties $\mathcal{K}$.

At first we consider the M_D-hyperidentity. Since $M_D = \{\sigma_0, \sigma_5\}$, let $\mathcal{K}$ be any biregular leftmost graph variety and for any $s \approx t \in Id\mathcal{K}$. We see that if s and t are trivial terms, then $s \approx t$ is an M_D-hyperidentity with respect to $\mathcal{K}$. If $s \approx t$ is a non-trivial equation and $G(s) = G(t)$, then $s \approx t$ is an M_D-hyperidentity with respect to $\mathcal{K}$, too. For the case $s \approx t$ is a non-trivial equation and $G(s) \neq G(t)$. We have $s \approx t$ is an M_D-hyperidentity with respect to $\mathcal{K}$ if and only if $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t] \in Id\mathcal{K}$ (i.e. $s^d \approx t^d \in Id\mathcal{K}$).

For M_{Left}-hyperidentity. Since $M_{Left} = \{\sigma_0, \sigma_1, \sigma_3, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{14}, \sigma_{16}, \sigma_{18}\}$, let $\mathcal{K}$ be any biregular leftmost graph variety and for any $s \approx t \in Id\mathcal{K}$. We see that if s and t are trivial terms, then $s \approx t$ is an M_{Left}-hyperidentity with respect to $\mathcal{K}$ if and only if $L(s) = L(t)$. If $s \approx t$ is a non-trivial equation and $G(s) = G(t)$, then $s \approx t$ is an M_{Left}-hyperidentity with respect to $\mathcal{K}$, too. Now we consider the case $s \approx t$ is non-trivial equation and $G(s) \neq G(t)$. We characterize M_{Left}-hyperidentity with respect to all biregular leftmost graph varieties as the following theorems:

Theorem 5.1. *Let $s \approx t$ be a non-trivial equation with $G(s) \neq G(t)$. If $s \approx t \in Id\mathcal{K}$, then $s \approx t$ is an M_{Left}-hyperidentity with respect to $\mathcal{K}$ for all biregular leftmost graph varieties $\mathcal{K}$ with $\mathcal{K} \notin \{\mathcal{K}_4, \mathcal{K}_{13}, \mathcal{K}_{27}\}$.*

Proof. Consider for $\mathcal{K}_1$. If $\sigma \in \{\sigma_0, \sigma_6, \sigma_{12}\}$, then σ is a proper hypersubstitution. Hence $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_1$. Since $\hat{\sigma}_1[s] = L(s) = L(t) = \hat{\sigma}_1[t]$ and $\hat{\sigma}_3[s] = L(s)L(s) = L(t)L(t) = \hat{\sigma}_3[t]$, we have $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in Id\mathcal{K}_1$ and $\hat{\sigma}_3[s] \approx \hat{\sigma}_3[t] \in Id\mathcal{K}_1$. By Table 1, we have $\sigma_0 {\sim_{\mathcal{K}_1}} \sigma_8$, $\sigma_6 {\sim_{\mathcal{K}_1}} \sigma_{10}$ and $\sigma_{12} {\sim_{\mathcal{K}_1}} \sigma_{14} {\sim_{\mathcal{K}_1}} \sigma_{16} {\sim_{\mathcal{K}_1}} \sigma_{18}$. We get that $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t] \in Id\mathcal{K}_1$, $\hat{\sigma}_{10}[s] \approx \hat{\sigma}_{10}[t] \in Id\mathcal{K}_1$, $\hat{\sigma}_{14}[s] \approx \hat{\sigma}_{14}[t] \in Id\mathcal{K}_1$, $\hat{\sigma}_{16}[s] \approx \hat{\sigma}_{16}[t] \in Id\mathcal{K}_1$ and $\hat{\sigma}_{18}[s] \approx \hat{\sigma}_{18}[t] \in Id\mathcal{K}_1$. Hence $s \approx t$ is an M_{Left}-hyperidentity with respect to $\mathcal{K}_1$. In the same way, we can prove the statement for the other biregular leftmost graph variety $\mathcal{K}$ with $\mathcal{K} \notin \{\mathcal{K}_4, \mathcal{K}_{13}, \mathcal{K}_{27}\}$. ∎

Theorem 5.2. *Let $s \approx t$ be a non-trivial equation with $G(s) \neq G(t)$. If $s \approx t \in Id\mathcal{K}_{13}$, then $s \approx t$ is an M_{Left}-hyperidentity with respect to $\mathcal{K}_{13}$ if and only if for any $x \in V(s)$ there exists $y \in V(s)$ such that $(x,y) \in E(s)$ if and only if there exists $z \in V(s)$ such that $(x,z) \in E(t)$.*

Proof. For any $x \in V(s)$ suppose that there exists $y \in V(s)$ such that $(x,y) \in E(s)$. By Lemma 4.2, we have $(x,x) \in E(\hat{\sigma}_6[s])$. Since $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t] \in Id\mathcal{K}_{13}$, we get $(x,x) \in E(\hat{\sigma}_6[t])$. If $(x,x) \notin E(t)$, then there exists $z \in V(s)$ such that $(x,z) \in E(t)$. In the same way we prove the converse.

Conversely, assume that $s \approx t \in Id\mathcal{K}_{13}$ and for any $x \in V(s)$, there exists $y \in V(s)$ such that $(x,y) \in E(s)$ if and only if there exists $z \in V(s)$ such that $(x,z) \in E(t)$. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from M_{Left}.

If $\sigma \in PM_{\mathcal{K}_{13}} = \{\sigma_0, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{16}\}$, then $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_{13}$. Since $\hat{\sigma}_1[s] = L(s) = L(t) = \hat{\sigma}_1[t]$ and $\hat{\sigma}_3[s] = L(s)L(s) = L(t)L(t) = \hat{\sigma}_3[t]$, we have $\sigma_1[s] \approx \sigma_1[t] \in Id\mathcal{K}_{13}$ and $\sigma_3[s] \approx \sigma_3[t] \in Id\mathcal{K}_{13}$

For σ_6 and for any $x \in V(s)$ suppose that $(x,x) \in E(\hat{\sigma}_6[s])$. By Lemma 4.2, $(x,x) \in E(s)$ or there exists $y \in V(s)$ such that $(x,y) \in E(s)$. Then there exists $z \in V(s)$ such that $(x,z) \in E(t)$. Hence $(x,x) \in E(\hat{\sigma}_6[t])$. In the same way, we can prove the converse. For any $x, y \in V(s)$ with $x \neq y$ suppose that $(x,y) \in E(\hat{\sigma}_6[s])$ or $(y,x),(y,y) \in E(\hat{\sigma}_6[s])$. If $(x,y) \in E(\hat{\sigma}_6[s])$, then $(x,y) \in E(s)$. We have $(x,y) \in E(t)$ or $(y,x),(y,y) \in E(t)$. Hence we get $(x,y) \in E(\hat{\sigma}_6[t])$ or $(y,x),(y,y) \in E(\hat{\sigma}_6[t])$. If $(y,x),(y,y) \in E(\hat{\sigma}_6[s])$, then $(y,x) \in E(s)$. We have $(y,x) \in E(t)$ or $(x,y),(x,x) \in E(t)$. By Lemma 4.2, we get $(y,x),(y,y) \in E(\hat{\sigma}_6[t])$ or $(x,y) \in E(\hat{\sigma}_6[t])$. In the same way, we can prove the converse. By Table 1, we get $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t] \in Id\mathcal{K}_{13}$.

