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HYPERIDENTITIES IN MANY-SORTED ALGEBRAS

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Abstract

The theory of hyperidentities generalizes the equational theory of universal algebras and is applicable in several fields of science, especially in computers sciences (see e.g., [2, 1]). The main tool to study hyperidentities is the concept of a hypersubstitution. Hypersubstitutions of many-sorted algebras were studied in [3]. On the basis of hypersubstitutions one defines a pair of closure operators which turns out to be a conjugate pair. The theory of conjugate pairs of additive closure operators can be applied to characterize solid varieties, i.e., varieties in which every identity is satisfied as a hyperidentity (see [4]). The aim of this paper is to apply the theory of conjugate pairs of additive closure operators to many-sorted algebras.

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1. Preliminaries

Hyperidentities in one-based algebras were considered by many authors (for references see e.g., [4, 2]). An identity $s \approx t$ is satisfied as a hyperidentity in the one-based algebra $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i\in I})$ of type τ if after any replacements of the operation symbols occurring in s and t by terms of the same arity the arising equation is satisfied in $\mathcal{A}$. These replacements can be described by hypersubstitutions, i.e., mappings from the set of operation symbols into the set of all terms of type τ. Hypersubstitutions cannot only be applied to terms or equations but also to algebras. This gives a pair of additive closure operators which are related to each other by the so-called conjugate property and which form a conjugate pair of additive closure operators (see [4]). A variety of one-based algebras is called solid if every identity is satisfied as a hyperidentity. Characterizations of solid varieties are based on the theory of conjugate pairs of additive closure operators. For more background see [4].

In this paper we want to apply the theory of conjugate pairs of additive closure operators to many-sorted algebras and identities and want to define hyperidentities and solid varieties of many-sorted algebras.

Many-sorted algebras occur in various branches of mathematics. They have found their way into computer science through abstract data type specifications. Many-sorted algebras, varieties and quasivarieties of many-sorted algebras are the mathematical fundament of approaches to abstract data types in programming and specification languages. For basic concepts on many-sorted algebras we refer the reader to [5].

The concept of terms in many-sorted algebras was discussed in [5]. First we want to give a slightly different version of the definitions and results from [3].

Let I be a non-empty set, let $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$, $n \in \mathbb{N}^+$, let $I^* := \bigcup_{n\geq 1} I^n$ and $\Sigma \subseteq I^* \times I$. Then we define $\Sigma_n := \Sigma \cap I^{n+1}$. For $\gamma \in \Sigma$ let $\gamma(l)$ denote the l-th component of γ. Let K_γ be a set of indices with respect to γ. If $|K_\gamma| = 1$, we will drop the index.

Definition 1.1. Let $n \in \mathbb{N}^+$ and $X^{(n)} := (X_i^{(n)})_{i\in I}$ be an I-sorted set of variables, also called an n-element I-sorted alphabet, with $X_i^{(n)} := \{x_{i1}, \ldots, x_{in}\}, i \in I$ and let $((f_\gamma)_k)_{k\in K_\gamma, \gamma\in\Sigma}$ be an indexed set of Σ-sorted operation symbols. Then for each $i \in I$ a set $W_n(i)$ which is called the set of all n-ary Σ-terms of sort i, is inductively defined as follows:

(i) $W_0^n(i) := X_i^{(n)}, i \in I,$

(ii) $W^n_{l+1}(i) := W^n_l(i) \cup \{f_\gamma(t_{k_1}, \ldots, t_{k_n}) \mid \gamma = (k_1, \ldots, k_n; i) \in \Sigma, t_{k_j} \in W^n_l(k_j), 1 \leq j \leq n\}, l \in \mathbb{N}$. (Here we inductively assume that the sets $W^n_l(i)$ are already defined for all sorts $i \in I$).

Then $W_n(i) := \bigcup_{l=0}^{\infty} W^n_l(i)$ and we set $W(i) := \bigcup_{n\in\mathbb{N}^+} W_n(i)$. Let $X_i := \bigcup_{n\in\mathbb{N}^+} X_i^{(n)}$ and $X := (X_i)_{i\in I}$. Let $W_\Sigma(X) := (W(i))_{i\in I}$. The set $W_\Sigma(X)$ is called I-sorted set of all Σ-terms and its elements are called I-sorted Σ-terms.

For any $n \in \mathbb{N}^+, i \in I$ we set $\Lambda_n(i) := \{(w; i) \in I^{n+1} \mid w \in I^n, \exists\, m \in \mathbb{N}^+, \exists\, \alpha \in \Sigma_m, \exists\, j\ (1 \leq j \leq m)(\alpha(j) = i)\}$. Let $\Lambda(i) := \bigcup_{n=1}^{\infty} \Lambda_n(i)$ and we set $\Lambda := \bigcup_{i\in I} \Lambda(i)$.

To define many-sorted hypersubstitutions we need the following superposition operation for I-sorted Σ-terms.

Definition 1.2. Let $t \in W(i), t_j \in W(k_j)$ where $1 \leq j \leq n, n \in \mathbb{N}$. Then the superposition operation

$$S_\beta : W(i) \times W(k_1) \times \cdots \times W(k_n) \to W(i)$$

for $\beta = (k_1, \ldots, k_n; i) \in \Lambda$, is defined inductively as follows:

1. If $t = x_{ij} \in X_i$, then

   1.1 $S_\beta(x_{ij}, t_1, \ldots, t_n) := x_{ij}$ for $i \neq k_j$ and

   1.2 $S_\beta(x_{ij}, t_1, \ldots, t_n) := t_j$ for $i = k_j$.

2. If $t = f_\gamma(s_1, \ldots, s_m) \in W(i)$ for $\gamma = (i_1, \ldots, i_m; i) \in \Sigma$ and $s_q \in W_n(i_q), 1 \leq q \leq m, m \in \mathbb{N}$, and if we assume that $S_{\beta_q}(s_q, t_1, \ldots, t_n)$ with $\beta_q = (k_1, \ldots, k_n; i_q) \in \Lambda$ are already defined, then $S_\beta(f_\gamma(s_1, \ldots, s_m), t_1, \ldots, t_n) := f_\gamma(S_{\beta_1}(s_1, t_1, \ldots, t_n), \ldots, S_{\beta_m}(s_m, t_1, \ldots, t_n))$.

Definition 1.3. Let $i \in I$ and $((f_\gamma)_k)_{k\in K_\gamma,\gamma\in\Sigma}$ be an indexed set of Σ-sorted operation symbols. Let $\Sigma_m(i) := \{\gamma \in \Sigma_m \mid \gamma(m+1) = i\}, m \in \mathbb{N}^+$ and let

$$\Sigma(i) := \bigcup_{m\geq 1} \Sigma_m(i).$$

Any mapping

$$\sigma_i : \{(f_\gamma)_k \mid k \in K_\gamma, \gamma \in \Sigma(i)\} \to W(i), i \in I,$$

which preserves arities, is said to be a Σ-hypersubstitution of sort i. Let $\Sigma(i)$-*Hyp* be the set of all Σ-hypersubstitutions of sort i. The I-sorted mapping $\sigma := (\sigma_i)_{i\in I}$ is called an I-sorted Σ-hypersubstitution. Let Σ-*Hyp* be the set of all I-sorted Σ-hypersubstitutions. Any I-sorted Σ-hypersubstitution σ can inductively be extended to an I-sorted mapping $\hat{\sigma} := (\hat{\sigma}_i)_{i\in I}$. The I-sorted mapping

$$\hat{\sigma} : W_\Sigma(X) \to W_\Sigma(X)$$

is defined by the following steps: For each $i \in I$ we define

(i) $\hat{\sigma}_i[x_{ij}] := x_{ij}$ for any variable $x_{ij} \in X_i$.

(ii) $\hat{\sigma}_i[f_\gamma(t_1,\ldots,t_n)] := S_\gamma(\sigma_i(f_\gamma), \hat{\sigma}_{k_1}[t_1],\ldots,\hat{\sigma}_{k_n}[t_n])$, where $\gamma = (k_1,\ldots, k_n; i) \in \Sigma$ and $t_q \in W(k_q), 1 \leq q \leq n, n \in \mathbb{N}$, assumed that $\hat{\sigma}_{k_q}[t_q]$, are already defined.

