# REMARKS ON PSEUDO MV-ALGEBRAS* 

Ivan Chajda and Miroslav Kolařík<br>Department of Algebra and Geometry<br>Palacký University Olomouc<br>Tomkova 40, 77900 Olomouc, Czech Republic<br>e-mail: chajda@inf.upol.cz<br>e-mail: kolarik@inf.upol.cz


#### Abstract

Pseudo MV-algebras (see e.g., $[4,6,8]$ ) are non-commutative extension of MV-algebras. We show that every pseudo MV-algebra is isomorphic to the algebra of action functions where the binary operation is function composition, zero is $x \wedge y$ and unit is $x$. Then we define the so-called difference functions in pseudo MV-algebras and show how a pseudo MV-algebra can be reconstructed by them.


Keywords: pseudo MV-algebra, action function, guard function, difference functions.

2000 Mathematics Subject Classification: 06D35, 08A02, 03G25.

## 1. Introduction

In 1958, C.C. Chang introduced the notion of MV-algebra as an algebraic counterpart of the Łukasiewicz propositional calculus.

Non-commutative MV-algebras, named pseudo MV-algebras, were introduced by G. Georgescu and A. Iorgulescu in [8].

[^0]Independently, starting from intervals of (not necessarily commutative) lattice-ordered groups, J. Rachůnek established in [12] the concept of a GMV-algebra (generalized MV-algebra).

Definition 1. A Pseudo $M V$-algebra is an algebra $\mathcal{A}=(A ; \oplus, \neg, \sim, 0,1)$ of type ( $2,1,1,0,0$ ) satisfying the following axioms:
(A1) $x \oplus(y \oplus z)=(x \oplus y) \oplus z$;
(A2) $x \oplus 0=x=0 \oplus x$;
(A3) $x \oplus 1=1=1 \oplus x$;
(A4) $\neg 1=0=\sim 1$;
(A5) $\neg(\sim x \oplus \sim y)=\sim(\neg x \oplus \neg y)$;
(A6) $x \oplus(y \odot \sim x)=y \oplus(x \odot \sim y)=(\neg y \odot x) \oplus y=(\neg x \odot y) \oplus x$;
(A7) $\sim \neg x=x$
where the additional operation $\odot$ is defined via

$$
x \odot y=\sim(\neg x \oplus \neg y) .
$$

Note that this structure can be defined also (in term equivalent way) as an $\mathrm{FL}_{\omega}$-algebra that satisfies $x /(y \backslash x)=x \vee y=(x / y) \backslash x$; see [10].

If $\oplus$ is commutative, then $\sim$ coincides with $\neg$ and $(A ; \oplus, \neg, 0,1)$ becomes an MV-algebra. For basic properties of MV-algebras and pseudo MV-algebras we refer to [5, 8, 12].

Remark 1. The definition of a pseudo MV-algebra usually contains the following axiom (see e.g., $[4,8,12]$ ):
(A8) $y \odot(x \oplus \sim y)=(\neg x \oplus y) \odot x$.
However, F. Švrček shows recently in an unpublished preprint [11] that (A8) is redundant. More precisely, everyone can derive (A8) using the axioms of Definition 1 as follows.

We have

$$
\begin{aligned}
& y \odot(x \oplus \sim y)=\sim \neg y \odot(\sim \neg x \oplus \sim \sim \neg y) \\
& =\sim \neg y \odot \sim(\neg x \odot \sim \neg y)=\sim(\neg y \oplus(\neg x \odot \sim \neg y)) .
\end{aligned}
$$

Further, by (A6), we obtain

$$
\begin{aligned}
& \sim(\neg y \oplus(\neg x \odot \sim \neg y))=\sim((\neg \neg x \odot \neg y) \oplus \neg x \\
& =\sim(\neg \neg x \odot \neg y) \odot \sim \neg x=(\sim \neg \neg x \oplus \sim \neg y) \oplus x=(\neg x \oplus y) \odot x .
\end{aligned}
$$

Proposition 1 (see e.g., [8, 12]). Let $\mathcal{A}=(A ; \oplus, \neg, \sim, 0,1)$ be a pseudo MV-algebra, let

$$
x \odot y=\sim(\neg x \oplus \neg y) .
$$

Then the following identities hold:
(a) $\neg \sim x=x$;
(b) $(x \odot y) \odot z=x \odot(y \odot z)$;
(c) $x \odot \sim x=0=\neg x \odot x$;
(d) $x \odot 0=0=0 \odot x, \quad x \odot 1=x=1 \odot x ;$
(e) $\neg 0=1=\sim 0$.