Since $\sigma_6 \sim_{\mathcal{K}_{13}} \sigma_{14}$ and $\sigma_{10} \sim_{\mathcal{K}_{13}} \sigma_{18}$, we get that $\hat{\sigma}_{14}[s] \approx \hat{\sigma}_{14}[t] \in Id\mathcal{K}_{13}$ and $\hat{\sigma}_{18}[s] \approx \hat{\sigma}_{18}[t] \in Id\mathcal{K}_{13}$. ∎

Theorem 5.3. *Let $s \approx t$ be a non-trivial equation with $G(s) \neq G(t)$. Let $\mathcal{K} \in \{\mathcal{K}_4,\ \mathcal{K}_{27}\}$ and $s \approx t \in Id\mathcal{K}$. Then $s \approx t$ is an M_{Left}-hyperidentity with respect to $\mathcal{K}$ if and only if the following are satisfied:*

(i) *for any $x \in V(s)$, there exists $y \in V(s)$ such that $(x,y) \in E(s)$ if and only if there exists $z \in V(s)$ such that $(x,z) \in E(t)$,*

(ii) *for any $x \in V(s)$, there exists $y' \in V(s)$ such that $(y',x) \in E(s)$ if and only if there exists $z' \in V(s)$ such that $(z',x) \in E(t)$.*

Proof. For $\mathcal{K}_4$. Suppose that $s \approx t$ is M_{Left}-hyperidentity with respect to $\mathcal{K}_4$. To prove (i), for any $x \in V(s)$ suppose that there exists $y \in V(s)$ such that $(x,y) \in E(s)$. By Lemma 4.2, we have $(x,x) \in E(\hat{\sigma}_6[s])$. Since $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t] \in Id\mathcal{K}_4$, we have $(x,x) \in E(\hat{\sigma}_6[t])$. If $(x,x) \notin E(t)$, then there exists $z \in V(s)$ such that $(x,z) \in E(t)$. In the same way, we prove the converse. Similarly, since $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t] \in Id\mathcal{K}_4$, we can prove (ii).

Conversely, assume that $s \approx t \in Id\mathcal{K}_4$ and that (i) and (ii) are satisfied. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from M_{Left}.

If $\sigma \in PM_{\mathcal{K}_4} = \{\sigma_0, \sigma_{10}\}$, then $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_4$.

For σ_1, σ_3, we have $\hat{\sigma}_1[s] = L(s) = L(t) = \hat{\sigma}_1[t]$ and $\hat{\sigma}_3[s] = L(s)L(s) = L(t)L(t) = \hat{\sigma}_3[t]$. We have $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in Id\mathcal{K}_4$ and $\hat{\sigma}_3[s] \approx \hat{\sigma}_3[t] \in Id\mathcal{K}_4$.

For σ_6 and for any $x \in V(s)$ suppose that $(x,x) \in E(\hat{\sigma}_6[s])$. By Lemma 4.2, $(x,x) \in E(s)$ or there exists $y \in V(s)$ such that $(x,y) \in E(s)$. Then by (i) there exists $z \in V(s)$ such that $(x,z) \in E(t)$. Hence $(x,x) \in E(\hat{\sigma}_6[t])$. In the same way, we prove the converse. For any $x, y \in V(s)$ with $x \neq y$ suppose that $(x,y) \in E(\hat{\sigma}_6[s])$ or $(y,x) \in E(\hat{\sigma}_6[s])$. By Lemma 4.2, $(y,x) \in E(s)$ or $(x,y) \in E(s)$. Then $(y,x) \in E(t)$ or $(x,y) \in E(t)$ and thus $(x,y) \in E(\hat{\sigma}_6[t])$ or $(y,x) \in E(\hat{\sigma}_6[t])$. In the same way, we can prove the converse. Hence we get $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t] \in Id\mathcal{K}_4$.

Similarly, by (ii) we can prove that $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t] \in Id\mathcal{K}_4$.

Since $\sigma_0 \sim_{\mathcal{K}_4} \sigma_{12}$, $\sigma_6 \sim_{\mathcal{K}_4} \sigma_{14}$, $\sigma_8 \sim_{\mathcal{K}_4} \sigma_{16}$ and $\sigma_{10} \sim_{\mathcal{K}_4} \sigma_{18}$, we get that $\hat{\sigma}_{12}[s] \approx \hat{\sigma}_{12}[t] \in Id\mathcal{K}_4$, $\hat{\sigma}_{14}[s] \approx \hat{\sigma}_{14}[t] \in Id\mathcal{K}_4$, $\hat{\sigma}_{16}[s] \approx \hat{\sigma}_{16}[t] \in Id\mathcal{K}_4$ and $\hat{\sigma}_{18}[s] \approx \hat{\sigma}_{18}[t] \in Id\mathcal{K}_4$.

In the similarly way, we can prove the statement for $\mathcal{K}_{27}$. ■

For M_{Out}-hyperidentity. Since $M_{Out} = \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{16}, \sigma_{18}\}$, let $\mathcal{K}$ be any biregular leftmost graph variety and for any $s \approx t \in Id\mathcal{K}$. We see that if s and t are trivial terms, then $s \approx t$ is an M_{Out}-hyperidentity with respect to $\mathcal{K}$. If $s \approx t$ is a non-trivial equation and $G(s) = G(t)$, then $s \approx t$ is an M_{Out}-hyperidentity with respect to $\mathcal{K}$, too. For the case $s \approx t$ is non-trivial equation and $G(s) \neq G(t)$. Since $M_{Out} \subset M_{Left}$, so we can check that it has the same results as M_{Left}-hyperidentity.

For M_{Reg}-hyperidentity. Since $M_{Reg} = \{\sigma_0, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$, let $\mathcal{K}$ be any biregular leftmost graph variety and for any $s \approx t \in Id\mathcal{K}$. We see that if s and t are trivial terms, then $s \approx t$ is an M_{Reg}-hyperidentity with respect to $\mathcal{K}$. If $s \approx t$ is a non-trivial equation and $G(s) = G(t)$, then $s \approx t$ is an M_{Reg}-hyperidentity with respect to $\mathcal{K}$, too. For the case $s \approx t$ is non-trivial equation and $G(s) \neq G(t)$. We get the same result as hyperidentity. That is we have the following theorems:

Theorem 5.4. *Let $s \approx t$ be a non-trivial equation with $G(s) \neq G(t)$ and let $\mathcal{K}$ be a biregular leftmost graph variety with $\mathcal{K} \notin \{\mathcal{K}_4, \mathcal{K}_{13}, \mathcal{K}_{27}\}$. If $s \approx t \in Id\mathcal{K}$ and $s^d \approx t^d \in Id\mathcal{K}$, then $s \approx t$ is an M_{Reg}-hyperidentity with respect to $\mathcal{K}$.*

Theorem 5.5. *Let $s \approx t$ be a non-trivial equation with $G(s) \neq G(t)$ and let $s \approx t \in Id\mathcal{K}_{13}$. Then $s \approx t$ is an M_{Reg}-hyperidentity with respect to $\mathcal{K}_{13}$ if and only if the following are satisfied:*

(i) $s^d \approx t^d \in Id\mathcal{K}_{13}$,

(ii) *for any $x \in V(s)$, there exists $y \in V(s)$ such that $(x, y) \in E(s)$ if and only if there exists $z \in V(s)$ such that $(x, z) \in E(t)$,*

(iii) *for any $x \in V(s)$, there exists $y \in V(s)$ such that $(x, y) \in E(s^d)$ if and only if there exists $z \in V(s)$ such that $(x, z) \in E(t^d)$.*

Theorem 5.6. *Let $s \approx t$ be a non-trivial equation with $G(s) \neq G(t)$ and let $\mathcal{K}$ be a biregular leftmost graph variety with $\mathcal{K} \in \{\mathcal{K}_4, \mathcal{K}_{27}\}$. Then $s \approx t$ is an M_{Reg}- hyperidentity with respect to $\mathcal{K}$ if and only if the following are satisfied:*

(i) $s^d \approx t^d \in Id\mathcal{K}$,

(ii) *for any $x \in V(s)$, there exists $y \in V(s)$ such that $(x, y) \in E(s)$ if and only if there exists $z \in V(s)$ such that $(x, z) \in E(t)$,*

(iii) *for any $x \in V(s)$, there exists $y \in V(s)$ such that $(y, x) \in E(s)$ if and only if there exists $z \in V(s)$ such that $(z, x) \in E(t)$,*

(iv) *for any $x \in V(s)$, there exists $y \in V(s)$ such that $(x, y) \in E(s^d)$ if and only if there exists $z \in V(s)$ such that $(x, z) \in E(t^d)$,*

(v) *for any $x \in V(s)$, there exists $y \in V(s)$ such that $(y, x) \in E(s^d)$ if and only if there exists $z \in V(s)$ such that $(z, x) \in E(t^d)$.*

For M_{Pre}-hyperidentity. Since $M_{Pre} = \{\sigma_0, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$, let $\mathcal{K}$ be any biregular leftmost graph variety and for any $s \approx t \in Id\mathcal{K}$. We see that if s and t are trivial terms, then $s \approx t$ is an M_{Pre}-hyperidentity with respect to $\mathcal{K}$ if and only if they have the same leftmost and the same rightmost. If $s \approx t$ is a non-trivial equation and $G(s) = G(t)$, then $s \approx t$ is an M_{Pre}-hyperidentity with respect to $\mathcal{K}$, too. For the case $s \approx t$ is non-trivial equation and $G(s) \neq G(t)$. Since $M_{Reg} = M_{Pre} - \{\sigma_3, \sigma_4\}$, we have the same results as M_{Reg}-hyperidentity.