Using the extension $\hat{\sigma}_i$, we define $(\sigma_1)_i \circ_i (\sigma_2)_i := (\hat{\sigma}_1)_i \circ (\sigma_2)_i$. Then we have $((\sigma_1)_i \circ_i (\sigma_2)_i)\hat{} = (\hat{\sigma}_1)_i \circ (\hat{\sigma}_2)_i$. Together with the identity mapping $(\sigma_{id})_i$ the set $\Sigma(i)$-*Hyp* forms a monoid (see [3]).

Now we want to describe the connection between heterogeneous algebras and Σ-terms.

Let A be an I-sorted set. Then $\mathcal{A}$ is said to be a Σ-algebra if it has the form

$$\mathcal{A} = \Big(A; \big(((f_\gamma)_k\big)^{\mathcal{A}}\big)_{k\in K_\gamma,\gamma\in\Sigma}\Big)$$

where $((f_\gamma)_k)^{\mathcal{A}} : A_{k_1} \times \cdots \times A_{k_n} \to A_i$ if $\gamma = (k_1, \ldots, k_n; i) \in \Sigma$. Let $Alg(\Sigma)$ be the collection of all Σ-algebras. To connect Σ-terms with Σ-algebras we need to consider operations on I-sorted sets. Let A be an I-sorted set, $n \in \mathbb{N}^+$, $(\omega; i) \in I^* \times I$. Then ω is called input sequence on A and i is called output sort.

Definition 1.4. Let A be an I-sorted set, let $\omega = (k_1, \ldots, k_n) \in I^n, n \in \mathbb{N}^+$ be an input sequence on A. Then we define the q-th n-ary projection operation

$$e_q^{\omega,A} : A_{k_1} \times \cdots \times A_{k_n} \to A_{k_q}, 1 \leq q \leq n$$

of the input sequence ω on A by

$$e_q^{\omega,A}(a_1, \ldots, a_n) := a_q.$$

We denote by

$$O^{(\omega,i)}(A) := \{f \mid f : A_{k_1} \times \cdots \times A_{k_n} \to A_i\}$$

the set of all n-ary operations on A with input sequence ω and output sort i.

In particular we denote by

$$O^\omega(A) := (O^{(\omega,i)}(A))_{i \in I}$$

the I-sorted set of all n-ary operations on A with the same input sequence ω.

Finally we introduce

$$O(A) := \bigcup_{\omega \in I^*} O^\omega(A)$$

as the I-sorted set of all finitary operations on the I-sorted set A.

Definition 1.5. Let A be an I-sorted set and let $\omega = (s_1, \ldots, s_n), \omega' = (s'_1, \ldots, s'_m)$ be input sequences on A. Then the superposition operation

$$S^{\omega,i}_{\omega'} : O^{(\omega,i)}(A) \times O^{(\omega',s_1)}(A) \times \cdots \times O^{(\omega',s_n)}(A) \to O^{(\omega',i)}(A)$$

is defined by

$$S^{\omega,i}_{\omega'}(f, g_1, \ldots, g_n) := f[g_1, \ldots, g_n], \text{ with}$$

$$f[g_1, \ldots, g_n](a_1, \ldots, a_m) := f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m))$$

for all $(a_1, \ldots, a_m) \in A_{s'_1} \times \cdots \times A_{s'_m}$.

Using these composition operations we may consider a many-sorted algebra, which satisfies similar identities as clones in the one-sorted case.

Theorem 1.6. *Let A be an I-sorted set. Then the many-sorted algebra*

$$\left((O^{\omega}(A))_{\omega \in I^*}; \left(S^{\omega,i}_{\omega'}\right)_{(\omega,i),(\omega',i) \in I^* \times I}, \left(e^{\omega,A}_{j}\right)_{\omega \in I^*, 1 \le j \le |\omega|}\right)$$

(where $|\omega|$ is the length of the sequence ω) satisfies the following identities:

1) $S^{\omega,i}_{\omega''}\left(f, S^{\omega',s_1}_{\omega''}(g_1, h_1, \ldots, h_m), \ldots, S^{\omega',s_n}_{\omega''}(g_n, h_1, \ldots, h_m)\right)$

$$= S^{\omega',i}_{\omega''}\left(S^{\omega,i}_{\omega'}(f, g_1, \ldots g_n), h_1, \ldots, h_m\right) \text{ where}$$

$$\omega = (s_1, \ldots, s_n) \in I^n,\ \omega' = (s'_1, \ldots, s'_m) \in I^m,\ \omega'' = (s''_1, \ldots, s''_p) \in\ I^p,$$

and

$f \in O^{(\omega,i)}(A), \quad g_j \in O^{(\omega',s_j)}(A), \quad h_k \in O^{(\omega'',s'_k)}(A) \quad for \quad 1 \leq j \leq n,$

$1 \leq k \leq m,\ m, n \in \mathbb{N}.$

2) $S^{\omega,s_j}_{\omega'}\Big(e^{\omega,A}_j, g_1, \ldots, g_n\Big) = g_j \quad where \quad \omega = (s_1, \ldots, s_n) \in I^n, \omega' \in I^m,$

and

$g_j \in O^{(\omega',s_j)}(A),\ 1 \leq j \leq n, m, n \in \mathbb{N}^+.$

3) $S^{\omega,i}_{\omega}\Big(f, e^{\omega,A}_1, \ldots, e^{\omega,A}_n\Big) = f$ *where* $f \in O^{(\omega,i)}(A),\ \omega \in I^n, n \in \mathbb{N}^+.$

The proofs are similar to the proofs of the corresponding propositions for Σ-terms (see [3]).

2. I-Sorted Identities and Model Classes

Definition 2.1. Let $n \in \mathbb{N}^+$ and $X^{(n)}$ be an n-element I-sorted alphabet and let A be an I-sorted set. Let $\mathcal{A} \in Alg(\Sigma)$ be a Σ-algebra, and $t \in W_n(i), i \in I$. Let $f := (f_i)_{i \in I}$, where $f_i : X^{(n)}_i \to A_i$ is an I-sorted evaluation mapping of variables from $X^{(n)}$ by elements in A. Each mapping f_i can be extended in a canonical way to a mapping $\bar{f}_i : W_n(i) \to A_i$. Then $t^{\mathcal{A}} : A^{X^{(n)}} \to A_i$ is defined by

$$t^{\mathcal{A}}(f) := \bar{f}_i(t) \text{ for all } f \in A^{X^{(n)}},$$

where $\bar{f}_i$ is the extension of the evaluation mapping $f_i : X^{(n)}_i \to A_i$. The operation $t^{\mathcal{A}}$ is called the n-ary Σ-term operation on $\mathcal{A}$ induced by the n-ary Σ-term t of sort i. We have $x^{\mathcal{A}}_{k_q q} = e^{\omega,A}_q, 1 \leq q \leq n$, where $\omega = (k_1, \ldots, k_n)$, since for $f \in A^{X^{(n)}}$ we have

$$
\begin{aligned}
x_{k_q q}^{\mathcal{A}}(f) &= \bar{f}_{k_q}(x_{k_q q}) \\
&= f_{k_q}(x_{k_q q}) \\
&= e_q^{\omega,A}(a_1, \ldots, a_{q-1}, f_{k_q}(x_{k_q q}), a_{q+1}, \ldots, a_n)
\end{aligned}
$$

for all $a_j \in A_{k_j}$ such that $j \in \{1, \ldots, q-1, q+1, \ldots, n\}$.

Let $W^A(i)$ be the set of all Σ-term operations on $\mathcal{A}$ induced by the Σ-terms of sort i. We set $W_\Sigma^{\mathcal{A}}(X) := (W^A(i))_{i\in I}$ and call it the I-sorted set of Σ-term operations on $\mathcal{A}$ induced by the Σ-terms.

Definition 2.2. Let $t \in W(i)$, $t_j \in W(k_j)$ where $1 \leq j \leq n, n \in \mathbb{N}$. Then the superposition operation

$$S_\alpha^A : W^A(i) \times W^A(k_1) \times \cdots \times W^A(k_n) \to W^A(i)$$

where $\alpha = (k_1, \ldots, k_n; i) \in \Lambda$, is inductively defined in the following way:

1) If $t = x_{ij} \in X_i$, then

 1.1) $S_\alpha^A\Big(x_{ij}^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}\Big) := x_{ij}^{\mathcal{A}}$ for $i \neq k_j$ and

 1.2) $S_\alpha^A\Big(x_{ij}^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}\Big) := t_j^{\mathcal{A}}$ for $i = k_j$.