Proposition 2 (see [8]). Let $\mathcal{A}=(A ; \oplus, \neg, \sim, 0,1)$ be a pseudo $M V$-algebra. For term functions

$$
\begin{aligned}
& x \vee y=x \oplus \sim(\neg y \oplus x)=\neg(x \oplus \sim y) \oplus y, \\
& x \wedge y=x \odot \sim(\neg y \odot x)=\neg(x \odot \sim y) \odot x,
\end{aligned}
$$

the algebra $\mathcal{L}(A)=(A ; \vee, \wedge, 0,1)$ is a bounded distributive lattice and the mappings $x \mapsto \neg x$ and $x \mapsto \sim x$ are mutually inverse antitone bijections on $A$.

The lattice order of $\mathcal{L}(A)$ is called the induced order of a given pseudo MV-algebra $\mathcal{A}$. Of course,

$$
x \leq y \quad \text { iff } \quad \neg x \oplus y=1 \quad \text { iff } \quad y \oplus \sim x=1
$$

Proposition 3 (see [4]). Let $\mathcal{A}=(A ; \oplus, \neg, \sim, 0,1)$ be a pseudo MV-algebra. For each $a \in A$, the mapping $f_{a}(x)=\neg x \oplus a$ is an antitone bijection on the interval $[a, 1]$ and its inverse mapping is given by $f_{a}^{-1}(x)=a \oplus \sim x$. Moreover, $x \oplus y=f_{x}^{-1}(\neg y \vee x)$.

Note that the first part of proposition is obvious as $f_{a}(x)=x \backslash a, f_{a}^{-1}(x)=$ $a / x$ and $a /(x \backslash a)=a \vee x=(a / x) \backslash a$.

## 2. A REpresentation of pseudo MV-ALgebras by means of BINARY FUNCTIONS

Cayley's Theorem provides a well-known representation of groups by means of certain unary functions (i.e. permutations) with composition as its binary operation. It was shown by W.C. Holland [9] that also $\ell$-groups can be represented by monotonous permutations. A representation of Boolean algebras by binary functions, the so-called "guard functions" was presented in [1]. Also MV-algebras were represented by binary functions which are an extended version of action functions (see [3]). The aim of this section is to give a representation of pseudo MV-algebras by certain binary functions which can be considered as action functions.

At first we define an action function on pseudo MV-algebra.
By an action we mean any process which starts whenever certain conditions are satisfied. Formally, if $a$ is a condition and $x$ a process, by an action we mean the proposition

$$
\text { IF } a=1 \quad \text { DO } \quad x, \quad \text { IF } a=0 \quad \text { DO NOTHING. }
$$

Consider a binary function

$$
h_{a}(x, y)=x \wedge(a \oplus y)
$$

defined on a pseudo MV-algebra $\mathcal{A}=(A ; \oplus, \neg, \sim, 0,1)$. One can easily see that

$$
h_{1}(x, 0)=x \wedge(1 \oplus 0)=x
$$

and

$$
h_{0}(x, 0)=x \wedge(0 \oplus 0)=0 .
$$

Hence, $h_{a}(x, 0)$ describes the aforementioned concept of action and hence $h_{a}(x, y)$ will be called an action function (cf. [3]).

For two action functions on $\mathcal{A}$ we define the following composition

$$
\left(h_{a} \circ h_{b}\right)(x, y)=h_{a}\left(x, h_{b}(x, y)\right) .
$$

Of course, $h_{a}(x, x)=x$ for each $a \in A$, thus this composition can be read as $\left(h_{a} \circ h_{b}\right)(x, y)=h_{a}\left(h_{b}(x, x), h_{b}(x, y)\right)$.

The following result can be found in [6]. Note that it actually holds for all involutive residuated lattices (for details see [7]).

