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ON THE MATRIX NEGATIVE PELL EQUATION*

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Abstract

Let N be a set of natural numbers and Z be a set of integers. Let $M_2(Z)$ denotes the set of all $2x2$ matrices with integer entries.

We give necessary and suficient conditions for solvability of the matrix negative Pell equation

$$\text{(P)} \qquad X^2 - dY^2 = -I \quad \text{with } d \in N$$

for nonsingular X, Y belonging to $M_2(Z)$ and his generalization

$$\text{(Pn)} \qquad \sum_{i=1}^{n} X_i^2 - d\sum_{i=1}^{n} Y_i^2 = -I \quad \text{with } d \in N$$

for nonsingular $X_i, Y_i \in M_2(Z), i = 1, ..., n$.

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1. Introduction

Let N be a set of natural numbers and Z be a set of integers. Let $M_2(Z)$ denotes the set of all $2x2$ matrices with integer entries.

We consider the matrix negative Pell equation $X^2 - dY^2 = -I$ with $d \in N$ for nonsingular X, Y belonging to $M_2(Z)$ as an analogue of the classical Diophantine equation

$$x^2 - dy^2 = -1 \tag{$\star$}$$

called negative Pell's equation. In 2000 A. Grytczuk, F. Luca and M. Wójtowicz [7] showed that the equation ($\star$) has a solution in integers x, y if and only if there exist a primitive Pythagorean triple (A, B, C) (ie. A, B, C are positive integers such that $A^2 + B^2 = C^2$ and $gcd(A, B) = 1$) and natural numbers a, b such that $d = a^2 + b^2$ and $|aA - bB| = 1$. In [9] we give an explicit form of the criterion for the solvability in integers x, y of the negative Pell equation ($\star$), where $d \equiv 1 (\mathrm{mod}\, 4)$.

We also study the generalization of the matrix negative Pell equation

$$\sum_{i=1}^{n} X_i^2 - d \sum_{i=1}^{n} Y_i^2 = -I$$

with $d \in N$ for nonsingular $X_i, Y_i \in M_2(Z)$, $i = 1, ..., n$.

Some generalizations of the classical Diophantine equations to matrix equations were studied by a number of autors; see [1, 2, 3, 4, 5, 8, 10, 11, 12, 13, 14]. The results presented in this paper extend that list a little.

2. BASIC LEMMA

Lemma 1 ([6]). *Let*

$$A = A(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$$

be the matrix with $a = a(x), b = b(x), c = c(x), d = d(x)$ *which are non-zero and real-valued functions defined on the interval* $J = (x_1, x_2) \subset R$, *where* R *is the set of real numbers, and let* $\det A(x) \neq 0$ *on* J. *Let us define the numbers* r, s, u_n $(n = 0, 1, ...)$ *by formulas:*

$$r = r(x) = a(x) + d(x) = TrA(x)$$

$$s = s(x) = -\det A(x)$$

$$u_0 = r,\ u_1 = ru_0 + s$$

$$u_n(x) = ru_{n-1}(x) + su_{n-2}(x) \quad \text{for } n \geq 2.$$

Then for every natural number $n \geq 2$ *we have*

$$\begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}^n = \begin{pmatrix} a(x)u_{n-2}(x) + v_{n-2}(x) & b(x)u_{n-2}(x) \\ c(x)u_{n-2}(x) & d(x)u_{n-2}(x) + v_{n-2}(x) \end{pmatrix}$$

where

$$v_{n-2}(x) = s(x)u_{n-3(x)},\ u_{-1}(x) = 1 \text{ for } x \in J.$$

3. Result

Theorem 1. *Let*

$$X_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, \quad Y_i = \begin{pmatrix} e_i & f_i \\ g_i & h_i \end{pmatrix}$$

be nonsingular integral matrices, $i = 1, ..., n$.

