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HORIZONTAL SUMS OF BASIC ALGEBRAS*

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Abstract

The variety of basic algebras is closed under formation of horizontal sums. We characterize when a given basic algebra is a horizontal sum of chains, MV-algebras or Boolean algebras.

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It is important in mathematics to construct relative complicated objects from simple ones or, conversely, to decompose complicated things into simple ones. For bounded lattices, there is known a construction of forming horizontal sums. This is just pasting distinct lattices at 0 and 1. Since every basic algebra has its alter ego as a bounded lattice with section antitone involutions (see e.g., [1, 2]), we can extend this construction also for these algebras. When decomposing a given basic algebra into a horizontal sum of more simple basic algebras, we can ask that these more simple objects would be of a special sort. In our paper we will treat the cases when the components are either chains or MV-algebras (or, in particular, Boolean algebras).

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Let us recall (see e.g., [2]) that by a *basic algebra* is meant an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type (2,1,0) satisfying the identities

(BA1) $x \oplus 0 = x$;

(BA2) $\neg\neg x = x$;

(BA3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$;

(BA4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1$.

Having a basic algebra, we can introduce an order $\leq$ on A by the rule

$$x \leq y \quad \text{if and only if} \quad \neg x \oplus y = 1.$$

It is easy to prove (see e.g., [2]) that $(A; \leq)$ is a bounded lattice where 0 is the least and 1 the greatest element and

$$x \vee y = \neg(\neg x \oplus y) \oplus y \quad \text{and} \quad x \wedge y = \neg(\neg x \vee \neg y). \tag{1}$$

The lattice $\mathcal{L}(A) = (A; \vee, \wedge, 0, 1)$ will be called the *assigned lattice* of $\mathcal{A}$.

If $\mathcal{A}$ is a basic algebra and its assigned lattice $\mathcal{L}(A)$ is a chain, $\mathcal{A}$ will be called a *chain basic algebra.*

For our study, the following result is important (see Theorem 8.5.7 in [2]):

Correspondence Theorem. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra. Define the operations $\vee$ and $\wedge$ by (1). Then $\mathcal{L}(A) = (A; \vee, \wedge, 0, 1)$ is a bounded lattice where for each $a \in A$ the mapping*

$$x \mapsto x^a = \neg x \oplus a$$

is an antitone involution on the interval $[a, 1]$ (i.e. $x^{aa} = x$ and $x \leq y \Rightarrow y^a \leq x^a$ for each $x, y \in [a, 1]$).

Conversely, let $\mathcal{L} = (L; \vee, \wedge, 0, 1)$ be a bounded lattice where for each $a \in A$ there exists an antitone involution $x \mapsto x^a$ on the interval $[a, 1]$. Define $x \oplus y = (\neg x \vee y)^y$ and $\neg x = x^0$. Then the algebra $\mathcal{A}(L) = (L; \oplus, \neg, 0)$ is a basic algebra.

Moreover, $\mathcal{A}(\mathcal{L}(A)) = \mathcal{A}$ and $\mathcal{L}(\mathcal{A}(L)) = \mathcal{L}$.

Due to the Correspondence Theorem, basic algebras can be identified with bounded lattices having antitone involutions on the upper intervals. This enable us to introduce the horizontal sum of basic algebras.

Let $\mathcal{L}_\gamma = (L_\gamma; \vee, \wedge, 0, 1)$ for $\gamma \in \Gamma$ be bounded lattices such that the sets $L_\gamma \setminus \{0, 1\}$ are mutually disjoint for distinct indices $\gamma \in \Gamma$.

Recall that a *horizontal sum of bounded lattices* $\mathcal{L}_\gamma = (L_\gamma; \vee, \wedge, 0, 1)$ is a lattice $\mathcal{L} = (\bigcup\{L_\gamma; \gamma \in \Gamma\}; \vee, \wedge, 0, 1)$ where we identify all 0's and all 1's and for $x, y \in L$ we have

$$x \vee y = \begin{cases} x \vee y & \text{in } \mathcal{L}_\gamma \text{ if } x, y \in L_\gamma \\ 1 & \text{otherwise} \end{cases}$$

$$x \wedge y = \begin{cases} x \wedge y & \text{in } \mathcal{L}_\gamma \text{ if } x, y \in L_\gamma \\ 0 & \text{otherwise.} \end{cases}$$

Due to the Correspondence Theorem and the fact, that every upper interval $[a, 1]$ in $\mathcal{L}$ is an upper interval of $\mathcal{L}_\gamma$ for $a \in L_\gamma$ we can define:

Definition. Let $\mathcal{A}_\gamma = (A_\gamma; \oplus, \neg, 0)$ be basic algebras for $\gamma \in \Gamma$. Then $\mathcal{A}(L)$ is a *horizontal sum* of $\mathcal{A}_\gamma$ if $\mathcal{L}$ is a horizontal sum of the assigned lattices $\mathcal{L}(A_\gamma)$ for $\gamma \in \Gamma$.