Lemma 1. Let $\mathcal{A}=(A ; \oplus, \neg, \sim, 0,1)$ be a pseudo $M V$-algebra, let $a, b, c \in A$. Then

$$
a \oplus(b \wedge c)=(a \oplus b) \wedge(a \oplus c)
$$

Lemma 2. Let $\mathcal{A}=(A ; \oplus, \neg, \sim, 0,1)$ be a pseudo $M V$-algebra and $h_{a}, h_{b}$ be action functions on $\mathcal{A}$. Then $h_{a} \circ h_{b}=h_{a \oplus b}$.

Proof. By Theorem 3 in [12], $0 \leq a$ implies $x=0 \oplus x \leq a \oplus x$. According to this fact and the previous definitions and Lemma 1, we have:

$$
\begin{aligned}
& \left(h_{a} \circ h_{b}\right)(x, y)=h_{a}\left(x, h_{b}(x, y)\right)=x \wedge(a \oplus(x \wedge(b \oplus y))) \\
= & x \wedge(a \oplus x) \wedge(a \oplus b \oplus y)=x \wedge(a \oplus b \oplus y)=h_{a \oplus b}(x, y) .
\end{aligned}
$$

Theorem 1. Let $\mathcal{A}=(A ; \oplus, \neg, \sim, 0,1)$ be a pseudo $M V$-algebra and $H(A)=$ $\left\{h_{a}(x, y) ; a \in A\right\}$ the set of all action functions on $\mathcal{A}$. Define

$$
\neg h_{a}(x, y)=h_{\neg a}(x, y), \quad \sim h_{a}(x, y)=h_{\sim a}(x, y)
$$

Then $\mathcal{H}=(H(A) ; \circ, \neg, \sim, x \wedge y, x)$ is a pseudo $M V$-algebra which is isomorphic to $\mathcal{A}$ via $h_{a}(x, y) \mapsto a$.

Proof. We compute $h_{0}(x, y)=x \wedge(0 \oplus y)=x \wedge y$ and $h_{1}(x, y)=x \wedge(1 \oplus y)=$ $x$. By the previous results the map $a \mapsto h_{a}$ is clearly a homomorphism of a pseudo MV-algebra $\mathcal{A}$ onto $H(A)$ and, therefore, $\mathcal{H}$ is also a psedo MValgebra. Moreover, if $h_{a}=h_{b}$ then $x \wedge(a \oplus y)=x \wedge(b \oplus y)$ for all $x, y$ of $A$. In particular, for $x=1$ and $y=0$ we obtain $a=b$, i.e. this mapping is one-to-one and hence $\mathcal{H}$ is isomorphic to $\mathcal{A}$.

Similarly as for Boolean algebras [1], for $q$-algebras [2] and for MV-algebras [3], we can define guard functions $g_{a}(x, y)$ on a pseudo MV-algebra by

$$
g_{a}(x, y)=(a \oplus x) \wedge(\neg a \oplus y)
$$

The meaning of this is as follows. The element $a$ is considered as a "condition" similarly as for action functions. Hence, if $a=0$ then

$$
g_{0}(x, y)=(0 \oplus x) \wedge(1 \oplus y)=x \wedge 1=x
$$

and if $a=1$ then

$$
g_{1}(x, y)=(1 \oplus x) \wedge(0 \oplus y)=1 \wedge y=y
$$

Thus $g_{a}(x, y)$ makes decisions on the action $x$ or $y$ in dependency of the value of $a$. Of course, in many-valued logic we have also another values of $a$ distinct from 0 and 1 and hence this guard function is much more complex. However, guard functions can be constructed by means of action functions as follows

$$
g_{a}(x, y)=h_{0}\left(h_{a}(1, x), h_{\neg a}(1, y)\right)
$$

On the contrary, action functions are not expressible in guard functions. Nevertheless, we can use a guard function to express an action function and its negations $\neg, \sim$ in the function algebra $\mathcal{H}$ as follows

$$
h_{a}(x, y)=h_{0}\left(x, g_{a}(y, 1)\right)
$$

and

$$
\neg h_{a}(x, y)=h_{0}\left(x, g_{a}(1, y)\right), \quad \sim h_{a}(x, y)=h_{0}\left(x, g_{\sim \sim a}(1, y)\right) .
$$

Of course, we can define another sort of guard functions by

$$
k_{a}(x, y)=(a \oplus x) \wedge(\sim a \oplus y)
$$

whose behavior is similar.