2) If $t = f_\gamma(s_1, \ldots, s_m) \in W(i)$ where $\gamma = (i_1, \ldots, i_m; i) \in \Sigma$, $s_q \in W(i_q), 1 \leq q \leq m, m \in \mathbb{N}$ and assume that $S_{\alpha_q}^A(s_q^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}})$, where $\alpha_q = (k_1, \ldots, k_n; i_q) \in \Lambda$, are already defined, then

$$S_{\alpha}^{\mathcal{A}}\Big((f_{\gamma}(s_1,\ldots,s_m))^{\mathcal{A}},t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}}\Big)$$

$$:= f_{\gamma}^{\mathcal{A}}\Big(S_{\alpha_1}^{\mathcal{A}}(s_1^{\mathcal{A}},t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}}),\ldots,S_{\alpha_m}^{\mathcal{A}}(s_m^{\mathcal{A}},t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}})\Big).$$

Example 2.3. Let $I = \{1,2\}$, $X^{(2)} = (X_i^{(2)})_{i\in I}$, $\Sigma = \{(1,2;1),(2,1;2)\}$. Let $\mathcal{A}$ be a Σ-algebra and let $t = f_{(1,2;1)}(f_{(1,2;1)}(x_{11},x_{21}),f_{(2,1;2)}(x_{22},x_{11})) \in W(1)$, $t_1 \in W(2)$, and $t_2 \in W(1)$. Then

$$S_{(2,1;1)}^{\mathcal{A}}\Big(t^{\mathcal{A}}t_1^{\mathcal{A}}t_2^{\mathcal{A}}\Big) = S_{(2,1;1)}^{\mathcal{A}}\Big((f_{(1,2;1)}(f_{(1,2;1)}(x_{11},x_{21}),f_{(2,1;2)}(x_{22},x_{11})))^{\mathcal{A}}t_1^{\mathcal{A}}t_2^{\mathcal{A}}\Big)$$

$$= f_{(1,2;1)}^{\mathcal{A}}\Big(S_{(2,1;1)}^{\mathcal{A}}((f_{(1,2;1)}(x_{11},x_{21}))^{\mathcal{A}}t_1^{\mathcal{A}}t_2^{\mathcal{A}}\Big),$$

$$S_{(1,2;2)}^{\mathcal{A}}\Big((f_{(2,1;2)}(x_{22},x_{11}))^{\mathcal{A}},t_1^{\mathcal{A}},t_2^{\mathcal{A}}\Big)\Big)$$

$$= f_{(1,2;1)}^{\mathcal{A}}\Big(f_{(1,2;1)}^{\mathcal{A}}\Big(S_{(2,1;1)}^{\mathcal{A}}\Big(x_{11}^{\mathcal{A}},t_1^{\mathcal{A}},t_2^{\mathcal{A}}\Big),S_{(2,1;2)}^{\mathcal{A}}\Big(x_{21}^{\mathcal{A}},t_1^{\mathcal{A}},t_2^{\mathcal{A}}\Big)\Big),$$

$$f_{(2,1;2)}^{\mathcal{A}}\Big(S_{(2,1;2)}^{\mathcal{A}}\Big(x_{22}^{\mathcal{A}},t_1^{\mathcal{A}},t_2^{\mathcal{A}}\Big),S_{(2,1;1)}^{\mathcal{A}}\Big(x_{11}^{\mathcal{A}},t_1^{\mathcal{A}},t_2^{\mathcal{A}}\Big)\Big)\Big)$$

$$= f_{(1,2;1)}^{\mathcal{A}}\Big(f_{(1,2;1)}^{\mathcal{A}}\Big(x_{11}^{\mathcal{A}},t_1^{\mathcal{A}}\Big),f_{(2,1;2)}^{\mathcal{A}}\Big(x_{22}^{\mathcal{A}},x_{11}^{\mathcal{A}}\Big)\Big).$$

Proposition 2.4. *Let $\mathcal{A}$ be a Σ-algebra and $f_{\gamma}(t_1,\ldots,t_n) \in W_n(i)$ where $\gamma = (i_1,\ldots,i_n,i) \in \Sigma$, $t_q \in W_n(i_q), 1 \leq q \leq n, n \in \mathbb{N}$. Then*

$$\Big(f_{\gamma}\Big(t_1,\ldots,t_n\Big)\Big)^{\mathcal{A}} = f_{\gamma}^{\mathcal{A}}\Big(t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}}\Big).$$

Proof. Let $f \in A^{X^{(n)}}$, then

$$\begin{aligned}\Big(f_\gamma\big(t_1,\ldots,t_n\big)\Big)^{\mathcal{A}}(f) &= \bar{f}_i\Big(f_\gamma\big(t_1,\ldots,t_n\big)\Big)\\ &= f_\gamma^{\mathcal{A}}\Big(\bar{f}_{i_1}(t_1),\ldots,\bar{f}_{i_n}(t_n)\Big)\\ &= f_\gamma^{\mathcal{A}}\Big(t_1^{\mathcal{A}}(f),\ldots,t_n^{\mathcal{A}}(f)\Big)\\ &= f_\gamma^{\mathcal{A}}\Big(t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}}\Big)(f).\end{aligned}$$ ■

Lemma 2.5. *Let $\mathcal{A}$ be a Σ-algebra. For $t \in W(i), t_j \in W(k_j)$, $1 \leq j \leq n, n \in \mathbb{N}$ we have:*

$$S_\alpha^{\mathcal{A}}\Big(t^{\mathcal{A}}, t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}}\Big) = \Big(S_\alpha(t,t_1,\ldots,t_n)\Big)^{\mathcal{A}}$$

where $\alpha = (k_1,\ldots,k_n;i) \in \Lambda$.

Proof. We will give a proof by induction on the complexity of the Σ-term t.

1) If $t = x_{ij} \in X_i$, then

1.1) for $i \neq k_j$,

$$\begin{aligned}S_\alpha^{\mathcal{A}}\Big(t^{\mathcal{A}}, t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}}\Big) &= S_\alpha^{\mathcal{A}}\Big(x_{ij}^{\mathcal{A}}, t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}}\Big)\\ &= x_{ij}^{\mathcal{A}}\\ &= \Big(S_\alpha(x_{ij},t_1,\ldots,t_n)\Big)^{\mathcal{A}}\\ &= \Big(S_\alpha(t,t_1,\ldots,t_n)\Big)^{\mathcal{A}},\end{aligned}$$

1.2) and for $i = k_j$,

$$S_\alpha^A\Big(t^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}\Big) = S_\alpha^A\Big(x_{ij}^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}\Big)$$

$$= t_j^{\mathcal{A}}$$

$$= \Big(S_\alpha(x_{ij}, t_1, \ldots, t_n)\Big)^{\mathcal{A}}$$

$$= \Big(S_\alpha(t, t_1, \ldots, t_n)\Big)^{\mathcal{A}}.$$

2) If $t = f_\gamma(s_1, \ldots, s_m) \in W(i)$, where $\gamma = (i_1, \ldots, i_m; i) \in \Sigma$ and $s_q \in W(i_q), 1 \leq q \leq m, m \in \mathbb{N}$, and if we assume that the equations

$$S_{\alpha_q}^{\mathcal{A}}\Big(s_q^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}\Big) = \Big(S_{\alpha_q}(s_q, t_1, \ldots, t_n)\Big)^{\mathcal{A}},$$

where $\alpha_q = (k_1, \ldots, k_n; i_q) \in \Lambda$, are satisfied, then for $f \in A^{X^{(n)}}$ we have

$$S_\alpha^A\Big(t^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}\Big)(f)$$

$$= S_\alpha^A\Big(\Big(f_\gamma(s_1, \ldots, s_m)\Big)^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}\Big)(f)$$

$$= f_\gamma^{\mathcal{A}}\Big(S_{\alpha_1}^A\Big((s_1^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}\Big)(f), \ldots, S_{\alpha_m}^A\Big(s_m^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}\Big)(f)\Big)$$

$$= f_\gamma^{\mathcal{A}}\Big(\Big(S_{\alpha_1}(s_1, t_1, \ldots, t_n)\Big)^{\mathcal{A}}(f), \ldots, \Big(S_{\alpha_m}(s_m, t_1, \ldots, t_n)\Big)^{\mathcal{A}}(f)\Big)$$

$$= f_\gamma^{\mathcal{A}}\Big(\bar{f}_{i_i}\Big(S_{\alpha_1}(s_1, t_1, \ldots, t_n)\Big), \ldots, \bar{f}_{i_m}\Big(S_{\alpha_m}(s_m, t_1, \ldots, t_n)\Big)\Big)$$

$$= \bar{f}_i\Big(f_\gamma\Big(S_{\alpha_1}(s_1,t_1,\dots,t_n),\dots,S_{\alpha_m}(s_m,t_1,\dots,t_n)\Big)\Big)$$

$$= \Big(f_\gamma\Big(S_{\alpha_1}(s_1,t_1,\dots,t_n),\dots,S_{\alpha_m}(s_m,t_1,\dots,t_n)\Big)\Big)^{\mathcal{A}}(f)$$

$$= \Big(S_\alpha\Big(f_\gamma(s_1,\dots,s_m),t_1,\dots,t_n\Big)\Big)^{\mathcal{A}}(f)$$

$$= \Big(S_\alpha\Big(t,t_1,\dots,t_n\Big)\Big)^{\mathcal{A}}(f). \qquad \blacksquare$$

Now we can define equations and identities.