For M_{Right}-hyperidentity. Since $M_{Right} = \{\sigma_0, \sigma_2, \sigma_4, \sigma_6, \sigma_7, \sigma_8, \sigma_{10}, \sigma_{11}, \sigma_{13}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$, let $\mathcal{K}$ be any biregular leftmost graph variety and for any $s \approx t \in Id\mathcal{K}$. We see that if s and t are trivial terms, then $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}$ if and only if they have the same rightmost variables. If $s \approx t$ is a non-trivial equation and $G(s) = G(t)$, then $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}$, too. Now we consider the case $s \approx t$ is non-trivial equation and $G(s) \neq G(t)$. Since $\sigma_2 \in M_{Right}$, hence the first property of s and t is $R(s) = R(t)$. Further since $\hat{\sigma}_6[t^d] = \hat{\sigma}_7[t]$, $\hat{\sigma}_8[t^d] = \hat{\sigma}_9[t]$, $\hat{\sigma}_{10}[t^d] = \hat{\sigma}_{11}[t]$, $\hat{\sigma}_{12}[t^d] = \hat{\sigma}_{13}[t]$, $\hat{\sigma}_{14}[t^d] = \hat{\sigma}_{15}[t]$, $\hat{\sigma}_{16}[t^d] = \hat{\sigma}_{17}[t]$, and $\hat{\sigma}_{18}[t^d] = \hat{\sigma}_{19}[t]$ for all terms t. Then use the properties of hypersubstitution that (a) if $\sigma_1(f) \approx \sigma_2(f) \in Id\mathcal{K}$, then $\sigma_1 \sim_{\mathcal{K}} \sigma_2$ and (b) if $\sigma_1 \sim_{\mathcal{K}} \sigma_2$ and $\hat{\sigma}_1(s) \approx \hat{\sigma}_1(t) \in Id\mathcal{K}$, then $\hat{\sigma}_2(s) \approx \hat{\sigma}_2(t) \in Id\mathcal{K}$. Hence we only find the properties of s and t such that $\hat{\sigma}(s) \approx \hat{\sigma}(t) \in Id\mathcal{K}$ and $\hat{\sigma}(s^d) \approx \hat{\sigma}(t^d) \in Id\mathcal{K}$ for all $\sigma \in \{\sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{14}, \sigma_{16}, \sigma_{18}\}$. The proof of each next theorem are the same pattern, so we will prove only some theorems and skip the others. Then we characterize M_{Right}-hyperidentity with respect to each biregular leftmost graph variety as the following theorems:

Theorem 5.7. *Let $s \approx t$ be a non-trivial equation with $G(s) \neq G(t)$ and let $s \approx t \in Id\mathcal{K}$. Then $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K} \in \{\mathcal{K}_1, \mathcal{K}_{25}, \mathcal{K}_{28}\}$ if and only if the following are satisfied:*

(i) $R(s) = R(t)$,

(ii) *for any $x, y \in V(s)$, $x \neq y$, $(x, y) \in E(s^d)$ if and only if $(x, y) \in E(t^d)$.*

Proof. Suppose that $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_1$. Since $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{K}_1$, we have $R(s) = R(t)$. For any $x, y \in V(s)$ with $x \neq y$, suppose that $(x, y) \in E(s^d)$. By Lemma 4.2, we have $(x, y) \in E(\hat{\sigma}_6[s^d])$. Since $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t] \in Id\mathcal{K}_1$ and $\hat{\sigma}_6[t'^d] = \hat{\sigma}_7[t']$, for all terms t', we get $\hat{\sigma}_6[s^d] \approx \hat{\sigma}_6[t^d] \in Id\mathcal{K}_1$. Hence $(x, y) \in E(\hat{\sigma}_6[t^d])$. By Lemma 4.2, $(x, y) \in E(t^d)$. In the same way, we can prove the converse.

Conversely, assume that $s \approx t \in Id\mathcal{K}_1$ and that (i) and (ii) are satisfied. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from M_{Right}.

If σ is proper, then $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_1$. We get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_1$ for all $\sigma \in \{\sigma_0, \sigma_6, \sigma_{12}\}$.

Since $\sigma_0 \sim_{\mathcal{K}_1} \sigma_8$, $\sigma_6 \sim_{\mathcal{K}_1} \sigma_{10}$ and $\sigma_{12} \sim_{\mathcal{K}_1} \sigma_{14} \sim_{\mathcal{K}_1} \sigma_{16} \sim_{\mathcal{K}_1} \sigma_{18}$, by Lemma 4.1, we get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_1$ for all $\sigma \in \{\sigma_8, \sigma_{10}, \sigma_{16}, \sigma_{18}\}$.

For σ_2 and σ_4, we have $\hat{\sigma}_2[s] = R(s) = R(t) = \hat{\sigma}_2[t]$ and $\hat{\sigma}_4[s] = R(s)R(s) = R(t)R(t) = \hat{\sigma}_4[t]$. We have $\sigma_2[s] \approx \sigma_2[t] \in Id\mathcal{K}_1$ and $\sigma_4[s] \approx \sigma_4[t] \in Id\mathcal{K}_1$.

Next we will show that $\hat{\sigma}_6[s^d] \approx \hat{\sigma}_6[t^d] \in Id\mathcal{K}_1$ and $\hat{\sigma}_{12}[s^d] \approx \hat{\sigma}_{12}[t^d] \in Id\mathcal{K}_1$. For $\hat{\sigma}_6$, $V(\hat{\sigma}_6[s^d]) = V(s) = V(t) = V(\hat{\sigma}_6[t^d])$, $L(\hat{\sigma}_6[s^d]) = R(s) = R(t) = L(\hat{\sigma}_6[t^d])$. For any $x \in V(s)$, suppose that $x = L(s^d) = L(t^d)$. Since $s \approx t$ is a non-trivial equation and $G(s) \neq G(t)$, we get $(x,x) \in E(\hat{\sigma}_6[s^d])$ and $(x,x) \in E(\hat{\sigma}_6[t^d])$. If $x \neq L(s^d)$, then there exist $y, y' \in V(s)$ such that $(y,x) \in E(\hat{\sigma}_6[s^d])$ and $(y',x) \in E(\hat{\sigma}_6[t^d])$.

For any $x, y \in V(s)$ with $x \neq y$ suppose that $(x,y) \in E(\hat{\sigma}_6[s^d])$. By Lemma 4.2, $(x,y) \in E(s^d)$. By (ii), $(x,y) \in E(t^d)$. We have $(x,y) \in E(\hat{\sigma}_6[t^d])$. In the same way, we can prove the converse. Hence we get that $\hat{\sigma}_6[s^d] \approx \hat{\sigma}_6[t^d] \in Id\mathcal{K}_1$ and thus $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t] \in Id\mathcal{K}_1$.

For $\hat{\sigma}_{12}$, $V(\hat{\sigma}_{12}[s^d]) = V(s) = V(t) = V(\hat{\sigma}_{12}[t^d])$, $L(\hat{\sigma}_{12}[s^d]) = R(s) = R(t) = L(\hat{\sigma}_{12}[t^d])$. For any $x \in V(s)$, suppose that $x = L(s^d) = L(t^d)$. Since $s \approx t$ is a non-trivial equation with $G(s) \neq G(t)$ and $G(s^d)$, $G(t^d)$ are the rooted graphs, then there exist $y, y' \in V(s)$ such that $(x,y) \in E(s^d)$ and $(x,y') \in E(t^d)$. By Lemma 4.2, we have $(y,x) \in E(\hat{\sigma}_{12}[s^d])$ and $(y',x) \in E(\hat{\sigma}_{12}[t^d])$. If $x \neq L(s^d)$, then there exist $y, y' \in V(s)$ with $x \neq y, x \neq y'$ such that $(y,x) \in E(s^d)$ and $(y',x) \in E(t^d)$. By Lemma 4.2 again, we get that $(y,x) \in E(\hat{\sigma}_{12}[s^d])$ and $(y',x) \in E(\hat{\sigma}_{12}[t^d])$.