## 3. Difference functions in pseudo MV-Algebras

The concept of difference function was already introduced for MV-algebras. Due to the fact that the binary operation of pseudo MV-algebra is not commutative and there are two unary operations $\neg$ and $\sim$, we should define two difference functions as follows.

Definition 2. Let $\mathcal{A}=(A ; \oplus, \neg, \sim, 0,1)$ be a pseudo MV-algebra. Define so-called difference functions $+_{1},+_{2}$ as follows:

$$
\begin{aligned}
& x+{ }_{1} y=(x \odot \sim y) \oplus(y \odot \sim x), \\
& x+{ }_{2} y=(\neg y \odot x) \oplus(\neg x \odot y) .
\end{aligned}
$$

Of course, $+_{1},+_{2}$ need not be commutative since $\oplus$ does not have this property either. By (c) of Proposition 1 and (A2), the difference functions satisfy the expected properties:

$$
x+{ }_{1} x=0, \quad x+{ }_{2} x=0 .
$$

Now, we give some basic properties of difference functions.

Lemma 3. Let $\mathcal{A}=(A ; \oplus, \neg, \sim, 0,1)$ be a pseudo $M V$-algebra and $+_{1},{ }_{+}$ difference functions. Then
(i) $\neg x+{ }_{1} \neg y=y+{ }_{2} x$;
(ii) $\sim x+{ }_{2} \sim y=y+{ }_{1} x$;
(iii) $x+{ }_{1} y=0 \Leftrightarrow x=y ;$
(iv) $x+2 y=0 \Leftrightarrow x=y$.

## Proof.

(i): $\neg x+{ }_{1} \neg y=(\neg x \odot \sim \neg y) \oplus(\neg y \odot \sim \neg x)=(\neg x \odot y) \oplus(\neg y \odot x)=y+{ }_{2} x$.
(ii): $\left.\sim x+{ }_{2} \sim y=(\neg \sim y \odot \sim x) \oplus \neg \sim x \odot \sim y\right)=(y \odot \sim x) \oplus$ $(x \odot \sim y)=y+{ }_{1} x$.
(iii): If $x=y$ then $x+{ }_{1} x=0$. Conversely, let $x+{ }_{1} y=0$. Then

$$
(x \odot \sim y) \oplus(y \odot \sim x)=0
$$

thus (according to Theorem 3 in [12])

$$
(x \odot \sim y) \vee(y \odot \sim x) \leq(x \odot \sim y) \oplus(y \odot \sim x)=0
$$

whence $x \odot \sim y=0$ and $y \odot \sim x=0$. By Theorem 5 in [12], $x \leq y$ and $y \leq x$, i.e. $x=y$.
(iv): In the same way as (iii).

Moreover, we can express the operation $\oplus$ by means of $+_{1},+_{2}$ and $\odot$ as follows.

Lemma 4. Let $\mathcal{A}=(A ; \oplus, \neg, \sim, 0,1)$ be a pseudo $M V$-algebra and $+{ }_{1},+_{2}$ difference functions. Then $1+1 x=\sim x, x+21=\neg x$ and

$$
x \oplus y=\left(1+{ }_{1} x\right) \odot\left(1+{ }_{1} y\right)+{ }_{2} 1 .
$$

## Proof.

$$
\begin{gathered}
1+{ }_{1} x=(1 \odot \sim x) \oplus(x \odot \sim 1)=\sim x \oplus 0=\sim x, \\
x+{ }_{2} 1=(\neg 1 \odot x) \oplus(\neg x \odot 1)=0 \oplus \neg x=\neg x .
\end{gathered}
$$

Further,

$$
\begin{aligned}
& \left(1+{ }_{1} x\right) \odot\left(1+{ }_{1} y\right)+{ }_{2} 1=(\sim x \odot \sim y)+{ }_{2} 1 \\
& =\neg(\sim x \odot \sim y)=\neg \sim x \oplus \neg \sim y=x \oplus y .
\end{aligned}
$$

Our next task is to set up axioms characterizing these difference functions and, further, to show that also conversely, a pseudo MV-algebra can be reconstructed by means of these difference functions and the binary operation $\odot$.