Definition 2.6. A Σ-equation of sort i in X is a pair (s_i,t_i) of elements from $W(i), i \in I$. Such pairs are more commonly written as $s_i \approx_i t_i$. The Σ-equation $s_i \approx_i t_i$ is said to be a Σ-identity of sort i in the Σ-algebra $\mathcal{A}$ if $s_i^{\mathcal{A}} = t_i^{\mathcal{A}}$, that is, if the Σ-term operations induced by s_i and t_i, respectively, on the Σ-algebra $\mathcal{A}$ are equal.

In this case we also say that the Σ-equation $s_i \approx_i t_i$ is satisfied or modelled by the Σ-algebra $\mathcal{A}$, and write $\mathcal{A} \models_i s_i \approx_i t_i$. If the Σ-equation $s_i \approx_i t_i$ is satisfied by every Σ-algebra $\mathcal{A}$ of a class K_0 of Σ-algebras, we write $K_0 \models_i s_i \approx_i t_i$. For a set $F(i)$ of equations of sort i we write $\mathcal{A} \models_i F(i)$ if $\mathcal{A} \models_i s_i \approx_i t_i$ for all $(s_i,t_i) \in F(i)$.

Example 2.7. Let $I = \{1,2\}$, $X^{(2)} := (X_i^{(2)})_{i\in I}$ be a 2-element I-sorted alphabet, and $\Sigma = \{(1,1;1),(2,1;1)\}$. Let $\mathcal{V} = (A; f^{\mathcal{V}}_{(2,1;1)}, f^{\mathcal{V}}_{(1,1;1)})$ where $f^{\mathcal{V}}_{(2,1;1)}, f^{\mathcal{V}}_{(1,1;1)}$ correspond to $\circ$, $+$, respectively, and $A := (V,\mathbb{R})$ is the universe of a real vector space. Then the Σ-equation

$$f_{(2,1;1)}\Big(x_{21}, f_{(1,1;1)}(x_{11},x_{12})\Big)$$

$$\approx_1 f_{(1,1;1)}\Big(f_{(2,1;1)}(x_{21},x_{11}), f_{(2,1;1)}(x_{21},x_{12})\Big) \in W(1)^2$$

is a Σ-identity of sort 1 in $\mathcal{V}$, that is,

$$\mathcal{V} \models_1 f_{(2,1;1)}\Big(x_{21}, f_{(1,1;1)}(x_{11}, x_{12})\Big)$$

$$\approx_1 f_{(1,1;1)}\Big(f_{(2,1;1)}(x_{21}, x_{11}), f_{(2,1;1)}(x_{21}, x_{12})\Big)$$

since for $f \in A^{X^{(2)}}$ we have

$$\begin{aligned} f_{(2,1;1)}\Big(x_{21}, f_{(1,1;1)}(x_{11}, x_{12})\Big)^{\mathcal{V}}(f) &= \bar{f}_1\Big(f_{(2,1;1)}(x_{21}, f_{(1,1;1)}(x_{11}, x_{12}))\Big) \\ &= f^{\mathcal{V}}_{(2,1;1)}\Big(\bar{f}_2(x_{21}), \bar{f}_1\Big(f_{(1,1;1)}(x_{11}, x_{12})\Big)\Big) \\ &= f^{\mathcal{V}}_{(2,1;1)}\Big(\bar{f}_2(x_{21}), f^{\mathcal{V}}_{(1,1;1)}\Big(\bar{f}_1(x_{11}), \bar{f}_1(x_{12})\Big)\Big) \\ &= f^{\mathcal{V}}_{(2,1;1)}\Big(f_2(x_{21}), f^{\mathcal{V}}_{(1,1;1)}\Big(f_1(x_{11}), f_1(x_{12})\Big)\Big) \end{aligned}$$

and

$$\begin{aligned} &f_{(1,1;1)}\Big(f_{2,1;1}(x_{21}, x_{11}), f_{(2,1;1)}(x_{21}, x_{12})\Big)^{\mathcal{V}}(f) \\ &\quad= \bar{f}_1\Big(f_{(1,1;1)}\Big(f_{(2,1;1)}(x_{21}, x_{11}),\ f_{(2,1;1)}(x_{21}, x_{12})\Big)\Big) \\ &\quad= f^{\mathcal{V}}_{(1,1;1)}\Big(\bar{f}_1\Big(f_{(2,1;1)}(x_{21}, x_{11})\Big),\ \bar{f}_1\Big(f_{(2,1;1)}(x_{21}, x_{12})\Big)\Big) \\ &\quad= f^{\mathcal{V}}_{(1,1;1)}\Big(f^{\mathcal{V}}_{(2,1;1)}\Big(\bar{f}_2(x_{21}), \bar{f}_1(x_{11})\Big),\ f^{\mathcal{V}}_{(2,1;1)}\Big(\bar{f}_2(x_{21}), \bar{f}_1(x_{12})\Big)\Big) \\ &\quad= f^{\mathcal{V}}_{(1,1;1)}\Big(f^{\mathcal{V}}_{(2,1;1)}\Big(f_2(x_{21}), f_1(x_{11})\Big),\ f^{\mathcal{V}}_{(2,1;1)}\Big(f_2(x_{21}), f_1(x_{12}))\Big)\Big). \end{aligned}$$

Therefore,

$$\Big(f_{(2,1;1)}\Big(x_{21}, f_{(1,1;1)}(x_{11}, x_{12})\Big)\Big)^{\mathcal{V}}$$
$$= \Big(f_{(1,1;1)}\Big(f_{2,1;1}(x_{21}, x_{11}), f_{(2,1;1)}(x_{21}, x_{12})\Big)\Big)^{\mathcal{V}}.$$

Now we extend the usual Galois-connection between identities and algebras to the many-sorted case.

Let $K_0 \subseteq Alg(\Sigma)$ and $L(i) \subseteq W(i)^2$. Then a mapping

$$\Sigma(i)\text{-}Id : P(Alg(\Sigma)) \to P\Big(W(i)^2\Big)$$

is defined by

$$\Sigma(i)\text{-}IdK_0 := \Big\{(s_i, t_i) \in W(i)^2 \mid (\forall \mathcal{A} \in K_0)(\mathcal{A} \models_i s_i \approx_i t_i)\Big\}$$

and a mapping $\Sigma(i)\text{-}Mod : P(W(i)^2) \to P(Alg(\Sigma))$ is defined by

$$\Sigma(i)\text{-}ModL(i) := \{\mathcal{A} \in Alg(\Sigma) \mid (\forall (s_i, t_i) \in L(i))(\mathcal{A} \models_i s_i \approx_i t_i)\}.$$

In the next propositions, we will show that these two mappings satisfy the Galois-connection properties.

Proposition 2.8. *Let $i \in I$ and let $K_0, K_1, K_2 \subseteq Alg(\Sigma)$. Then*

(1) $K_1 \subseteq K_2 \Rightarrow \Sigma(i)\text{-}IdK_2 \subseteq \Sigma(i)\text{-}IdK_1$,

(2) $K_0 \subseteq \Sigma(i)\text{-}Mod\Sigma(i)\text{-}IdK_0$.

Proof.