For any $x, y \in V(s)$ with $x \neq y$ suppose that $(x,y) \in E(\hat{\sigma}_{12}[s^d])$. By Lemma 4.2, $(x,y) \in E(s^d)$ or $(y,x) \in E(s^d)$. By (ii), $(x,y) \in E(t^d)$ or $(y,x) \in E(t^d)$. We have $(x,y) \in E(\hat{\sigma}_{12}[t^d])$. In the same way, we prove the converse. Therefore we get that $\hat{\sigma}_{12}[s^d] \approx \hat{\sigma}_{12}[t^d] \in Id\mathcal{K}_1$, and thus $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t] \in Id\mathcal{K}_1$,

Since $\sigma_7 \sim_{\mathcal{K}_1} \sigma_{11}$ and $\sigma_{13} \sim_{\mathcal{K}_1} \sigma_{15} \sim_{\mathcal{K}_1} \sigma_{17} \sim_{\mathcal{K}_1} \sigma_{19}$, we get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_1$ for all $\sigma \in \{\sigma_{11}, \sigma_{15}, \sigma_{17}, \sigma_{19}\}$.

In the similar way, we can prove the statement for K_{25} and K_{28}. ■

Theorem 5.8. *Let $s \approx t$ be a non-trivial equation with $G(s) \neq G(t)$ and let $s \approx t \in Id\mathcal{K}$. Then $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K} \in \{\mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_5, \mathcal{K}_6, \mathcal{K}_7, \mathcal{K}_{10}, \mathcal{K}_{11}, \mathcal{K}_{13}, \mathcal{K}_{21}, \mathcal{K}_{22}\}$ if and only if $s^d \approx t^d \in Id\mathcal{K}$.*

Proof. Suppose that $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_2$. Since $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t] \in Id\mathcal{K}_2$ and $\sigma_5 \sim_{\mathcal{K}_2} \sigma_7$, we get that $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t] \in Id\mathcal{K}_2$, i.e., $s^d \approx t^d \in Id\mathcal{K}_2$.

Conversely, assume that $s \approx t \in Id\mathcal{K}_2$ and $s^d \approx t^d$ is an identity in $\mathcal{K}_2$. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from M_{Right}.

If σ is proper, then $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_2$. We get $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_2$ for all $\sigma \in \{\sigma_0, \sigma_8, \sigma_{12}\}$.

For σ_2 and σ_4, we have $\hat{\sigma}_2[s] = L(s^d) = L(t^d) = \hat{\sigma}_2[t]$ and $\hat{\sigma}_4[s] = L(s^d)L(s^d) = L(t^d)L(t^d) = \hat{\sigma}_4[t]$. We have $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{K}_2$ and $\hat{\sigma}_4[s] \approx \hat{\sigma}_4[t] \in Id\mathcal{K}_2$.

Since $\sigma_0 \sim_{\mathcal{K}_2} \sigma_6$, $\sigma_8 \sim_{\mathcal{K}_2} \sigma_{10}$ and $\sigma_{12} \sim_{\mathcal{K}_2} \sigma_{16} \sim_{\mathcal{K}_2} \sigma_{18}$, we get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_2$ for all $\sigma \in \{\sigma_6, \sigma_{10}, \sigma_{16}, \sigma_{18}\}$.

Since $s^d \approx t^d \in Id\mathcal{K}_2$, $\sigma_0 \sim_{\mathcal{K}_2} \sigma_6$, $\sigma_8 \sim_{\mathcal{K}_2} \sigma_{10}$ and σ_0, σ_8, σ_{12} are proper graph hypersubstitutions, we get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_2$ for all $\sigma \in \{\sigma_7, \sigma_{11}, \sigma_{13}\}$.

Since $\sigma_{13} \sim_{\mathcal{K}_2} \sigma_{15} \sim_{\mathcal{K}_2} \sigma_{17} \sim_{\mathcal{K}_2} \sigma_{19}$, we get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_2$ for all $\sigma \in \{\sigma_{15}, \sigma_{17}, \sigma_{19}\}$.

In the similar way, we can prove for the others $K_i, i \in \{3, 5, 6, 7, 10, 11, 13, 21, 22\}$. ■

Theorem 5.9. *Let $s \approx t$ be a non-trivial equation with $G(s) \neq G(t)$ and let $s \approx t \in Id\mathcal{K}_4$. Then $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_4$ if and only if the following are satisfied:*

(i) $s^d \approx t^d \in Id\mathcal{K}_4$,

(ii) *for any $x \in V(s)$, there exists $y \in V(s)$ such that $(x, y) \in E(s)$ if and only if there exists $z \in V(s)$ such that $(x, z) \in E(t)$,*

(iii) *for any $x \in V(s)$, there exists $y \in V(s)$ such that $(y, x) \in E(s)$ if and only if there exists $z \in V(s)$ such that $(z, x) \in E(t)$,*

(iv) *for any $x \in V(s)$, there exists $y \in V(s)$ such that $(x, y) \in E(s^d)$ if and only if there exists $z \in V(s)$ such that $(x, z) \in E(t^d)$,*

(v) *for any $x \in V(s)$, there exists $y \in V(s)$ such that $(y, x) \in E(s^d)$ if and only if there exists $z \in V(s)$ such that $(z, x) \in E(t^d)$.*

Proof. Suppose that $s \approx t$ is M_{Right}-hyperidentity with respect to $\mathcal{K}_4$. Since $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t] \in Id\mathcal{K}_4$ and $\sigma_5 \sim_{\mathcal{K}_4} \sigma_{13}$, we get that $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t] \in Id\mathcal{K}_4$, i.e., $s^d \approx t^d \in Id\mathcal{K}_4$. To prove (ii), for any $x \in V(s)$ suppose that there exists $y \in V(s)$ such that $(x, y) \in E(s)$. By Lemma 4.2, we have

$(x,x) \in E(\hat{\sigma}_6[s])$. Since $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t] \in Id\mathcal{K}_4$, we get $(x,x) \in E(\hat{\sigma}_6[t])$. If $(x,x) \notin E(t)$, then by Lemma 4.2, we have there exists $z \in V(s)$ such that $(x,z) \in E(t)$. In the same way, we can prove the converse. Similarly, since $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t] \in Id\mathcal{K}_4$, we can prove (iii). To prove (iv), for any $x \in V(s)$ suppose that there exists $y \in V(s)$ such that $(x,y) \in E(s^d)$. By Lemma 4.2, we have $(x,x) \in E(\hat{\sigma}_6[s^d])$. Since $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t] \in Id\mathcal{K}_4$, we get $\hat{\sigma}_6[s^d] \approx \hat{\sigma}_6[t^d] \in Id\mathcal{K}_4$ and thus $(x,x) \in E(\hat{\sigma}_6[t^d])$. If $(x,x) \notin E(t^d)$, then there exists $z \in V(s)$ such that $(x,z) \in E(t^d)$. In the same way, we prove the converse. Similarly, since $\hat{\sigma}_{17}[s] \approx \hat{\sigma}_{17}[t] \in Id\mathcal{K}_4$ and $\sigma_9 \sim_{\mathcal{K}_4} \sigma_{17}$, we get $\hat{\sigma}_8[s^d] \approx \hat{\sigma}_8[t^d] \in Id\mathcal{K}_4$. Then, we can prove (v).

Conversely, assume that $s \approx t$ is an identity in $\mathcal{K}_4$ and that (i), (ii), (iii), (iv) and (v) are satisfied. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from M_{Right}.

If $\sigma \in \{\sigma_0, \sigma_{10}\}$, then σ is proper and we get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_4$.

For σ_2 and σ_4, we have $\hat{\sigma}_2[s] = L(s^d) = L(t^d) = \hat{\sigma}_2[t]$ and $\hat{\sigma}_4[s] = L(s^d)L(s^d) = L(t^d)L(t^d) = \hat{\sigma}_4[t]$. We have $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{K}_4$ and $\hat{\sigma}_4[s] \approx \hat{\sigma}_4[t] \in Id\mathcal{K}_4$.