Theorem 2. Let $\mathcal{A}=(A ; \oplus, \neg, \sim, 0,1)$ be a pseudo $M V$-algebra and $+_{1}$, $+_{2}$ be the difference functions. Then the following identities (D1)-(D5), (M1)-(M3) are satisfied:
(D1) $\quad\left(\left(1+{ }_{1} x\right) \odot\left(1+{ }_{1} y\right)+{ }_{2} 1\right)+{ }_{2} 1=\left(x+{ }_{2} 1\right) \odot\left(y+{ }_{2} 1\right)$,

$$
1+{ }_{1}\left(\left(1+{ }_{1} x\right) \odot\left(1+{ }_{1} y\right)+{ }_{2} 1\right)=\left(1+{ }_{1} x\right) \odot\left(1+{ }_{1} y\right)
$$

(D2) $\left(1+{ }_{1} x\right)+{ }_{2} 1=x, \quad 1+1\left(x+{ }_{2} 1\right)=x$;
(D3) $1+{ }_{1} 1=0, \quad 1+{ }_{2} 1=0 ;$
(D4) $0+{ }_{1} x=x=x+{ }_{1} 0, \quad x+{ }_{2} 0=x=0+{ }_{2} x ;$
(D5) $\quad\left(1+{ }_{1} x\right) \odot\left(1+{ }_{1}\left(y \odot\left(1+{ }_{1} x\right)\right)\right)+{ }_{2} 1=\left(1+{ }_{1} y\right) \odot\left(1+{ }_{1}\left(x \odot\left(1+{ }_{1} y\right)\right)\right)+{ }_{2} 1$ $=\left(1+_{1}\left(\left(y+_{2} 1\right) \odot x\right) \odot\left(1+{ }_{1} y\right)\right)+_{2} 1=\left(1+_{1}\left(\left(x+_{2} 1\right) \odot y\right)\right.$

$$
\left.\odot\left(1+{ }_{1} x\right)\right)+_{2} 1
$$

$(\mathrm{M} 1) \quad 1 \odot x=x=x \odot 1 ;$
$(\mathrm{M} 2) \quad 0 \odot x=0=x \odot 0 ;$
$(\mathrm{M} 3) \quad(x \odot y) \odot z=x \odot(y \odot z)$.

## Proof.

(D1): $\quad\left(\left(1+{ }_{1} x\right) \odot\left(1+{ }_{1} y\right)+{ }_{2} 1\right)+{ }_{2} 1=\neg(\neg(\sim x \odot \sim y))$

$$
\begin{aligned}
& =\neg(\neg(\sim(\neg \sim x \oplus \neg \sim y))) \stackrel{(a)}{=} \neg(x \oplus y) \stackrel{(A 7)}{=} \neg(\sim \neg x \oplus \sim \neg y) \\
& \stackrel{(A 5)}{=} \sim(\neg \neg x \oplus \neg \neg y)=\neg x \odot \neg y=\left(x+{ }_{2} 1\right) \odot\left(y+{ }_{2} 1\right), \\
& 1+{ }_{1}\left(\left(1+{ }_{1} x\right) \odot\left(1+{ }_{1} y\right)+_{2} 1\right)=\sim(\neg(\sim x \odot \sim y)) \\
& \stackrel{(A 7)}{=} \sim x \odot \sim y=\left(1+_{1} x\right) \odot\left(1+{ }_{1} y\right)
\end{aligned}
$$

(D2): $\left(1+{ }_{1} x\right)+{ }_{2} 1=\neg \sim x \stackrel{(a)}{=} x$, $1+1(x+21)=\sim \neg x=x \quad$ by $(\mathrm{A} 7) ;$
$(\mathrm{D} 3): \quad 1+{ }_{1} 1=(1 \odot \sim 1) \oplus(1 \odot \sim 1) \stackrel{(c)}{=} 0 \oplus 0 \stackrel{(A 2)}{=} 0$, $1+21=(\neg 1 \odot 1) \oplus(\neg 1 \odot 1) \stackrel{(A 4)}{=}(0 \odot 1) \oplus(0 \odot 1) \stackrel{(d)}{=} 0 \oplus 0 \stackrel{(A 2)}{=} 0 ;$
$(\mathrm{D} 4): 0+_{1} x=(0 \odot \sim x) \oplus(x \odot \sim 0) \stackrel{(d),(e)}{=} 0 \oplus(x \odot 1) \stackrel{(d)}{=} 0 \oplus x \stackrel{(A 2)}{=} x$, analogously $x+{ }_{1} 0=x$,