An immediate reflexion of the definition of basic algebra reveals that the class of all basic algebras is a variety. An immediate consequence of the Correspondence Theorem is the following fact:

- The variety of basic algebras is closed under formation of horizontal sums.

1. Horizontal sums of chain basic algebras

Due to the previous fact, every horizontal sum of basic algebras is a basic algebra. In what follows, we will study the structure of basic algebras which are horizontal sums of chain basic algebras.

Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra. We say that A satisfies the condition (P) if the following formula

(P) $x \oplus x = 1$ or $\neg x \oplus \neg x = 1$

holds for each $x \in A$. It is straightforward that (P) is equivalent to

(P′) $\neg x \leq x$ or $x \leq \neg x$.

For $y \in A$ we denote by $C(y)$ the subset of A defined by $C(y) = \{z \in A; z \leq y$ or $y \leq z\}$. It is evident that $C(y)$ is a 0–1 sublattice of the induced lattice $\mathcal{L}(A)$.

Lemma 1. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra satisfying* (P) *and $y \in A$; $0 \neq y \neq 1$. If $C(y)$ is a chain and $z \in C(y)$ then also $\neg z \in C(y)$.*

Proof. By (P′), either $\neg y \leq y$ or $y \leq \neg y$ thus $\neg y \in C(y)$. Assume that $C(y)$ is a chain and $z \in C(y)$. Then either $z \leq \neg y$ or $z \geq \neg y$, i.e. either $\neg z \geq y$ or $\neg z \leq y$ whence $\neg z \in C(y)$. ■

We say that a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ satisfies the condition (Q) if the following formula holds for each $x, y \in A$:

(Q) $\neg x \oplus y = 1$ or $\neg y \oplus x = 1$ or ($x \vee y = 1$ and $\neg x \vee \neg y = 1$).

It is evident that (Q) is equivalent to

(Q′) $x \parallel y \Rightarrow x \vee y = 1$ and $x \wedge y = 0$

since $\neg x \oplus y = 1$ is equivalent to $x \leq y$ and $\neg x \vee \neg y = 1$ yields $\neg(x \wedge y) = 1$, i.e. $x \wedge y = 0$.

Lemma 2. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra satisfying* (Q) *and $y \in A$; $0 \neq y \neq 1$. Then C(y) is a chain.*

Proof. Let $a, b \in C(y)$ for $0 \neq y \neq 1$ and assume $a \parallel b$. Then either $y \leq a, b$ or $y \geq a, b$. In the first case we have $a \wedge b \geq y \neq 0$ which contradicts to (Q'). In the second case we have $a \vee b \leq y \neq 1$, a contradiction again. ■

Corollary 1. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra satisfying* (Q). *Then the assigned lattice $\mathcal{L}(A)$ is a horizontal sum of chains.*

Proof. Let $y \in A, 0 \neq y \neq 1$. By Lemma 2, $C(y)$ is a chain. Let $z \in A, z \parallel y$. Then clearly $0 \neq z \neq 1$ thus also $C(z)$ is a chain. Since $z \parallel y$, we have $C(y) \neq C(z)$. Assume that there exists $a \in C(y) \cap C(z), 0 \neq a \neq 1$. If $a \leq y$ and $a \leq z$ then $0 \neq a \leq y \wedge z$, i.e. $\neg y \vee \neg z \neq 1$, a contradiction with (Q). If $a \geq y$ and $a \geq z$ then $1 \neq a \geq y \vee z$, a contradiction again. If $a \leq y$ and $a \geq z$ then $z \leq y$, a contradiction with $y \parallel z$. Hence, we have shown $C(y) \cap C(z) = \{0, 1\}$ for any $y, z \notin \{0, 1\}$ with $y \parallel z$. It yields immediately that $\mathcal{L}(A)$ is a horizontal sum of $C(y), y \in A, 0 \neq y \neq 1$. ∎