For σ_6, for any $x \in V(s)$ suppose that $(x,x) \in E(\hat{\sigma}_6[s])$. By Lemma 4.2, $(x,x) \in E(s)$ or there exists $y \in V(s)$ such that $(x,y) \in E(s)$. Then by (ii), there exists $z \in V(s)$ such that $(x,z) \in E(t)$. Hence $(x,x) \in E(\hat{\sigma}_6[t])$. In the same way, we can prove the converse. For any $x,y \in V(s)$ with $x \neq y$ suppose that $(x,y) \in E(\hat{\sigma}_6[s])$ or $(y,x) \in E(\hat{\sigma}_6[s])$. If $(x,y) \in E(\hat{\sigma}_6[s])$, then $(x,y) \in E(s)$. We have $(x,y) \in E(\hat{\sigma}_6[t])$ or $(y,x) \in E(\hat{\sigma}_6[t])$. Suppose that $(y,x) \in E(\hat{\sigma}_6[s])$. We see that $(y,x) \in E(s)$. Then we get $(y,x) \in E(t)$ or $(x,y) \in E(t)$ and thus $(x,y) \in E(\hat{\sigma}_6[t])$ or $(y,x) \in E(\hat{\sigma}_6[t])$. In the same way, we prove the converse. Hence we get $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t] \in Id\mathcal{K}_4$.

Similarly, by (iii) we can prove $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t] \in Id\mathcal{K}_4$.

Next we will show that $\hat{\sigma}_6[s^d] \approx \hat{\sigma}_6[t^d] \in Id\mathcal{K}_4$. For any $x \in V(s)$ suppose that $(x,x) \in E(\hat{\sigma}_6[s^d])$. By Lemma 4.2, $(x,x) \in E(s^d)$ or there exists $y \in V(s)$ such that $(x,y) \in E(s^d)$. Then by (iv), there exists $z \in V(s)$ such that $(x,z) \in E(t^d)$. Hence $(x,x) \in E(\hat{\sigma}_6[t^d])$. In the same way, we can prove the converse. For any $x,y \in V(s)$ with $x \neq y$ suppose that $(x,y) \in E(\hat{\sigma}_6[s^d])$ or $(y,x) \in E(\hat{\sigma}_6[s^d])$. If $(x,y) \in E(\hat{\sigma}_6[s^d])$, then $(x,y) \in E(s^d)$. We have $(x,y) \in E(\hat{\sigma}_6[t^d])$ or $(y,x) \in E(\hat{\sigma}_6[t^d])$. Suppose that $(y,x) \in E(\hat{\sigma}_6[s^d])$. We see that $(y,x) \in E(s^d)$. Then we get that $(y,x) \in E(t^d)$ or $(x,y) \in E(t^d)$ and thus $(x,y) \in E(\hat{\sigma}_6[t^d])$ or $(y,x) \in E(\hat{\sigma}_6[t^d])$. In the same way, we can prove the converse. Therefore we get $\hat{\sigma}_6[s^d] \approx \hat{\sigma}_6[t^d] \in Id\mathcal{K}_4$ and thus $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t] \in Id\mathcal{K}_4$.

Similarly, by (v), we prove $\hat{\sigma}_8[s^d] \approx \hat{\sigma}_8[t^d] \in Id\mathcal{K}_4$ and thus $\hat{\sigma}_9[s] \approx \hat{\sigma}_9[t] \in Id\mathcal{K}_4$.

Since $s^d \approx t^d \in Id\mathcal{K}_4$, σ_{10} is proper, we have $\hat{\sigma}_{10}[s^d] \approx \hat{\sigma}_{10}[t^d] \in Id\mathcal{K}_4$ and thus $\hat{\sigma}_{11}[s] \approx \hat{\sigma}_{11}[t] \in Id\mathcal{K}_4$.

Since $\hat{\sigma}_7[t'] = \hat{\sigma}_6[t'^d]$, $\hat{\sigma}_9[t'] = \hat{\sigma}_8[t'^d]$ and $\hat{\sigma}_{11}[t'] = \hat{\sigma}_{10}[t'^d]$ for all $t' \in T(X)$, we have that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_4$ for all $\sigma \in \{\sigma_7, \sigma_9, \sigma_{11}\}$.

Since $\sigma_5 \sim_{\mathcal{K}_4} \sigma_{13}$, $\sigma_7 \sim_{\mathcal{K}_4} \sigma_{15}$, $\sigma_8 \sim_{\mathcal{K}_4} \sigma_{16}$, $\sigma_9 \sim_{\mathcal{K}_4} \sigma_{17}$, $\sigma_{10} \sim_{\mathcal{K}_4} \sigma_{18}$ and $\sigma_{11} \sim_{\mathcal{K}_4} \sigma_{19}$, by Lemma 4.1, we get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_4$ for all $\sigma \in \{\sigma_{13}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$. ∎

Theorem 5.10. *Let $s \approx t$ be a non-trivial equation with $G(s) \neq G(t)$ and let $s \approx t \in Id\mathcal{K}$. Then $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K} \in \{\mathcal{K}_8, \mathcal{K}_{12}, \mathcal{K}_{20}, \mathcal{K}_{23}\}$ if and only if the following are satisfied:*

(i) $R(s) = R(t)$,

(ii) *for any $x, y \in V(s)$, $x \neq y$, $(x, y) \in E(s^d)$ or $(y, x) \in E(s^d)$ if and only if $(x, y) \in E(t^d)$ or $(y, x) \in E(t^d)$.*

Theorem 5.11. *Let $s \approx t$ be a non-trivial equation with $G(s) \neq G(t)$ and let $s \approx t \in Id\mathcal{K}_9$. Then $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_9$ if and only if the following are satisfied:*

(i) $R(s) = R(t)$,

(ii) *for any $x \in V(s)$ there exists $y \in V(s)$ such that $(x, y) \in E(s^d)$ if and only if there exists $z \in V(s)$ such that $(x, z) \in E(t^d)$,*

(iii) *for any $x, y \in V(s)$, $x \neq y$, $(x, y) \in E(s^d)$ or $(y, x) \in E(s^d)$, and there exists $z \in V(s)$ such that $(x, z) \in E(s^d)$ if and only if $(x, y) \in E(t^d)$ or $(y, x) \in E(t^d)$, and there exists $z' \in V(s)$ such that $(x, z') \in E(t^d)$,*

(iv) *there exists $x \in V(s)$ such that $(x, x) \in E(s^d)$ if and only if there exists $x' \in V(s)$ such that $(x', x') \in E(t^d)$.*

Proof. Suppose that $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_9$. To prove (i), since $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{K}_1$, we have $R(s) = R(t)$.

To prove (ii), for any $x \in V(s)$ suppose that there exists $y \in V(s^d)$ such that $(x, y) \in E(s^d)$. By Lemma 4.2, we have $(x, x) \in E(\hat{\sigma}_6[s^d])$. Since $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t] \in Id\mathcal{K}_9$ and $\hat{\sigma}_6[t'^d] = \sigma_7[t']$ for all $t' \in T(X)$, we get that $\hat{\sigma}_6[s^d] \approx \hat{\sigma}_6[t^d] \in Id\mathcal{K}_9$. Hence there exists $y' \in V_x(t^d)$ such that $(y', y') \in E(\hat{\sigma}_6[t^d])$, we get that there exists $z \in V(s)$ such that $(x, z) \in E(t^d)$. In the same way, we can prove the converse.

To prove (iii), for any $x, y \in V(s)$, $x \neq y$. Suppose that $(x, y) \in E(s^d)$ or $(y, x) \in E(s^d)$, and there exists $z \in V(s)$ such that $(x, z) \in E(s^d)$. We get $(x, y) \in E(\hat{\sigma}_6[s^d])$ or $(y, x) \in E(\hat{\sigma}_6[s^d])$, and there exists $z' \in V_x(\hat{\sigma}_6[s^d])$ such that $(x, z') \in E(\hat{\sigma}_6[s^d])$. Since $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t] \in Id\mathcal{K}_9$, and $\hat{\sigma}_6[t'^d] = \sigma_7[t']$ for all $t' \in T(X)$, we get that $\hat{\sigma}_6[s^d] \approx \hat{\sigma}_6[t^d] \in Id\mathcal{K}_9$. Hence $(x, y) \in E(\hat{\sigma}_6[t^d])$ or $(y, x) \in E(\hat{\sigma}_6[t^d])$, and there exists $w \in V_x(\hat{\sigma}_6[t^d])$ such that $(x, w) \in E(\hat{\sigma}_6[t^d])$. By Lemma 4.2, we have $(x, y) \in E(t^d)$ or $(y, x) \in E(t^d)$, and there exists $w' \in V(s)$ such that $(x, w') \in E(t^d)$. In the same way, we can prove the converse.