(1) Assume that $K_1 \subseteq K_2$ and let $s_i \approx_i t_i \in \Sigma(i)\text{-}IdK_2$. Then for all $\mathcal{A} \in K_2$, we have $\mathcal{A} \models_i s_i \approx_i t_i$. Because of $K_1 \subseteq K_2$, we obtain $\mathcal{A} \models_i s_i \approx_i t_i$, for all $\mathcal{A} \in K_1$. This means that $s_i \approx_i t_i \in \Sigma(i)\text{-}IdK_1$, and then $\Sigma(i)\text{-}IdK_2 \subseteq \Sigma(i)\text{-}IdK_1$.

(2) Let $\mathcal{A} \in K_0$. Then $\mathcal{A} \models_i \Sigma(i)\text{-}IdK_0$, means that $\mathcal{A} \in \Sigma(i)\text{-}Mod\Sigma(i)\text{-}IdK_0$ and then $K_0 \subseteq \Sigma(i)\text{-}Mod\Sigma(i)\text{-}IdK_0$.

■

Proposition 2.9. *Let $L(i), L_1(i), L_2(i) \subseteq W(i)^2$ be subsets of the set of all Σ-equations of sort $i \in I$. Then*

(1) $L_1(i) \subseteq L_2(i) \Rightarrow \Sigma(i)\text{-}ModL_2(i) \subseteq \Sigma(i)\text{-}ModL_1(i)$,

(2) $L(i) \subseteq \Sigma(i)\text{-}Id\Sigma(i)\text{-}ModL(i)$.

Proof.

(1) Assume that $L_1(i) \subseteq L_2(i)$ and let $\mathcal{A} \in \Sigma(i)\text{-}ModL_2(i)$. Then $\mathcal{A} \models_i s_i \approx_i t_i$ for all $s_i \approx_i t_i \in L_2(i)$, but we have $L_1(i) \subseteq L_2(i)$, so that $\mathcal{A} \models_i s_i \approx_i t_i$ for all $s_i \approx_i t_i \in L_1(i)$. It follows that $\mathcal{A} \in \Sigma(i)\text{-}ModL_1(i)$ and then $\Sigma(i)\text{-}ModL_2(i) \subseteq \Sigma(i)\text{-}ModL_1(i)$.

(2) Let $s_i \approx_i t_i \in L(i)$. Then we have $\Sigma(i)\text{-}ModL(i) \models_i s_i \approx_i t_i$, that is $s_i \approx_i t_i \in \Sigma(i)\text{-}Id\Sigma(i)\text{-}ModL(i)$ and then $L(i) \subseteq \Sigma(i)\text{-}Id\Sigma(i)\text{-}ModL(i)$.

■

From both propositions, we have that $(\Sigma(i)\text{-}Mod, \Sigma(i)\text{-}Id)$ is a Galois connection between $Alg(\Sigma)$ and $W(i)^2$ with respect to the relation

$$\models_i := \Big\{(\mathcal{A}, (s_i, t_i)) \in Alg(\Sigma) \times W(i)^2 \mid \mathcal{A} \models_i s_i \approx_i t_i\Big\}.$$

The fixed points with respect to the closure operator $\Sigma(i)\text{-}Mod\Sigma(i)\text{-}Id$ are called Σ-varieties of sort i and the fixed points with respect to the closure operator $\Sigma(i)\text{-}Id\Sigma(i)\text{-}Mod$ are called Σ-equational theories of sort i.

3. Application of Σ-Hypersubstitutions

Now we apply Σ-hypersubstitutions to many-sorted algebras and to many-sorted equations.

Definition 3.1. Let A be an I-sorted set, let $\mathcal{A} := (A; (((f_\gamma)_k)^{\mathcal{A}})_{k\in K_\gamma,\gamma\in\Sigma})$ be a Σ-algebra and let $\sigma \in \Sigma$-Hyp. Then we define the Σ-algebra

$$\sigma(\mathcal{A}) := \left(A; \left((\sigma_i((f_\gamma)_k))^{\mathcal{A}}\right)_{k\in K_\gamma,\gamma\in\Sigma(i),i\in I}\right).$$

This Σ-algebra is called the Σ-algebra derived from $\mathcal{A}$ and σ, for short derived Σ-algebra.

For illustration we consider the following example.

Example 3.2. Let $I = \{1,2\}$, $\Sigma = \{(1,2,1),(2,1,2)\}$, $K_{(1,2,1)} = \{1,2\}$, $A = (A_1, A_2)$, $\mathcal{A} = ((A_1,A_2); ((f_{(1,2,1)})_1)^{\mathcal{A}}, ((f_{(1,2,1)})_2)^{\mathcal{A}}, f^{\mathcal{A}}_{(2,1,2)})$. Let $\sigma = (\sigma_1, \sigma_2) \in \Sigma$-$Hyp$. Then we have

$$\begin{aligned}&\sigma(\mathcal{A})\\ &= \Big((A_1,A_2);\ \Big(\sigma_1((f_{(1,2,1)}))_1\Big)^{\mathcal{A}},\ \Big(\sigma_1((f_{(1,2,1)}))_2\Big)^{\mathcal{A}},\ \Big(\sigma_2(f_{(2,1,2)})\Big)^{\mathcal{A}}\Big).\end{aligned}$$

Theorem 3.3. *Let A be an I-sorted set and $\mathcal{A} := (A; (((f_\gamma)_k)^{\mathcal{A}})_{k\in K_\gamma,\gamma\in\Sigma})$ be a Σ-algebra. Let $\sigma \in \Sigma$-Hyp and $t \in W(i), i \in I$. Then $t^{\sigma(\mathcal{A})} = (\hat{\sigma}_i[t])^{\mathcal{A}}$.*

Proof. We will give a proof by induction on the complexity of the Σ-term t.

1) If $t = x_{ij} \in X_i$ where $1 \le j \le n, n \in \mathbb{N}$, then for $f \in A^{X^{(n)}}$ we have

$$\begin{aligned}
t^{\sigma(\mathcal{A})}(f) &= x_{ij}^{\sigma(\mathcal{A})}(f) \\
&= \bar{f}_i(x_{ij}) \\
&= x_{ij}^{\mathcal{A}}(f) \\
&= (\hat{\sigma}_i[x_{ij}])^{\mathcal{A}}(f) \\
&= (\hat{\sigma}_i[t])^{\mathcal{A}}(f).
\end{aligned}$$

2) If $t = f_\gamma(s_1, \ldots, s_m) \in W(i)$ where $\gamma = (i_i, \ldots, i_m; i) \in \Sigma, s_q \in W(i_q)$, $1 \leq q \leq m, m \in \mathbb{N}$ and assume that $s_q^{\sigma(\mathcal{A})} = \hat{\sigma}_{i_q}[s_q]^{\mathcal{A}}$ are satisfied, then for $f \in A^{X^{(n)}}$ we have

$$\begin{aligned}
t^{\sigma(\mathcal{A})}(f) &= (f_\gamma(s_1, \ldots, s_m))^{\sigma(\mathcal{A})}(f) \\
&= \bar{f}_i(f_\gamma(s_1, \ldots, s_m)) \\
&= f_\gamma^{\sigma(\mathcal{A})}(\bar{f}_{i_1}(s_1), \ldots, \bar{f}_{i_m}(s_m)) \\
&= f_\gamma^{\sigma(\mathcal{A})}(s_1^{\sigma(\mathcal{A})}(f), \ldots, s_m^{\sigma(\mathcal{A})}(f)) \\
&= \sigma_i(f_\gamma)^{\mathcal{A}}(\hat{\sigma}_{i_1}[s_1]^{\mathcal{A}}(f), \ldots, \hat{\sigma}_{i_m}[s_m]^{\mathcal{A}}(f)) \\
&= \sigma_i(f_\gamma)^{\mathcal{A}}(\hat{\sigma}_{i_1}[s_1]^{\mathcal{A}}, \ldots, \hat{\sigma}_{i_m}[s_m]^{\mathcal{A}})(f) \\
&= S_\gamma^{A}(\sigma_i(f_\gamma)^{\mathcal{A}}, \hat{\sigma}_{i_1}[s_1]^{\mathcal{A}}, \ldots, \hat{\sigma}_{i_m}[s_m]^{\mathcal{A}})(f) \\
&= (S_\gamma(\sigma_i(f_\gamma), \hat{\sigma}_{i_1}[s_1], \cdots, \hat{\sigma}_{i_m}[s_m]))^{\mathcal{A}}(f) \text{ by Lemma 2.5} \\
&= (\hat{\sigma}_i[f_\gamma(s_1, \ldots, s_m)])^{\mathcal{A}}(f) \\
&= (\hat{\sigma}_i[t])^{\mathcal{A}}(f).
\end{aligned}$$

■

Lemma 3.4. *Let $\mathcal{A} \in Alg(\Sigma)$, $\sigma_1, \sigma_2 \in \Sigma$-$Hyp$. Then we have*

$$\Big((\sigma_1)_i(f_\gamma)\Big)^{\sigma_2(\mathcal{A})} = \Big(((\sigma_2)_i \circ_i (\sigma_1)_i)(f_\gamma)\Big)^{\mathcal{A}},$$

for $\gamma \in \Sigma(i), i \in I$.