The equation

$$\sum_{i=1}^{n} X_i^2 - d \sum_{i=1}^{n} Y_i^2 = -I \quad \textit{with} \quad d \in N \tag{Pn}$$

is satisfied if and only if

$$\sum_{i=1}^{n} (TrX_i)^2 - d \sum_{i=1}^{n} (TrY_i)^2 = 2 \left(\sum_{i=1}^{n} \det X_i - d \sum_{i=1}^{n} \det Y_i - 1 \right), \tag{1}$$

and

$$\begin{gathered} \sum_{i=1}^{n} b_i TrX_i - d \sum_{i=1}^{n} f_i TrY_i = 0 \\ \sum_{i=1}^{n} c_i Tr_i - d \sum_{i=1}^{n} g_i TrY_i = 0. \end{gathered} \tag{2}$$

Proof. Suppose that the equation (Pn) holds. Set

$$r_i = TrX_i, \;\; r_i' = TrY_i, \;\; s_i = -\det X_i, \;\; s_i' = -\det Y_i, \;\; i = 1, ..., n,$$

and

$$B = \sum_{i=1}^{n} X_i^2 - d \sum_{i=1}^{n} Y_i^2.$$

Then from Lemma 1 we obtain

$$B = \sum_{i=1}^{n} \begin{pmatrix} a_i r_i + s_i & b_i r_i \\ c_i r_i & d_i r_i + s_i \end{pmatrix} - d \sum_{i=1}^{n} \begin{pmatrix} e_i r_i' + s_i' & f_i r_i' \\ g_i r_i' & h_i r_i' + s_i' \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{n} (a_i r_i + s_i) - d \sum_{i=1}^{n} (e_i r_i' + s_i') & \sum_{i=1}^{n} b_i r_i - d \sum_{i=1}^{n} f_i r_i' \\ \sum_{i=1}^{n} c_i r_i - d \sum_{i=1}^{n} g_i r_i' & \sum_{i=1}^{n} (d_i r_i + s_i) - d \sum_{i=1}^{n} (h_i r_i' + s_i') \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore

$$\sum_{i=1}^{n} (a_i r_i + s_i) - d \sum_{i=1}^{n} (e_i r_i' + s_i') = -1$$

$$\sum_{i=1}^{n} (d_i r_i + s_i) - d \sum_{i=1}^{n} (h_i r_i' + s_i') = -1$$

$$\sum_{i=1}^{n} b_i r_i - d \sum_{i=1}^{n} f_i r_i' = 0$$

$$\sum_{i=1}^{n} c_i r_i - d \sum_{i=1}^{n} g_i r_i' = 0.$$

Now we have

$$TrB = \sum_{i=1}^{n}[(a_i r_i + s_i) + (d_i r_i + s_i)] - d\sum_{i=1}^{n}[(e_i r_i' + s_i') + (h_i r_i' + s_i')]$$

$$= \sum_{i=1}^{n} TrX_i^2 - d\sum_{i=1}^{n} TrY_i^2$$

$$= \sum_{i=1}^{n}\left(\left(\lambda_i^{(1)}\right)^2 + \left(\lambda_i^{(2)}\right)^2\right) - d\sum_{i=1}^{n}\left(\left(\lambda_i'^{(1)}\right)^2 + \left(\lambda_i'^{(2)}\right)^2\right),$$

where $\lambda_i^{(1)}$, $\lambda_i^{(2)}$ are the characteristic roots of X_i, $\lambda_i'^{(1)}$, $\lambda_i'^{(2)}$ are the characteristic roots of Y_i, $i = 1, ..., n$. On the other hand $TrB = -2$ and consequently we have

$$(TrX_i)^2 = \left(\lambda_i^{(1)} + \lambda_i^{(2)}\right)^2, \ (TrY_i)^2 = \left(\lambda_i'^{(1)} + \lambda_i'^{(2)}\right)^2, \ \ i = 1, ..., n,$$

and

$$\sum_{i=1}^{n}(TrX_i)^2 - d\sum_{i=1}^{n}(TrY_i)^2 = 2\left(\sum_{i=1}^{n}\lambda_i^{(1)}\lambda_i^{(2)} - d\sum_{i=1}^{n}\lambda_i'^{(1)}\lambda_i'^{(2)} - 1\right),$$

and this impies that (1).