To prove (iv), suppose that there exists $x \in V(s)$ such that $(x, x) \in E(s^d)$. By Lemma 4.2, we have $(x, x) \in E(\hat{\sigma}_{12}[s^d])$. Hence there exists $y \in V(s)$ which there exists $x \in V_y(\hat{\sigma}_{12}[s^d])$ such that $(x, x) \in E(\hat{\sigma}_{12}[s^d])$. Since $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t] \in Id\mathcal{K}_9$ and $\hat{\sigma}_{12}[t'^d] = \sigma_{13}[t']$ for all $t' \in T(X)$, we get that $\hat{\sigma}_{12}[s^d] \approx \hat{\sigma}_{12}[t^d] \in Id\mathcal{K}_9$. Therefore there exists $x' \in V_y(\hat{\sigma}_{12}[t^d])$ such that $(x', x') \in E(\hat{\sigma}_{12}[t^d])$. By Lemma 4.2, $(x', x') \in E(t^d)$. In the same way, we can prove the converse.

Conversely, assume that $s \approx t \in Id\mathcal{K}_9$ and that (i), (ii), (iii) and (iv) are satisfied. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from M_{Right}.

If σ is proper, then $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_9$. We get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_9$ for all $\sigma \in \{\sigma_0, \sigma_6, \sigma_8, \sigma_{12}\}$.

Since $\sigma_8 \sim_{\mathcal{K}_9} \sigma_{10} \sim_{\mathcal{K}_9} \sigma_{14} \sim_{\mathcal{K}_9} \sigma_{16} \sim_{\mathcal{K}_9} \sigma_{18}$, we get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_9$ for all $\sigma \in \{\sigma_{10}, \sigma_{16}, \sigma_{18}\}$.

For σ_2 and σ_4, we have $\hat{\sigma}_2[s] = R(s) = R(t) = \hat{\sigma}_2[t]$ and $\hat{\sigma}_4[s] = R(s)R(s) = R(t)R(t) = \hat{\sigma}_4[t]$. We have $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{K}_9$ and $\hat{\sigma}_4[s] \approx \hat{\sigma}_4[t] \in Id\mathcal{K}_9$.

We will show that $\hat{\sigma}_6[s^d] \approx \hat{\sigma}_6[t^d] \in Id\mathcal{K}_9$. By (ii) and Lemma 4.2, we see that for any $x \in V(s)$ there exists $y \in V_x(\hat{\sigma}_6[s^d])$ such that $(y, y) \in E(\hat{\sigma}_6[s^d])$ if and only if there exists $z \in V_x(\hat{\sigma}_6[t^d])$ such that $(z, z) \in E(\hat{\sigma}_6[t^d])$. For any $x, y \in V(s)$ with $x \neq y$ suppose that $(x, y) \in E(\hat{\sigma}_6[s^d])$ or $(y, x) \in E(\hat{\sigma}_6[s^d])$, and there exists $z \in V_x(\hat{\sigma}_6[s^d])$ such that $(z, z) \in E(\hat{\sigma}_6[s^d])$. By Lemma 4.2, we have $(x, y) \in E(s^d)$ or $(y, x) \in E(s^d)$, and there exists $z' \in V(s)$ such that $(x, z') \in E(s^d)$. By (iii), we get $(x, y) \in E(t^d)$ or $(y, x) \in E(t^d)$, and there exists $w \in V(s)$ such that $(x, w) \in E(t^d)$. We have $(x, y) \in E(\hat{\sigma}_6[t^d])$ or $(y, x) \in E(\hat{\sigma}_6[t^d])$, and there exists $w' \in V_x(\hat{\sigma}_6[t^d])$ such that $(w', w') \in E(\hat{\sigma}_6[t^d])$. In the same way, we can prove the converse. Therefore we get $\hat{\sigma}_6[s^d] \approx \hat{\sigma}_6[t^d] \in Id\mathcal{K}_9$ and thus $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t] \in Id\mathcal{K}_9$.

Similarly, we can prove $\hat{\sigma}_8[s^d] \approx \hat{\sigma}_8[t^d] \in Id\mathcal{K}_9$ and thus $\hat{\sigma}_9[s] \approx \hat{\sigma}_9[t] \in Id\mathcal{K}_9$.

Next we will show that $\hat{\sigma}_{12}[s^d] \approx \hat{\sigma}_{12}[t^d] \in Id\mathcal{K}_9$. For any $x \in V(s)$ suppose that there exists $y \in V_x(\hat{\sigma}_{12}[s^d])$ such that $(y, y) \in E(\hat{\sigma}_{12}[s^d])$. By Lemma 4.2, we have $(y, y) \in E(s^d)$. By (iv), we get that there exists $y' \in V(s)$ such that $(y', y') \in E(t^d)$. By Lemma 4.2, we have for any $x \neq y, (x, y) \in E(\hat{\sigma}_{12}[t^d])$ if and only if $(y, x) \in E(\hat{\sigma}_{12}[t^d])$. Hence $y' \in V_x(\hat{\sigma}_{12}[t^d])$ such that $(y', y') \in E(\hat{\sigma}_{12}[t^d])$. In the same way, we can prove the converse. For any $x, y \in V(s)$ with $x \neq y$ suppose that $(x, y) \in E(\hat{\sigma}_{12}[s^d])$ or $(y, x) \in E(\hat{\sigma}_{12}[s^d])$, and there exists $z \in V_x(\hat{\sigma}_{12}[s^d])$ such that $(z, z) \in E(\hat{\sigma}_{12}[s^d])$. By Lemma 4.2, we have $(z, z) \in E(s^d)$ and $z \in V_{x'}(\hat{\sigma}_{12}[s^d])$ for all $x' \in V(s^d)$. Therefore $(x, y) \in E(\hat{\sigma}_{12}[s^d])$ or $(y, x) \in E(\hat{\sigma}_{12}[s^d])$, and $z \in V_x(\hat{\sigma}_{12}[t^d])$ such that $(z, z) \in E(\hat{\sigma}_{12}[t^d])$. In the same way, we can prove the converse. Therefore we get $\hat{\sigma}_{12}[s^d] \approx \hat{\sigma}_{12}[t^d] \in Id\mathcal{K}_9$ and thus $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t] \in Id\mathcal{K}_9$.

Since $\sigma_9 \sim_{\mathcal{K}_9} \sigma_{11} \sim_{\mathcal{K}_9} \sigma_{15} \sim_{\mathcal{K}_9} \sigma_{17} \sim_{\mathcal{K}_9} \sigma_{19}$, we get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_9$ for all $\sigma \in \{\sigma_{11}, \sigma_{15}, \sigma_{17}, \sigma_{19}\}$. ■

Theorem 5.12. *Let $s \approx t$ be a non-trivial equation with $G(s) \neq G(t)$ and let $s \approx t \in Id\mathcal{K}_{14}$. Then $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_{14}$ if and only if the following are satisfied:*

(i) *$R(s) = R(t)$,*

(ii) *for any $x, y \in V(s)$, $x \neq y$, $(x, y) \in E(s^d)$ or $(y, x) \in E(s^d)$, and there exists $z \in V(s)$ such that $(x, z) \in E(s^d)$ if and only if $(x, y) \in E(t^d)$ or $(y, x) \in E(t^d)$, and there exists $z' \in V(s)$ such that $(x, z') \in E(t^d)$,*

(iii) *there exists $x \in V(s)$ such that $(x, x) \in E(s^d)$ if and only if there exists $x' \in V(s)$ such that $(x', x') \in E(t^d)$.*

Theorem 5.13. *Let $s \approx t$ be a non-trivial equation with $G(s) \neq G(t)$ and let $s \approx t \in Id\mathcal{K}_{15}$. Then $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_{15}$ if and only if the following are satisfied:*