Proof. By Theorem 3.3, we have

$$\begin{aligned}\Big((\sigma_1)_i(f_\gamma)\Big)^{\sigma_2(\mathcal{A})} &= \Big((\hat{\sigma}_2)_i[(\sigma_1)_i(f_\gamma)]\Big)^{\mathcal{A}} \\ &= \Big(((\hat{\sigma}_2)_i \circ (\sigma_1)_i)(f_\gamma)\Big)^{\mathcal{A}} \\ &= \Big(((\sigma_2)_i \circ_i (\sigma_1)_i)(f_\gamma)\Big)^{\mathcal{A}}.\end{aligned}$$

■

Let σ_1, σ_2 be elements in Σ-Hyp. Then we set $\sigma_1 \diamond \sigma_2 := ((\sigma_1)_i \circ_i (\sigma_2)_i)_{i \in I}$.

Lemma 3.5. *Let A be an I-sorted set, let $\mathcal{A} = (A; (((f_\gamma)_k)^{\mathcal{A}})_{k \in K_\gamma, \gamma \in \Sigma})$ be a Σ-algebra, and $\sigma_1, \sigma_2 \in \Sigma$-$Hyp$. Then we have*

$$\sigma_1(\sigma_2(\mathcal{A})) = (\sigma_2 \diamond \sigma_1)(\mathcal{A}).$$

Proof. By Lemma 3.4, we have

$$\begin{aligned}\sigma_1(\sigma_2(\mathcal{A})) &= \Big(A; \Big(((\sigma_1)_i((f_\gamma)_k)^{\sigma_2(\mathcal{A})}\Big)_{k \in K_\gamma, \gamma \in \Sigma(i), i \in I}\Big) \\ &= \Big(A; \Big((((\sigma_2)_i \circ_i (\sigma_1)_i)((f_\gamma)_k)^{\mathcal{A}}\Big)_{k \in K_\gamma, \gamma \in \Sigma(i), i \in I}\Big) \\ &= (\sigma_2 \diamond \sigma_1)(\mathcal{A}).\end{aligned}$$

■

Theorem 3.6. *Let A be an I-sorted set, $\mathcal{A} := (A; (((f_\alpha)_k)^{\mathcal{A}})_{k\in K_\alpha,\alpha\in\Sigma})$, and $\sigma_{id} \in \Sigma$-Hyp. Then we have*

$$\sigma_{id}(\mathcal{A}) = \mathcal{A}.$$

Proof. We will show that $((\sigma_{id})_i(f_\alpha)_k)^{\mathcal{A}} = f_\alpha^{\mathcal{A}}$ for all $k \in K_\alpha, \alpha \in \Sigma$. Assume that $\alpha = (k_1, \ldots, k_n; i) \in \Sigma$ and $\omega = (k_1, \ldots, k_n) \in I^n$. Then

$$\begin{aligned}\Big((\sigma_{id})_i(f_\alpha)\Big)^{\mathcal{A}} &= \Big(f_\alpha(x_{k_1 1}, \ldots, x_{k_n n})\Big)^{\mathcal{A}} \\ &= f_\alpha^{\mathcal{A}}\Big(x_{k_1 1}^{\mathcal{A}}, \ldots, x_{k_n n}^{\mathcal{A}}\Big) \\ &= f_\alpha^{\mathcal{A}}\Big(e_1^{\omega,A}, \ldots, e_n^{\omega,A}\Big) \\ &= f_\alpha^{\mathcal{A}}.\end{aligned}$$

■

Definition 3.7. A Σ-algebra $\mathcal{A}$ is said to hypersatisfy the Σ-identity $s_i \approx_i t_i$ of sort $i \in I$, if for every Σ-hypersubstitution of sort i, i.e., $\sigma_i \in \Sigma(i)$-Hyp, the Σ-identity $\hat{\sigma}_i[s_i] \approx_i \hat{\sigma}_i[t_i]$ holds in $\mathcal{A}$.

In this case we say that the Σ-identity $s_i \approx_i t_i$ of sort i is satisfied as a Σ-hyperidentity of sort i in $\mathcal{A}$ and write $\mathcal{A} \underset{\Sigma-hyp}{\models_i} s_i \approx_i t_i$, that is

$$\mathcal{A} \underset{\Sigma-hyp}{\models_i} s_i \approx_i t_i :\Leftrightarrow \forall \sigma_i \in \Sigma(i)\text{-}Hyp \ \ (\mathcal{A} \models_i \hat{\sigma}_i[s_i] \approx_i \hat{\sigma}_i[t_i]).$$

Let us consider the following example.

Example 3.8. Let $I=\{1,2\}$, $X^{(2)}:=(X_i^{(2)})_{i\in I}$ and let $\Sigma=\{(1,1;1),(2,2;2)\}$. Let $\mathcal{B}_i := (B_i; \circ_i)$ be bands. Then $f_{(i,i,i)}(x_{ij}, x_{ij}) \approx_i x_{ij}$ are hyperidentities in $\mathcal{B}_i, i \in I$. Let $\mathcal{B} := (B; \circ)$ be a double band, where $B := (B_i)_{i\in I}$, $\circ := (\circ_i)_{i\in I}$. Then $f_{(i,i,i)}(x_{ij}, x_{ij}) \approx_i x_{ij}$ are Σ-hyperidentities of sort i in $\mathcal{B}$.

Let $K_0 \subseteq Alg(\Sigma)$ be a set of Σ-algebras, and let $L(i) \subseteq W(i)^2$ be a set of Σ-equations of sort i. Then we define a mapping

$$H\Sigma(i)\text{-}Id : P(Alg(\Sigma)) \to P\Big(W(i)^2\Big)$$

by

$$H\Sigma(i)\text{-}IdK_0 := \Big\{(s_i, t_i) \in W(i)^2 \Big| (\forall \mathcal{A} \in K_0)\Big(\mathcal{A} \underset{\Sigma\text{-}hyp}{\models_i} s_i \approx_i t_i\Big)\Big\}$$

and a mapping $H\Sigma(i)\text{-}Mod : P(W(i)^2) \to P(Alg(\Sigma))$ by

$$H\Sigma(i)\text{-}ModL(i) := \Big\{\mathcal{A} \in Alg(\Sigma) \Big| (\forall (s_i, t_i) \in L(i))\Big(\mathcal{A} \underset{\Sigma\text{-}hyp}{\models_i} s_i \approx_i t_i\Big)\Big\}.$$

We get that $(H\Sigma(i)\text{-}Mod, H\Sigma(i)\text{-}Id)$ is also a Galois connection between $Alg(\Sigma)$ and $W(i)^2$ with respect to the relation

$$\underset{\Sigma-hyp}{\models_i} := \Big\{(\mathcal{A}, (s_i, t_i)) \in Alg(\Sigma) \times W(i)^2 \Big| \mathcal{A} \underset{\Sigma-hyp}{\models_i} s_i \approx_i t_i\Big\}.$$

Definition 3.9. Let $K_0 \subseteq Alg(\Sigma)$ be a subclass of Σ-algebras and let $L(i) \subseteq W(i)^2$ be a set of Σ-equations of sort i. Then we set

$$\chi^{\Sigma\text{-}E(i)}[s_i \approx_i t_i] := \{\hat{\sigma}_i[s_i] \approx_i \hat{\sigma}_i[t_i] \mid \sigma_i \in \Sigma(i)\text{-}Hyp\}$$

and

$$\chi^{\Sigma\text{-}A}[\mathcal{A}] := \{\sigma(\mathcal{A}) \mid \sigma \in \Sigma\text{-}Hyp\}.$$