$$
x+{ }_{2} 0=(\neg 0 \odot x) \oplus(\neg x \odot x) \stackrel{(d),(e)}{=} x \oplus 0 \stackrel{(A 2)}{=} x
$$

analogously $x+20=x$;
(D5): Using Lemma 4 we have:

$$
\begin{aligned}
& \left(1+{ }_{1} x\right) \odot\left(1+{ }_{1}\left(y \odot\left(1+{ }_{1} x\right)\right)\right)+{ }_{2} 1=x \oplus(y \odot \sim x) \\
& \left(1+{ }_{1} y\right) \odot\left(1+{ }_{1}\left(x \odot\left(1+{ }_{1} y\right)\right)\right)+{ }_{2} 1=y \oplus(x \odot \sim y) \\
& \left(1+{ }_{1}\left(\left(y+{ }_{2} 1\right) \odot x\right) \odot\left(1+{ }_{1} y\right)\right)+{ }_{2} 1=(\neg y \odot x) \oplus y \\
& \left(1+{ }_{1}\left(\left(x+{ }_{2} 1\right) \odot y\right) \odot\left(1+{ }_{1} x\right)\right)+{ }_{2} 1=(\neg x \odot y) \oplus x
\end{aligned}
$$

The rest of the proof of (D5) follows directly by (A6).
(M1), (M2) and (M3) follows immediately by (b) and (d) of Proposition 1.

Now, we can prove the converse.
Theorem 3. Let $A$ be a non-void set, $1 \in A$ and $+_{1},+_{2}, \odot$ be binary operations on $A$ satisfying the identities (D1)-(D5) and (M1), (M2), (M3). Then for

$$
\begin{gathered}
x \oplus y=\left(1+{ }_{1} x\right) \odot\left(1+{ }_{1} y\right)+{ }_{2} 1 \\
\sim x=1+{ }_{1} x \\
\neg x=x+{ }_{2} 1 \\
0=1+{ }_{1} 1
\end{gathered}
$$

the algebra $\mathcal{A}=(A ; \oplus, \neg, \sim, 0,1)$ is a pseudo $M V$-algebra.

Proof. We must verify the axioms (A1)-(A7).
(A1): $\quad(x \oplus y) \oplus z=\left(1+{ }_{1}(x \oplus y)\right) \odot\left(1+{ }_{1} z\right)+{ }_{2} 1$

$$
\begin{aligned}
& =\left(1+_{1}\left(\left(1+{ }_{1} x\right) \odot\left(1+_{1} y\right)+_{2} 1\right)\right) \odot\left(1+{ }_{1} z\right)+_{2} 1 \\
& \stackrel{(D 2)}{=}\left(\left(1+_{1} x\right) \odot\left(1+{ }_{1} y\right)\right) \odot\left(1+_{1} z\right)+_{2} 1 \\
& \stackrel{(M 3)}{=}\left(1+_{1} x\right) \odot\left(\left(1+_{1} y\right) \odot\left(1+_{1} z\right)\right)+_{2} 1 \\
& \stackrel{(D 2)}{=}\left(1+_{1} x\right) \odot\left(1+{ }_{1}\left(\left(1+{ }_{1} y\right) \odot\left(1+{ }_{1} z\right)+_{2} 1\right)\right)+_{2} 1 \\
& =\left(1+{ }_{1} x\right) \odot\left(1+_{1}(y \oplus z)\right)+{ }_{2} 1=x \oplus(y \oplus z)
\end{aligned}
$$