(i) *$R(s) = R(t)$,*

(ii) *for any $x \in V(s)$ there exists $y \in V(s)$ such that $(x, y) \in E(s^d)$ if and only if there exists $z \in V(s)$ such that $(x, z) \in E(t^d)$.*

(iii) *for any $x, y \in V(s)$ with $x \neq y$, $(x, y) \in E(s^d)$ or $(y, x) \in E(s^d)$, and there exists $z \in V(s)$ such that $(x, z) \in E(s^d)$ if and only if $(x, y) \in E(t^d)$ or $(y, x) \in E(t^d)$, and there exists $z' \in V(s)$ such that $(x, z') \in E(t^d)$.*

Theorem 5.14. *Let $s \approx t$ be a non-trivial equation with $G(s) \neq G(t)$ and let $s \approx t \in Id\mathcal{K}$. Then $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K} \in \{\mathcal{K}_{16}, \mathcal{K}_{17}\}$ if and only if the following conditions are satisfied:*

(i) $R(s) = R(t)$,

(ii) *for any $x \in V(s)$ there exists $y \in V(s)$ such that $(x, y) \in E(s^d)$ if and only if there exists $z \in V(s)$ such that $(x, z) \in E(t^d)$,*

(iii) *there exists $x \in V(s)$ such that $(x, x) \in E(s^d)$ if and only if there exists $y \in V(s)$ such that $(y, y) \in E(t^d)$,*

(iv) *for any $x, y \in V(s)$ with $x \neq y$, $(x, y) \in E(s^d)$ or $(y, x) \in E(s^d)$ if and only if $(x, y) \in E(t^d)$ or $(y, x) \in E(t^d)$.*

Theorem 5.15. *Let $s \approx t$ be a non-trivial equation with $G(s) \neq G(t)$ and let $s \approx t \in Id\mathcal{K}_{18}$. Then $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_{18}$ if and only if the following conditions are satisfied:*

(i) $R(s) = R(t)$,

(ii) *for any $x \in V(s)$ there exists $y \in V(s)$ such that $(x, y) \in E(s^d)$ if and only if there exists $z \in V(s)$ such that $(x, z) \in E(t^d)$,*

(iii) *for any $x, y \in V(s)$ with $x \neq y$, $(x, y) \in E(s^d)$ if and only if $(x, y) \in E(t^d)$.*

Theorem 5.16. *Let $s \approx t$ be a non-trivial equation with $G(s) \neq G(t)$ and let $s \approx t \in Id\mathcal{K}_{19}$. Then $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_{19}$ if and only if the following conditions are satisfied:*

(i) $R(s) = R(t)$,

(ii) *for any $x \in V(s)$ there exists $y \in V(s)$ such that $(x, y) \in E(s^d)$ if and only if there exists $z \in V(s)$ such that $(x, z) \in E(t^d)$,*

(iii) *there exists $x \in V(s)$ such that $(x, x) \in E(s^d)$ if and only if there exists $z \in V(s)$ such that $(z, z) \in E(t^d)$,*

(iv) *for any $x, y \in V(s)$, $x \neq y$, $(x, y) \in E(s^d)$ if and only if $(x, y) \in E(t^d)$.*

Theorem 5.17. *Let $s \approx t$ be a non-trivial equation with $G(s) \neq G(t)$ and let $s \approx t \in Id\mathcal{K}_{24}$. Then $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_{24}$ if and only if the following conditions are satisfied:*

(i) $R(s) = R(t)$,

(ii) *there exists* $x \in V(s)$ *such that* $(x,x) \in E(s^d)$ *if and only if there exists* $y \in V(s)$ *such that* $(y,y) \in E(t^d)$

(iii) *for any* $x, y \in V(s)$ *with* $x \neq y$, $(x,y) \in E(s^d)$ *or* $(y,x) \in E(s^d)$ *if and only if* $(x,y) \in E(t^d)$ *or* $(y,x) \in E(t^d)$.

Theorem 5.18. *Let* $s \approx t$ *be a non-trivial equation with* $G(s) \neq G(t)$ *and let* $s \approx t \in Id\mathcal{K}_{26}$. *Then* $s \approx t$ *is an* M_{Right}*-hyperidentity with respect to* $\mathcal{K}_{26}$ *if and only if the following conditions are satisfied:*

(i) $R(s) = R(t)$,

(ii) *for any* $x \in V(s)$ *there exists* $y \in V(s)$ *such that* $(x,y) \in E(s^d)$ *if and only if there exists* $z \in V(s)$ *such that* $(x,z) \in E(t^d)$,

(iii) *for any* $x, y \in V(s)$, $x \neq y$, $(x,y) \in E(s^d)$ *if and only if* $(x,y) \in E(t^d)$.

Theorem 5.19. *Let* $s \approx t$ *be a non-trivial equation with* $G(s) \neq G(t)$ *and let* $s \approx t \in Id\mathcal{K}_{27}$. *Then* $s \approx t$ *is an* M_{Right}*-hyperidentity with respect to* $\mathcal{K}_{27}$ *if and only if the following conditions are satisfied:*

(i) $R(s) = R(t)$,

(ii) *for any* $x \in V(s)$, $(x,x) \in E(s^d)$ *if and only if* $(x,x) \in E(t^d)$,

(iii) *for any* $x, y \in V(s)$ *with* $x \neq y$, $(x,y) \in E(s^d)$ *or* $(y,x) \in E(s^d)$ *if and only if* $(x,y) \in E(t^d)$ *or* $(y,x) \in E(t^d)$,

(iv) *for any* $x \in V(s)$, *there exists* $y \in V(s)$ *such that* $(x,y) \in E(s)$ *if and only if there exists* $z \in V(s)$ *such that* $(x,z) \in E(t)$,

(v) *for any* $x \in V(s)$, *there exists* $y \in V(s)$ *such that* $(y,x) \in E(s)$ *if and only if there exists* $z \in V(s)$ *such that* $(z,x) \in E(t)$,

(vi) *for any* $x \in V(s)$, *there exists* $y \in V(s)$ *such that* $(x,y) \in E(s^d)$ *if and only if there exists* $z \in V(s)$ *such that* $(x,z) \in E(t^d)$,

(vii) *for any* $x \in V(s)$, *there exists* $y \in V(s)$ *such that* $(y,x) \in E(s^d)$ *if and only if there exists* $z \in V(s)$ *such that* $(z,x) \in E(t^d)$.

Proof. Suppose that $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_{27}$. To prove (i), Since $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{K}_{27}$, we have $R(s) = R(t)$.

To prove (ii), For any $x \in V(s)$ suppose that $(x,x) \in E(s^d)$. By Lemma 4.2, we have $(x,x) \in E(\hat{\sigma}_{12}[s^d])$. Since $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t] \in Id\mathcal{K}_{27}$ and $\hat{\sigma}_{12}[t'^d] = \hat{\sigma}_{13}[t']$ for all $t' \in T(X)$, we get $\hat{\sigma}_{12}[s^d] \approx \hat{\sigma}_{12}[t^d] \in Id\mathcal{K}_{27}$. Hence $(x,x) \in E(\hat{\sigma}_{12}[t^d])$ and thus $(x,x) \in E(t^d)$. By the same way, we can prove the converse.

To prove (iii), for any $x, y \in V(s)$, $x \neq y$, suppose that $(x,y) \in E(s^d)$ or $(y,x) \in E(s^d)$. By Lemma 4.2, we have $(x,y) \in E(\hat{\sigma}_6[s^d])$ or $(y,x) \in E(\hat{\sigma}_6[s^d])$. Since $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t] \in Id\mathcal{K}_{27}$ and $\hat{\sigma}_6[t'^d] = \hat{\sigma}_7[t']$ for all $t' \in T(X)$, we get $\hat{\sigma}_6[s^d] \approx \hat{\sigma}_6[t^d] \in Id\mathcal{K}_{27}$. We see that $(x,y) \in E(t^d)$ or $(y,x) \in E(t^d)$. By the same way, we can prove the converse.

To prove (iv), for any $x \in V(s)$ suppose that there exists $y \in V(s)$ such that $(x,y) \in E(s)$. By Lemma 4.2, we have $(x,x) \in E(\hat{\sigma}_6[s])$. Since $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t] \in Id\mathcal{K}_{27}$, we get that $(x,x) \in E(\hat{\sigma}_6[t])$. Then, there exists $z \in V(s)$ such that $(x,z) \in E(t)$. In the same way, we can prove the converse. Similarly, since $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t] \in Id\mathcal{K}_{27}$, we prove (v).