We define two operators

$$\chi^{\Sigma\text{-}E(i)} : P\Big(W(i)^2\Big) \to P\Big(W(i)^2\Big)$$

by

$$\chi^{\Sigma\text{-}E(i)}[L(i)] := \bigcup_{s_i \approx_i t_i \in L(i)} \chi^{\Sigma\text{-}E(i)}[s_i \approx_i t_i]$$

and

$$\chi^{\Sigma\text{-}A} : P(Alg(\Sigma)) \to P(Alg(\Sigma))$$

by

$$\chi^{\Sigma\text{-}A}[K_0] := \bigcup_{\mathcal{A} \in K_0} \chi^{\Sigma\text{-}A}[\mathcal{A}].$$

Proposition 3.10. *Let $L(i), L_k(i) \subseteq W(i)^2$ be sets of Σ-equations of sort $i \in I$ with $k = 1, 2$. Then*

(i) $L(i) \subseteq \chi^{\Sigma-E(i)}[L(i)]$,

(ii) $L_1(i) \subseteq L_2(i) \Rightarrow \chi^{\Sigma-E(i)}[L_1(i)] \subseteq \chi^{\Sigma-E(i)}[L_2(i)]$,

(iii) $\chi^{\Sigma-E(i)}[L(i)] = \chi^{\Sigma-E(i)}[\chi^{\Sigma-E(i)}[L(i)]]$.

Proof.

(i) Let $s_i \approx_i t_i \in L(i)$. Then since $s_i = (\hat{\sigma}_{id})_i[s_i]$ and $t_i = (\hat{\sigma}_{id})_i[t_i]$, we have $(\hat{\sigma}_{id})_i[s_i] = s_i \approx_i t_i = (\hat{\sigma}_{id})_i[t_i] \in \chi^{\Sigma-E(i)}[L(i)]$ and then $L(i) \subseteq \chi^{\Sigma-E(i)}[L(i)]$.

(ii) Assume that $L_1(i) \subseteq L_2(i)$ and let $\hat{\sigma}[s_i] \approx_i \hat{\sigma}[t_i] \in \chi^{\Sigma-E(i)}[L_1(i)]$. Then $s_i \approx_i t_i \in L_1(i)$ but $L_1(i) \subseteq L_2(i)$, so that $s_i \approx_i t_i \in L_2(i)$ and $\hat{\sigma}_i[s_i] \approx_i \hat{\sigma}_i[t_i] \in \chi^{\Sigma-E(i)}[L_2(i)]$. We have $\chi^{\Sigma-E(i)}[L_1(i)] \subseteq \chi^{\Sigma-E(i)}$ $[L_2(i)]$.

(iii) By (i) we have $\chi^{\Sigma-E(i)}[L(i)] \subseteq \chi^{\Sigma-E(i)}[\chi^{\Sigma-E(i)}[L(i)]]$. Let $\hat{\sigma}_i[s_i] \approx_i \hat{\sigma}_i[t_i] \in \chi^{\Sigma-E(i)}[\chi^{\Sigma-E(i)}[L(i)]]$. Then $s_i \approx_i t_i \in \chi^{\Sigma-E(i)}[L(i)]$, and there exists $\rho_i \in \Sigma(i)$-Hyp and $u_i \approx_i v_i \in L(i)$ such that $s_i = \hat{\rho}_i[u_i]$ and $t_i = \hat{\rho}_i[v_i]$, and we have

$$\begin{aligned}
\hat{\sigma}_i[s_i] &= \hat{\sigma}_i[\hat{\rho}_i[u_i]] \\
&= (\hat{\sigma}_i \circ \hat{\rho}_i)\ [u_i] \\
&= (\sigma_i \circ_i \rho_i)\hat{}[u_i] \\
&= \hat{\lambda}_i[u_i],\ \text{ where } \lambda_i = \sigma_i \circ_i \rho_i \in \Sigma(i)\text{-}Hyp,
\end{aligned}$$

and

$$\begin{aligned}
\hat{\sigma}_i[t_i] &= \hat{\sigma}_i[\hat{\rho}_i[v_i]] \\
&= (\hat{\sigma}_i \circ \hat{\rho}_i)\ [v_i] \\
&= (\sigma_i \circ_i \rho_i)\hat{}[v_i] \\
&= \hat{\lambda}_i[v_i].
\end{aligned}$$

Then we set

$$\hat{\lambda}_i[u_i] = \hat{\sigma}_i[s_i] \approx_i \hat{\sigma}_i[t_i] = \hat{\lambda}_i[v_i] \in \chi^{\Sigma-E(i)}[L(i)],$$

and then

$$\chi^{\Sigma-E(i)}[\chi^{\Sigma-E(i)}[L(i)]] \subseteq \chi^{\Sigma-E(i)}[L(i)].$$

■

Proposition 3.11. *Let* $K_0, K_1, K_2 \subseteq Alg(\Sigma)$ *be classes of* Σ*-algebras. Then*

(i) $K_0 \subseteq \chi^{\Sigma-A}[K_0]$,

(ii) $K_1 \subseteq K_2 \Rightarrow \chi^{\Sigma-A}[K_1] \subseteq \chi^{\Sigma-A}[K_2]$,

(iii) $\chi^{\Sigma-A}[K_0] = \chi^{\Sigma-A}[\chi^{\Sigma-A}[K_0]]$.

Proof.

(i) Let $\mathcal{A} \in K_0$. Then since $\mathcal{A} = \sigma_{id}(\mathcal{A}) \in \chi^{\Sigma-A}[K_0]$, we have $K_0 \subseteq \chi^{\Sigma-A}[K_0]$.

(ii) Assume that $K_1 \subseteq K_2$ and let $\sigma(\mathcal{A}) \in \chi^{\Sigma-A}[K_1]$. Then $\mathcal{A} \in K_1$ by our assumption that $\mathcal{A} \in K_2$, with $\sigma(\mathcal{A}) \in \chi^{\Sigma-A}[K_2]$, and then $\chi^{\Sigma-A}[K_1] \subseteq \chi^{\Sigma-A}[K_2]$.

(iii) By (i), we have $\chi^{\Sigma-A}[K_0] \subseteq \chi^{\Sigma-A}[\chi^{\Sigma-A}[K_0]]$. We will show that $\chi^{\Sigma-A}[\chi^{\Sigma-A}[K_0]] \subseteq \chi^{\Sigma-A}[K_0]$. Let $\sigma(\mathcal{A}) \in \chi^{\Sigma-A}[\chi^{\Sigma-A}[K_0]]$. Then $\mathcal{A} \in \chi^{\Sigma-A}[K_0]$, and there exists $\rho \in \Sigma\text{-}Hyp$ and $\mathcal{B} \in K_0$ such that $\mathcal{A} = \rho(\mathcal{B})$. We have

$$\begin{aligned}\sigma(\mathcal{A}) &= \sigma(\rho(\mathcal{B}))\\ &= (\rho \diamond \sigma)(\mathcal{B})\\ &= \lambda(\mathcal{B}), \text{ where } \lambda = \rho \diamond \sigma \in \Sigma\text{-}Hyp.\end{aligned}$$

Thus we have $\sigma(\mathcal{A}) = \lambda(\mathcal{B}) \in \chi^{\Sigma-A}[K_0]$ and then $\chi^{\Sigma-A}[\chi^{\Sigma-A}[K_0]] \subseteq \chi^{\Sigma-A}[K_0]$.

∎

Lemma 3.12. *Let $\mathcal{A} \in Alg(\Sigma)$ be a Σ-algebra, let $s_i \approx_i t_i \in W(i)^2$ be a Σ-equation of sort $i \in I$, and $\sigma \in \Sigma$-Hyp. Then*

$$\sigma(\mathcal{A}) \models_i s_i \approx_i t_i \Longleftrightarrow \mathcal{A} \models_i \hat{\sigma}_i[s_i] \approx_i \hat{\sigma}_i[t_i].$$

Proof. We obtain

$$\begin{aligned} \sigma(\mathcal{A}) \models_i s_i \approx_i t_i &\Longleftrightarrow s_i^{\sigma(\mathcal{A})} = t_i^{\sigma(\mathcal{A})} \\ &\Longleftrightarrow \hat{\sigma}_i[s_i]^{\mathcal{A}} = \hat{\sigma}_i[t_i]^{\mathcal{A}} \\ &\Longleftrightarrow \mathcal{A} \models_i \hat{\sigma}_i[s_i] \approx_i \hat{\sigma}_i[t_i]. \end{aligned}$$

■

The next theorem needs the concept of a conjugate pair of additive closure operators (see [4]).