(A2): $\quad x \oplus 0=\left(1+{ }_{1} x\right) \odot\left(1+{ }_{1} 0\right)+21 \stackrel{(D 4)}{=}$

$$
\stackrel{(D 4)}{=}\left(1+_{1} x\right) \odot 1+_{2} 1 \stackrel{(M 1)}{=}\left(1+_{1} x\right)++_{2} 1 \stackrel{(D 2)}{=} x,
$$

analogously $0 \oplus x=x$.
(A3): $\quad x \oplus 1=\left(1+{ }_{1} x\right) \odot\left(1+{ }_{1} 1\right)+21 \stackrel{(D 3)}{=}$
$\stackrel{(D 3)}{=}\left(1+{ }_{1} x\right) \odot 0+{ }_{2} 1 \stackrel{(M 2)}{=} 0+{ }_{2} 1 \stackrel{(D 4)}{=} 1$,
analogously $1 \oplus x=1$.
(A4): $\neg 1=1++_{1} \stackrel{(D 3)}{=} 0, \quad \sim 1=1+11 \stackrel{(D 3)}{=} 0$.
(A5): $\quad \neg(\sim x \oplus \sim y)=\left(\left(1+{ }_{1} x\right) \oplus\left(1+{ }_{1} y\right)\right)+{ }_{2} 1$
$=\left(\left(1+{ }_{1}\left(1+{ }_{1} x\right)\right) \odot\left(1+{ }_{1}\left(1+{ }_{1} y\right)\right)+{ }_{2} 1\right)+{ }_{2} 1 \stackrel{(D 1)}{=}$
$\stackrel{(D 1)}{=}\left(\left(1+{ }_{1} x\right)+{ }_{2} 1\right) \odot\left(\left(1+{ }_{1} y\right)+{ }_{2} 1\right) \stackrel{(D 2)}{=} x \odot y \stackrel{(D 2)}{=}$
$\stackrel{(D 2)}{=}(1+1(x+21)) \odot\left(1+1\left(y+{ }_{2} 1\right)\right) \stackrel{(D 1)}{=}$
$\stackrel{(D 1)}{=} 1+{ }_{1}\left((1+1(x+21)) \odot(1+1(y+21))+{ }_{2} 1\right)$
$=1+{ }_{1}\left(\left(x+{ }_{2} 1\right) \oplus(y+21)\right)=\sim(\neg x \oplus \neg y)$.
(A6): Follows directly by (D5).
(A7): $\sim \neg x=1+1\left(x+{ }_{2} 1\right) \stackrel{(D 2)}{=} x$.

## References

[1] S.L. Bloom, Z. Ésik and E. Manes, A Cayley theorem for Boolean algebras, Amer. Math. Monthly 97 (1990), 831-833.
[2] I. Chajda, A representation of the algebra of quasiordered logic by binary functions, Demonstratio Mathem. 27 (1994), 601-607.
[3] I. Chajda and H. Länger, Action algebras, Italian J. of Pure Appl. Mathem., submitted.
[4] I. Chajda and J. Kühr, Pseudo MV-algebras and meet-semilattices with sectionally antitone permutations, Math. Slovaca 56 (2006), 275-288.
[5] R.L.O. Cignoli, I.M.L. D'Ottaviano and D. Mundici, Algebraic Foundations of Many-Valued Reasoning, Kluwer Acad. Publ., Dordrecht-Boston-London 2000.
[6] A. Dvurečenskij, Pseudo MV-algebras are intervals in $\ell$-groups, J. Aust. Math. Soc. 72 (3) (2002), 427-445.
[7] N. Galatos, P. Jipsen, T. Kowalski and H. Ono, Residuated Lattices, An Algebraic Glimpse at Substructural Logics, Elsevier 2007.
[8] G. Georgescu and A. Iorgelescu, Pseudo MV-algebras, Multiple Valued Log. 6 (2001), 95-135.
[9] A.M.W Glass and W.C. Holland, Lattice-Ordered Groups, Kluwer Acad. Publ., Dordrecht-Boston-London 1989.
[10] P. Jipsen and C. Tsinakis, A survey of Residuated Lattices, Ordered Agebraic Structures (Martinez J., editor), Kluwer Academic Publishers, Dordrecht, 2002, 19-56.
[11] J. Kühr and F. Švrček, Operators on unital $\ell$-groups, preprint, 2007.
[12] J. Rachůnek, A non-commutative generalization of MV-algebras, Czechoslovak Math. J. 52 (2002), 255-273.

Received 19 October 2007
Revised 1 April 2008


[^0]:    *This work is supported by the Research and Development Council of the Czech Government via the project MSM6198959214.