To prove (vi), for any $x \in V(s)$ suppose that there exists $y \in V(s)$ such that $(x,y) \in E(s^d)$. By Lemma 4.2, we have $(x,x) \in E(\hat{\sigma}_6[s^d])$. Since $\hat{\sigma}_6[s^d] \approx \hat{\sigma}_6[t^d] \in Id\mathcal{K}_{27}$, we get that $(x,x) \in E(\hat{\sigma}_6[t^d])$. Hence there exists $z \in V(s)$ such that $(x,z) \in E(t^d)$. In the same way, we can prove the converse. Similarly, since $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t] \in Id\mathcal{K}_{27}$, we can prove (vii).

Conversely, assume that $s \approx t$ is an identity in $\mathcal{K}_{27}$ and that (i), (ii), (iii), (iv), (v), (vi) and (vii) are satisfied. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from M_{Right}.

If $\sigma \in \{\sigma_0, \sigma_{10}, \sigma_{12}\}$, then σ is proper and we get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_{27}$.

For σ_2 and σ_4, we have $\hat{\sigma}_2[s] = L(s^d) = L(t^d) = \hat{\sigma}_2[t]$ and $\hat{\sigma}_4[s] = L(s^d)L(s^d) = L(t^d)L(t^d) = \hat{\sigma}_4[t]$. We have $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{K}_{27}$ and $\hat{\sigma}_4[s] \approx \hat{\sigma}_4[t] \in Id\mathcal{K}_{27}$.

For σ_6, for any $x \in V(s)$ suppose that $(x,x) \in E(\hat{\sigma}_6[s])$. By Lemma 4.2, we see that there exists $y \in V(s)$ such that $(x,y) \in E(s)$. Then by (iv), there exists $z \in V(s)$ such that $(x,z) \in E(t)$. Hence $(x,x) \in E(\hat{\sigma}_6[t])$. In the same way, we can prove the converse. For any $x, y \in V(s)$ with $x \neq y$ suppose that $(x,y) \in E(\hat{\sigma}_6[s])$ or $(y,x),(x,x) \in E(\hat{\sigma}_6[s])$ or $(y,x),(y,y) \in E(\hat{\sigma}_6[s])$. If $(x,y) \in E(\hat{\sigma}_6[s])$, then $(x,y) \in E(s)$. We have $(x,y) \in E(t)$ or $(y,x),(x,x) \in E(t)$ or $(y,x),(y,y) \in E(t)$. Hence $(x,y) \in E(\hat{\sigma}_6[t])$ or $(y,x),(x,x) \in E(\hat{\sigma}_6[t])$ or $(y,x),(y,y) \in E(\hat{\sigma}_6[t])$. If $(y,x),(x,x) \in E(\hat{\sigma}_6[s])$, then $(y,x) \in E(s)$ and there exists $z \in V(s)$ such that $(x,z) \in E(s)$.

By (iv) and $s \approx t \in Id\mathcal{K}_{27}$, we get that $(y,x),(x,z') \in E(t)$ or $(x,y),(x,x),(x,z') \in E(t)$ or $(x,y),(y,y),(x,z') \in E(t)$. We have $(x,y) \in E(\hat{\sigma}_6[t])$ or $(y,x),(x,x) \in E(\hat{\sigma}_6[t])$. Hence $(x,y) \in E(\hat{\sigma}_6[t])$ or $(y,x),(x,x) \in E(\hat{\sigma}_6[t])$ or $(y,x),(y,y) \in E(\hat{\sigma}_6[t])$. If $(y,x),(y,y) \in E(\hat{\sigma}_6[s])$, then $(y,x) \in E(s)$. We have $(y,x) \in E(t)$ or $(x,y),(x,x) \in E(t)$ or $(x,y),(y,y) \in E(t)$. Hence $(y,x),(y,y) \in E(\hat{\sigma}_6[t])$ or $(x,y) \in E(\hat{\sigma}_6[t])$. Therefore $(x,y) \in E(\hat{\sigma}_6[t])$ or $(y,x),(x,x) \in E(\hat{\sigma}_6[t])$ or $(y,x),(y,y) \in E(\hat{\sigma}_6[t])$. In the same way, we can prove the converse. We get $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t] \in Id\mathcal{K}_{27}$.

Similarly, by (v), we can prove $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t] \in Id\mathcal{K}_{27}$.

Next we will show that $\hat{\sigma}_6[s^d] \approx \hat{\sigma}_6[t^d] \in Id\mathcal{K}_{27}$. For any $x \in V(s)$ suppose that $(x,x) \in E(\hat{\sigma}_6[s^d])$. By Lemma 4.2, we see that there exists $y \in V(s)$ such that $(x,y) \in E(s^d)$. Then by (vi), there exists $z \in V(s)$ such that $(x,z) \in E(t^d)$. Hence $(x,x) \in E(\hat{\sigma}_6[t^d])$. In the same way, we can prove the converse. For any $x,y \in V(s)$ with $x \neq y$ suppose that $(x,y) \in E(\hat{\sigma}_6[s^d])$ or $(y,x),(x,x) \in E(\hat{\sigma}_6[s^d])$ or $(y,x),(y,y) \in E(\hat{\sigma}_6[s^d])$. We have $(x,y) \in E(s^d)$ or $(y,x) \in E(s^d)$. By (iii), we get $(x,y) \in E(t^d)$ or $(y,x) \in E(t^d)$. By Lemma 4.2, we get that $(x,y) \in E(\hat{\sigma}_6[t^d])$ or $(y,x),(y,y) \in E(\hat{\sigma}_6[t^d])$. That is $(x,y) \in E(\hat{\sigma}_6[t^d])$ or $(y,x),(x,x) \in E(\hat{\sigma}_6[t^d])$ or $(y,x),(y,y) \in E(\hat{\sigma}_6[t^d])$. In the same way, we can prove the converse. Hence $\hat{\sigma}_6[s^d] \approx \hat{\sigma}_6[t^d] \in Id\mathcal{K}_{27}$ and thus $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t] \in Id\mathcal{K}_{27}$.

Similarly, we can prove $\hat{\sigma}_8[s^d] \approx \hat{\sigma}_8[t^d] \in Id\mathcal{K}_{27}$ and thus $\hat{\sigma}_9[s] \approx \hat{\sigma}_9[t] \in Id\mathcal{K}_{27}$.

For σ_{10}, since $s \approx t$ is a non-trivial equation with $G(s) \neq G(t)$, we see that $(x,x) \in E(\hat{\sigma}_{10}[s^d])$ and $(x,x) \in E(\hat{\sigma}_{10}[t^d])$ for all $x \in V(s)$. Then by (iii), we can prove that $\hat{\sigma}_{10}[s^d] \approx \hat{\sigma}_{10}[t^d] \in Id\mathcal{K}_{27}$ and thus $\hat{\sigma}_{11}[s] \approx \hat{\sigma}_{11}[t] \in Id\mathcal{K}_{27}$.

For σ_{12}, since $s \approx t$ is a non-trivial equation with $G(s) \neq G(t)$, by (iii), we see that the graph $G(\hat{\sigma}_{12}[s^d])$ with deleted loops and the graph $G(\hat{\sigma}_{12}[t^d])$ with deleted loops are the same graphs and for any $x,y \in V(s)$ with $x \neq y$, $(x,y) \in E(\hat{\sigma}_{12}[s^d])$ if and only if $(y,x) \in E(\hat{\sigma}_{12}[s^d])$. We get that $\hat{\sigma}_{12}[s^d] \approx \hat{\sigma}_{12}[t^d] \in Id\mathcal{K}_{27}$ and thus $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t] \in Id\mathcal{K}_{27}$.

Since $\sigma_7 \sim_{\mathcal{K}_{27}} \sigma_{15}$, $\sigma_8 \sim_{\mathcal{K}_{27}} \sigma_{16}$, $\sigma_9 \sim_{\mathcal{K}_{27}} \sigma_{17}$, $\sigma_{10} \sim_{\mathcal{K}_{27}} \sigma_{18}$ and $\sigma_{11} \sim_{\mathcal{K}_{27}} \sigma_{19}$, we get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_{27}$ for all $\sigma \in \{\sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$. ■

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