Example 1. If a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ satisfies (Q) then it need not be a horizontal sum of chain basic algebras. For example, let $\mathcal{A} = (A; , \oplus, \neg, 0)$ where $A = \{0, x, \neg x, y, \neg y, 1\}$ and whose table is as follows:

$\oplus$	0	x	$\neg x$	y	$\neg y$	1
0	0	x	$\neg x$	y	$\neg y$	1
x	x	x	1	$\neg x$	$\neg y$	1
$\neg x$	$\neg x$	1	1	y	1	1
y	y	$\neg y$	$\neg x$	y	1	1
$\neg y$	$\neg y$	x	1	1	$\neg y$	1
1	1	1	1	1	1	1

Then the assigned lattice $\mathcal{L}(A)$ is as shown in Figure 1.

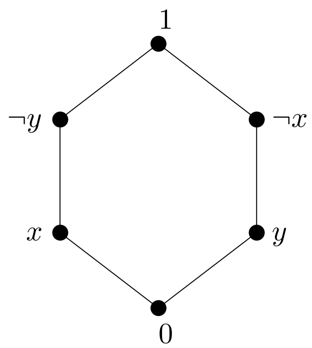

Figure 1

Evidently, $\mathcal{L}(A)$ is a horizontal sum of the chains $C(x) = \{0, x, \neg y, 1\}$ and $C(y) = \{0, y, \neg x, 1\}$ but none of them is a subalgebra of $\mathcal{A}$. $\diamondsuit$

Theorem 1. *A basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is a horizontal sum of chain basic algebras if and only if it satisfies* (P) *and* (Q).

Proof. It is easy to verify that if $\mathcal{A}$ is a horizontal sum of chain basic algebras $\mathcal{A}_i$ $(i \in I)$ then it satisfies both (P) and (Q).

Conversely, if $\mathcal{A}$ satisfies (Q) then, by Lemma 2 and Corollary 1, the assigned lattice $\mathcal{L}(A)$ is a horizontal sum of the chains $C(y)$ (for $y \in A; 0 \neq y \neq 1$). Clearly $0, 1 \in C(y)$ for each $y \in A$. Let $z \in C(y)$. Since $\mathcal{A}$ satisfies also (P), Lemma 1 yields $\neg z \in C(y)$. Assume $a, b \in C(y), a, b \notin \{0, 1\}$. Then, by Corollary 1, $C(a) = C(b) = C(y)$. Since $a \leq b \oplus a$ and $b \leq a \oplus b$, also $a \oplus b \in C(b)$ and $b \oplus a \in C(a)$ thus $C(y)$ is closed with respect to $\oplus$ and hence it is a subalgebra of $\mathcal{A}$. Hence $\mathcal{A}$ is a horizontal sum of chain basic algebras $C(y)$ for $y \in A, 0 \neq y \neq 1$. ■

2. Horizontal sums of MV-algebras and Boolean algebras

For the concept of an MV-algebra, the reader is referred to [5]. For us it is enough to say that an *MV-algebra* is a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ whose binary operation $\oplus$ is commutative and associative. Let us note that a Boolean algebra is an MV-algebra where for any element $a \in A$ its negation $\neg a$ is a complement of a in the assigned lattice $\mathcal{L}(A)$.

The concept of an effect algebra was introduced by D.J. Foulis and M.K. Bennett [6]. In fact, we will not deal with this original concept since this is only a partial algebra. However, it was shown in [3] that every lattice effect algebra can be organized into a total algebra which is a basic algebra (satisfying the condition (H1), see bellow). For the reader convenience, we get a list of necessary results taken from [3]. The following is taken from Lemma 4.9 and Propositions 4.8 and 4.10 of [3].