Conversely, assume that (1) and (2) are true. Then it is easy to see that the equation (Pn) is satisfied, and the proof of Theorem 2 is finished. ■

4. COROLLARIES

Now consider the equation

$$nX^2 - nd_1Y^2 = -I,$$

where X, Y are nonsingular integral matrices, and $d_1, n \in N$.

Let

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad Y = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

Then from (1) we obtain

$$n\,(TrX)^2 - d_1 n\,(TrY)^2 = 2\,(n\det X - d_1 n \det Y - 1)\,. \tag{3}$$

From (2) we have

$$bTrX - d_1 fTrY = 0, \quad cTrX - d_1 gTrY = 0. \tag{4}$$

Hence, from Theorem 1 we get the following corollary:

Corollary 1. *Let*

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad Y = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

be nonsingular integral matrices.

The equation

$$nX^2 - nd_1 Y^2 = -I \quad \textit{with } d_1, n \in N$$

has a solution if and only if

$$n\,(TrX)^2 - d_1 n\,(TrY)^2 = 2\,(n\det X - d_1 n \det Y - 1)$$

and

$$bTrX - d_1 fTrY = 0, \quad cTrX - d_1 gTrY = 0.$$

From Corollary 1 when $n = 1$ we get

Corollary 2. *Let be satisfied the assumptions of Corollary* 1. *The equation*

$$X^2 - d_1 Y^2 = -I, \text{ where } d_1 \in N \tag{5}$$

has a solution if and only if

$$(TrX)^2 - d_1(TrY)^2 = 2(\det X - d_1 \det Y - 1) \ .$$

and

$$bTrX - d_1 fTrY = 0, \quad cTrX - d_1 gTrY = 0 \ .$$

Let

$$X = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \ Y = \begin{pmatrix} e & f \\ f & e \end{pmatrix}$$

be integral nonsingular matrices.

Then from (4) we have

$$ab - d_1 ef = 0,$$

and for n=1 from (3) we get

$$4a^2 - 4d_1 e^2 = 2(\det X - d_1 \det Y - 1).$$

Hence we obtain the following corollary:

Corollary 3. *Let*

$$X = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \ Y = \begin{pmatrix} e & f \\ f & e \end{pmatrix}$$

be integral nonsingular matrices. The equation (5) *holds if and only if*

$$\det X - d_1 \det Y - 1 = 2(a^2 - d_1 e^2)$$

and

$$ab - d_1 ef = 0.$$

Example 1. Let $b, a = db - 1$ be non-zero integers and $d \in N$.

Let

$$X = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}.$$

It easy to see that the conditions (1) and (2) are satisfied.

We have

$$\begin{aligned} X^2 - dY^2 &= \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} - d \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \\ &= \begin{pmatrix} db-1 & 0 \\ 0 & db-1 \end{pmatrix} - d \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

Example 2. Let x, y be non-zero integers and $d \in N$.

We consider the following matrices:

$$X = \begin{pmatrix} 0 & 1 \\ x^2 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ y^2 & 0 \end{pmatrix}. \tag{6}$$

We have

$$\begin{aligned} X^2 - dY^2 &= \begin{pmatrix} x^2 & 0 \\ 0 & x^2 \end{pmatrix} - d \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix} \\ &= \begin{pmatrix} x^2 - dy^2 & 0 \\ 0 & x^2 - dy^2 \end{pmatrix}. \end{aligned}$$

It is easy to see that from the result given in the paper [7] we can generate infinitely many matrices of the form (6) satisfying the matrix negative Pell equation (P).

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