Theorem 3.13. *The pair $(\chi^{\Sigma-A}, \chi^{\Sigma-E(i)})$ is a conjugate pair of completely additive closure operators of sort i with respect to the relation $\models_i$.*

Proof. By Definition 3.9, Propositions 3.10–3.11, and Lemma 3.12. ■

Now we may apply the theory of conjugate pairs of additive closure operators (see e.g., [4]) and obtain the following propositions:

Lemma 3.14 ([4])**.** *For all $K_0 \subseteq Alg(\Sigma)$ and for all $L(i) \subseteq W(i)^2$ the following properties hold:*

(i) $H\Sigma(i)$-$ModL(i) = \Sigma(i)$-$Mod\chi^{\Sigma-E(i)}[L(i)]$,

(ii) $H\Sigma(i)$-$ModL(i) \subseteq \Sigma(i)$-$ModL(i)$,

(iii) $\chi^{\Sigma-A}[H\Sigma(i)$-$ModL(i)] = H\Sigma(i)$-$ModL(i)$,

(iv) $\chi^{\Sigma-E(i)}[\Sigma(i)\text{-}IdH\Sigma(i)\text{-}ModL(i)] = \Sigma(i)\text{-}IdH\Sigma(i)\text{-}ModL(i)$,

(v) $H\Sigma(i)\text{-}ModH\Sigma(i)\text{-}IdK_0 = \Sigma(i)\text{-}Mod\Sigma(i)\text{-}Id\chi^{\Sigma-A}[K_0]$, *and*

(i)′ $H\Sigma(i)\text{-}IdK_0 = \Sigma(i)\text{-}Id\chi^{\Sigma-A}[K_0]$,

(ii)′ $H\Sigma(i)\text{-}IdK_0 \subseteq \Sigma(i)\text{-}IdK_0$,

(iii)′ $\chi^{\Sigma-E(i)}[H\Sigma(i)\text{-}IdK_0] = H\Sigma(i)\text{-}IdK_0$,

(iv)′ $\chi^{\Sigma-A}[\Sigma(i)\text{-}ModH\Sigma(i)\text{-}IdK_0] = \Sigma(i)\text{-}ModH\Sigma(i)\text{-}IdK_0$,

(v)′ $H\Sigma(i)\text{-}IdH\Sigma(i)\text{-}ModL(i) = \Sigma(i)\text{-}Id\Sigma(i)\text{-}Mod\chi^{\Sigma-E(i)}[L(i)]$.

4. I-Sorted Solid Σ-Varieties

Definition 4.1. Let $K_0 \subseteq Alg(\Sigma)$ be a subclass of Σ-algebras. Then K_0 is called a solid model class of sort i or a solid Σ-variety of sort i if every Σ-identity of sort i is satisfied as a Σ-hyperidentity of sort i:

$$K_0 \underset{\Sigma-hyp}{\models_i} \Sigma(i)\text{-}IdK_0.$$

K_0 is called an I-sorted solid model class if every Σ-identity of sort i is satisfied as a Σ-hyperidentity of sort i for all $i \in I$, that is,

$$K_0 \underset{\Sigma-hyp}{\models_i} \Sigma(i)\text{-}IdK_0 \text{ for all } i \in I.$$

$L(i)$ is said to be a Σ-equational theory of sort i if there exists a class of Σ-algebras K_0 such that $L(i) = \Sigma(i)\text{-}IdK_0$. Then we set $L := (L(i))_{i\in I}$. This I-sorted set is called I-sorted Σ-equational theory.

Using the propositions of Lemma 3.14 one obtains the following characterization of solid Σ-varieties of sort i and solid Σ-equational theories of sort i (see e.g., [4]).

Theorem 4.2 ([4]). *Let K_0 be a Σ-variety of sort i. Then the following properties are equivalent:*

(i) $K_0 = H\Sigma(i)\text{-}ModH\Sigma(i)\text{-}IdK_0$,

(ii) $\chi^{\Sigma-A}[K_0] = K_0$,

(iii) $\Sigma(i)\text{-}IdK_0 = H\Sigma(i)\text{-}IdK_0$,

(iv) $\chi^{\Sigma-E(i)}[\Sigma(i)\text{-}IdK_0] = \Sigma(i)\text{-}IdK_0$.

Theorem 4.3 ([4]). *Let $L(i)$ be a Σ-equational theory of sort i. Then the following properties are equivalent:*

(i) $L(i) = H\Sigma(i)\text{-}IdH\Sigma(i)\text{-}ModL(i)$,

(ii) $\chi^{\Sigma-E(i)}[L(i)] = L(i)$,

(iii) $\Sigma(i)\text{-}ModL(i) = H\Sigma(i)\text{-}ModL(i)$,

(iv) $\chi^{\Sigma-A}[\Sigma(i)\text{-}ModL(i)] = \Sigma(i)\text{-}ModL(i)$.

5. I-sorted Complete Lattices

Let $\mathcal{H}(i)$ be the class of all fixed points with respect to the closure operator $\Sigma(i)\text{-}Mod\Sigma(i)\text{-}Id$:

$$\mathcal{H}(i) := \{K_0 \subseteq Alg(\Sigma) \mid K_0 = \Sigma(i)\text{-}Mod\Sigma(i)\text{-}IdK_0\},$$

that is, $\mathcal{H}(i)$ is the class of all Σ-varieties of sort i. Then $\mathcal{H}(i)$ forms a complete lattice of Σ-varieties of sort i. Let $\mathcal{H}y(i)$ be the class of all fixed points with respect to the closure operator $H\Sigma(i)$-$ModH\Sigma(i)$-Id:

$$\mathcal{H}y(i) := \{K_0 \subseteq Alg(\Sigma) \mid K_0 = H\Sigma(i)\text{-}ModH\Sigma(i)\text{-}IdK_0\},$$

that is, $\mathcal{H}y(i)$ is the class of all solid Σ-varieties of sort i. Then $\mathcal{H}y(i)$ forms a complete lattice of solid Σ-varieties of sort i and $\mathcal{H}y(i)$ is a complete sublattice of $\mathcal{H}(i)$. We set $\mathcal{H} := (\mathcal{H}(i))_{i\in I}$ and $\mathcal{H}y := (\mathcal{H}y(i))_{i\in I}$. $\mathcal{H}$ is called an I-sorted complete lattice. $\mathcal{H}y$ is called an I-sorted complete sublattice of $\mathcal{H}$, since for every $i \in I, \mathcal{H}y(i)$ is a complete sublattice of $\mathcal{H}(i)$. Dually

Let $\mathcal{L}(i)$ be the class of all fixed points with respect to the closure operator $\Sigma(i)$-$Id\Sigma(i)$-Mod:

$$\mathcal{L}(i) := \{L(i) \subseteq W(i)^2 \mid L(i) = \Sigma(i)\text{-}Id\Sigma(i)\text{-}ModL(i)\},$$

that is, $\mathcal{L}(i)$ is the class of all Σ-equational theories of sort i. Then $\mathcal{L}(i)$ forms a complete lattice of Σ-equational theories of sort i. Let $\mathcal{L}y(i)$ be the class of all fixed points with respect to the closure operator $H\Sigma(i)$-$IdH\Sigma(i)$-Mod:

$$\mathcal{L}y(i) := \{L(i) \subseteq W(i)^2 \mid L(i) = H\Sigma(i)\text{-}IdH\Sigma(i)\text{-}ModL(i)\},$$

that is, $\mathcal{L}y(i)$ is the class of all solid Σ-equational theories of sort i. Then $\mathcal{L}y(i)$ forms a complete lattice of solid Σ-equational theories of sort i and $\mathcal{L}y(i)$ is a complete sublattice of $\mathcal{L}(i)$. We set $\mathcal{L} := (\mathcal{L}(i))_{i\in I}$ and $\mathcal{L}y := (\mathcal{L}y(i))_{i\in I}$. $\mathcal{L}$ is called an I-sorted complete lattice. $\mathcal{L}y$ is called an I-sorted complete sublattice of $\mathcal{L}$, since for every $i \in I, \mathcal{L}y(i)$ is a complete sublattice of $\mathcal{L}(i)$.

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