Proposition 1. *A basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is a lattice effect algebra if and only if it satisfies the quasi-identity*

$$\text{(H1)} \qquad x \leq \neg y \quad \textit{and} \quad x \oplus y \leq \neg z \quad \Rightarrow \quad x \oplus (z \oplus y) = (x \oplus y) \oplus z$$

which is equivalent to the identity

$$\text{(H1}'\text{)} \qquad (x \wedge \neg y) \oplus [(\neg(x \oplus y) \wedge z) \oplus y] = (x \oplus y) \oplus (\neg(x \oplus y) \wedge z).$$

Due to Proposition 1, we will identify lattice effect algebra with a basic algebra satisfying (H1) and hence we can apply known concepts and results on effect algebras (see [5, 6, 7]) for basic algebras. The first useful concept is the compatibility which was introduced in [6]. We will adopt the equivalent condition from [5] which is more suitable for lattice effect algebras:

Elements a, b of $\mathcal{A} = (A; \oplus, \neg, 0)$ are *compatible* if $(a \wedge b) \oplus (a \vee b) = b \oplus a$. The following result is Theorem 4.5 in [3].

Proposition 2. *Elements a, b of a lattice effect algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ are compatible if and only if $a \oplus b = b \oplus a$.*

By a *block* of a basic algebra is meant a maximal subset of mutually compatible elements. The following result is a combination of that of Z. Riečanová [7] and Theorem 4.7 in [3].

Proposition 3. *Every lattice effect algebra is a set-theoretical union of its blocks, which are MV-algebras.*

Finally, we quote from Theorem 7.8 of [3].

Proposition 4. *For a basic algebra $\mathcal{A}$, the following are equivalent:*

(i) *every block of $\mathcal{A}$ is a subalgebra which is an MV-algebra;*

(ii) *$\mathcal{A}$ is a lattice effect algebra.*

Now, we are able to prove

Lemma 3. *Let a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ be a horizontal sum of MV-algebras $\mathcal{A}_\gamma = (A_\gamma; \oplus, \neg, 0)$ for $\gamma \in \Gamma$. Then $\mathcal{A}$ is a lattice effect algebra and $\mathcal{A}_\gamma$ are blocks of $\mathcal{A}$.*

Proof. By Proposition 1, we need only to verify the quasi-identity (H1). Let $x, y, z \in A$. If $z = 0$ or $y = 0$ then (H1) holds trivially.

Assume $y \neq 0 \neq z$. If $x \leq \neg y$ then $x, \neg y$ belong to the same A_γ. It is evident that every A_γ is a subalgebra of $\mathcal{A}$ and also x, y belong to A_γ thus $x \oplus y \in A_\gamma$. If $x \oplus y \leq \neg z$ then, analogously, also $x \oplus y$ and z belong to the same A_γ, i.e. $x, y, z \in A_\gamma$. Since every A_γ is an MV-algebra, the operation $\oplus$ on A_γ is associative and commutative thus (H1) holds. By Proposition 1, $\mathcal{A}$ is a lattice effect algebra. It is almost evident that A_γ are blocks of $\mathcal{A}$. ■

Lemma 4. *Let a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ be a horizontal sum of MV-algebras. Then $\mathcal{A}$ satisfies the condition*

(H2) $\quad x \oplus y \neq y \oplus x, x \oplus z = z \oplus x, y \oplus z = z \oplus y \Rightarrow z = 0$ *or* $z = 1$.

Proof. Assume that $\mathcal{A}$ is a horizontal sum of MV-algebras $A_\gamma, \gamma \in \Gamma$. By Lemma 3, $\mathcal{A}$ is a lattice effect algebra and A_γ are its blocks. Let $x, y, z \in A$ and $x \oplus y \neq y \oplus x$. Then clearly $x \in A_{\gamma_1}, y \in A_{\gamma_2}$ for $\gamma_1 \neq \gamma_2$ (and, trivially, $x, y \notin \{0, 1\}$). Assume $x \oplus z = z \oplus x$ and $y \oplus z = z \oplus y$. Then z is compatible both with x as well as with y thus x, z must be in the same block of $\mathcal{A}$ and y, z must be in the same block of $\mathcal{A}$. Since $\mathcal{A}$ is a horizontal sum of its blocks, the only possibility is $z \in A_{\gamma_1}$ and $z \in A_{\gamma_2}$, i.e. $z \in A_{\gamma_1} \cap A_{\gamma_2}$ whence $z = 0$ or $z = 1$. ■

Now, we are able to prove our second main result

Theorem 2. *For a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$, the following are equivalent*

- (a) *$\mathcal{A}$ is a horizontal sum of MV-algebras;*
- (b) *$\mathcal{A}$ satisfies* (H1) *and* (H2)*;*
- (c) *$\mathcal{A}$ satisfies* (H1) *and*

(H3) $\quad x \oplus y = y \oplus x, y \oplus z = z \oplus y$ *for* $0 \neq y \neq 1 \Rightarrow x \oplus z = z \oplus x$.

Proof. Let $\mathcal{A}$ be a horizontal sum of MV-algebras $A_\gamma, \gamma \in \Gamma$. By Lemma 3, Lemma 4 and Proposition 1, it satisfies (H1) and (H2) proving $(a) \Rightarrow (b)$. Using of the propositional calculus we have that (H2) is equivalent to (H3). Hence, the equivalence (a) $\Leftrightarrow$ (c) is evident.

Assume now that a basic algebra $\mathcal{A}$ satisfies (H1). By Proposition 1, $\mathcal{A}$ is a lattice effect algebra and, by Proposition 3, $\mathcal{A}$ is a set-theoretical union of MV-algebras $A_\gamma, \gamma \in \Gamma$, which are blocks of $\mathcal{A}$. Let $\gamma, \delta \in \Gamma, \gamma \neq \delta$. Then $A_\gamma \neq A_\delta$, i.e. there exist $x \in A_\gamma \setminus A_\delta$ and $z \in A_\delta \setminus A_\gamma$. Clearly $x, z \notin \{0, 1\}$. Assume $y \in A_\gamma \cap A_\delta$ for $0 \neq y \neq 1$. By Proposition 2, we have $x \oplus y = y \oplus x$ and $y \oplus z = z \oplus y$.

Assume that $\mathcal{A}$ satisfies (H2). Then we conclude $y = 0$ or $y = 1$, a contradiction. Thus $A_\gamma \cap A_\delta = \{0, 1\}$ for all $\gamma \neq \delta$ and hence $\mathcal{A}$ is a horizontal sum of $A_\gamma, \gamma \in \Gamma$. We have shown $(b) \Rightarrow (a)$. ■

We are going to get example of a basic algebra which is not commutative and hence not an MV-algebra but which is a horizontal sum of MV-algebras.

Example 2. Let $\mathcal{A} = \{0, x, \neg x, y, \neg y, a, \neg a, b, \neg b, 1\}$ and consider a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ whose operation table for $\oplus$ is as follows:

$\oplus$	0	x	$\neg x$	y	$\neg y$	a	$\neg a$	b	$\neg b$	1
0	0	x	$\neg x$	y	$\neg y$	a	$\neg a$	b	$\neg b$	1
x	x	x	1	y	$\neg y$	a	a	b	$\neg b$	1
$\neg x$	$\neg x$	1	$\neg x$	y	$\neg y$	1	$\neg x$	b	$\neg b$	1
y	y	x	$\neg x$	y	1	a	$\neg a$	b	b	1
$\neg y$	$\neg y$	x	$\neg x$	1	$\neg y$	a	$\neg a$	1	$\neg y$	1
a	a	a	1	y	$\neg y$	1	1	b	$\neg b$	1
$\neg a$	$\neg a$	a	$\neg x$	y	$\neg y$	1	$\neg x$	b	$\neg b$	1
b	b	x	$\neg x$	b	1	a	$\neg a$	1	1	1
$\neg b$	$\neg b$	x	$\neg x$	b	$\neg y$	a	$\neg a$	1	$\neg y$	1
1	1	1	1	1	1	1	1	1	1	1

It is easy to check that the assigned lattice $\mathcal{L}(A)$ is that of Figure 2.

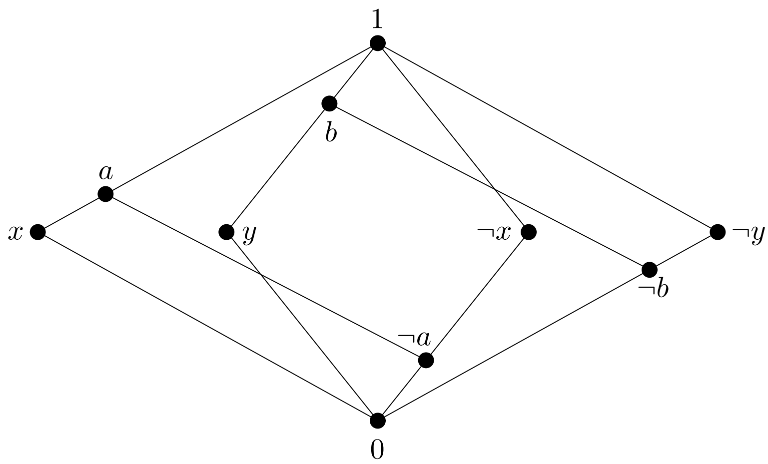

Figure 2

One can verify that $\mathcal{A}$ satisfies the conditions (H1) and (H2) and it is a horizontal sum of two copies of an MV-algebra whose (lattice) diagram is visualized in Figure 3.

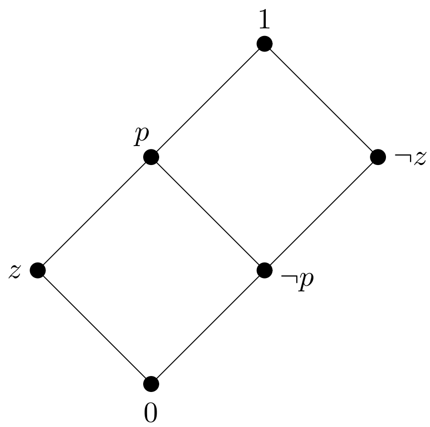

Figure 3

Of course, we have $x = z, a = p$ for $A_1 = \{0, x, \neg x, a, \neg a, 1\}$ and $y = z, b = p$ for $A_2 = \{0, y, \neg y, b, \neg b, 1\}$. $\diamondsuit$

Example 3. We can show that the conditions (H1) and (H2) are independent. For this, one can mention that the basic algebra of Example 1 satisfies (H2) but not (H1): for $y = x$ and $z = \neg y$ we have $x \leq x, x \oplus x = x \leq \neg y$ but

$$x \oplus (\neg y \oplus x) = x \oplus x = x \neq \neg y = x \oplus \neg y = (x \oplus x) \oplus \neg y.$$

Conversely, consider the basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ with $A = \{0, a, b, c, \neg a, \neg b, \neg c, 1\}$ whose table is as follows

$\oplus$	0	a	b	c	$\neg a$	$\neg b$	$\neg c$	1
0	0	a	b	c	$\neg a$	$\neg b$	$\neg c$	1
a	a	$\neg b$	$\neg a$	c	1	$\neg b$	$\neg c$	1
b	b	$\neg a$	b	$\neg c$	$\neg a$	1	$\neg c$	1
c	c	a	$\neg c$	$\neg b$	$\neg a$	$\neg b$	1	1
$\neg a$	$\neg a$	1	$\neg a$	$\neg c$	1	1	$\neg c$	1
$\neg b$	$\neg b$	$\neg b$	1	$\neg b$	1	$\neg b$	1	1
$\neg c$	$\neg c$	$\neg a$	$\neg c$	1	$\neg a$	1	1	1
1	1	1	1	1	1	1	1	1

Then $\mathcal{A}$ satisfies (H1) but not (H2) since e.g., $a \oplus c = c \neq a = c \oplus a, a \oplus b = \neg a = b \oplus a, c \oplus b = \neg c = b \oplus c$ but $b \notin \{0, 1\}$. The assigned lattice $\mathcal{L}(A)$ is depicted in Figure 4.

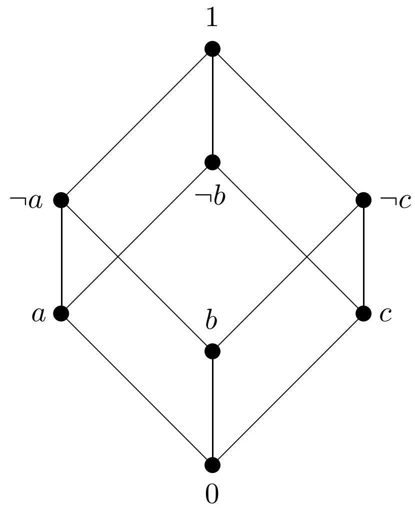

Figure 4

◇

Our last task is when a basic algebra is a horizontal sum of Boolean algebras. As already mentioned, a Boolean algebra is an MV-algebra where for every its element x the negation $\neg x$ is a complement of x. It is well-known (see e.g., [2] or [5]) that $\neg x$ is a complement of x if and only if $x \oplus x = x$. Hence, we obtain an immediate consequence of Theorem 2.

Corollary 2. *A basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is a horizontal sum of Boolean algebras if and only if it satisfies* (H1), (H2) *and the identity $x \oplus x = x$.*

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