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REMARKS ON PSEUDO MV-ALGEBRAS*

Ivan Chajda and Miroslav Kolařík

Department of Algebra and Geometry
Palacký University Olomouc
Tomkova 40, 779 00 Olomouc, Czech Republic
e-mail: chajda@inf.upol.cz
e-mail: kolarik@inf.upol.cz

Abstract

Pseudo MV-algebras (see e.g., [4, 6, 8]) are non-commutative extension of MV-algebras. We show that every pseudo MV-algebra is isomorphic to the algebra of action functions where the binary operation is function composition, zero is $x \wedge y$ and unit is x. Then we define the so-called difference functions in pseudo MV-algebras and show how a pseudo MV-algebra can be reconstructed by them.

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1. Introduction

In 1958, C.C. Chang introduced the notion of MV-algebra as an algebraic counterpart of the Łukasiewicz propositional calculus.

Non-commutative MV-algebras, named pseudo MV-algebras, were introduced by G. Georgescu and A. Iorgulescu in [8].

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Independently, starting from intervals of (not necessarily commutative) lattice-ordered groups, J. Rachůnek established in [12] the concept of a GMV-algebra (generalized MV-algebra).

Definition 1. A *Pseudo MV-algebra* is an algebra $\mathcal{A} = (A; \oplus, \neg, \sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ satisfying the following axioms:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(A2) $x \oplus 0 = x = 0 \oplus x$;

(A3) $x \oplus 1 = 1 = 1 \oplus x$;

(A4) $\neg 1 = 0 = \sim 1$;

(A5) $\neg(\sim x \oplus \sim y) = \sim(\neg x \oplus \neg y)$;

(A6) $x \oplus (y \odot \sim x) = y \oplus (x \odot \sim y) = (\neg y \odot x) \oplus y = (\neg x \odot y) \oplus x$;

(A7) $\sim \neg x = x$

where the additional operation $\odot$ is defined via

$$x \odot y = \sim(\neg x \oplus \neg y).$$

Note that this structure can be defined also (in term equivalent way) as an FL_ω-algebra that satisfies $x/(y\backslash x) = x \vee y = (x/y)\backslash x$; see [10].

If $\oplus$ is commutative, then $\sim$ coincides with $\neg$ and $(A; \oplus, \neg, 0, 1)$ becomes an MV-algebra. For basic properties of MV-algebras and pseudo MV-algebras we refer to [5, 8, 12].

Remark 1. The definition of a pseudo MV-algebra usually contains the following axiom (see e.g., [4, 8, 12]):

(A8) $y \odot (x \oplus \sim y) = (\neg x \oplus y) \odot x$.

However, F. Švrček shows recently in an unpublished preprint [11] that (A8) is redundant. More precisely, everyone can derive (A8) using the axioms of Definition 1 as follows.

We have

$$y \odot (x \oplus \sim y) = \sim \neg y \odot (\sim \neg x \oplus \sim\sim \neg y)$$

$$= \sim \neg y \odot \sim (\neg x \odot \sim \neg y) = \sim (\neg y \oplus (\neg x \odot \sim \neg y)).$$

Further, by (A6), we obtain

$$\sim (\neg y \oplus (\neg x \odot \sim \neg y)) = \sim ((\neg\neg x \odot \neg y) \oplus \neg x$$

$$= \sim (\neg\neg x \odot \neg y) \odot \sim \neg x = (\sim \neg\neg x \oplus \sim \neg y) \oplus x = (\neg x \oplus y) \odot x.$$

Proposition 1 (see e.g., [8, 12]). *Let $\mathcal{A} = (A; \oplus, \neg, \sim, 0, 1)$ be a pseudo MV-algebra, let*

$$x \odot y = \sim (\neg x \oplus \neg y).$$

Then the following identities hold:

(a) $\neg \sim x = x$;

(b) $(x \odot y) \odot z = x \odot (y \odot z)$;

(c) $x \odot \sim x = 0 = \neg x \odot x$;

(d) $x \odot 0 = 0 = 0 \odot x, \quad x \odot 1 = x = 1 \odot x$;

(e) $\neg 0 = 1 = \sim 0$.

Proposition 2 (see [8]). *Let $\mathcal{A} = (A; \oplus, \neg, \sim, 0, 1)$ be a pseudo MV-algebra. For term functions*

$$x \vee y = x \oplus \sim (\neg y \oplus x) = \neg(x \oplus \sim y) \oplus y,$$

$$x \wedge y = x \odot \sim (\neg y \odot x) = \neg(x \odot \sim y) \odot x,$$

the algebra $\mathfrak{L}(A) = (A; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the mappings $x \mapsto \neg x$ and $x \mapsto \sim x$ are mutually inverse antitone bijections on A.

The lattice order of $\mathcal{L}(A)$ is called the *induced order* of a given pseudo MV-algebra $\mathcal{A}$. Of course,

$$x \leq y \quad \text{iff} \quad \neg x \oplus y = 1 \quad \text{iff} \quad y \oplus \sim x = 1.$$

Proposition 3 (see [4]). *Let $\mathcal{A} = (A; \oplus, \neg, \sim, 0, 1)$ be a pseudo MV-algebra. For each $a \in A$, the mapping $f_a(x) = \neg x \oplus a$ is an antitone bijection on the interval $[a, 1]$ and its inverse mapping is given by $f_a^{-1}(x) = a \oplus \sim x$. Moreover, $x \oplus y = f_x^{-1}(\neg y \vee x)$.*

Note that the first part of proposition is obvious as $f_a(x) = x \backslash a$, $f_a^{-1}(x) = a/x$ and $a/(x \backslash a) = a \vee x = (a/x) \backslash a$.

2. A representation of pseudo MV-algebras by means of binary functions

Cayley's Theorem provides a well-known representation of groups by means of certain unary functions (i.e. permutations) with composition as its binary operation. It was shown by W.C. Holland [9] that also ℓ-groups can be represented by monotonous permutations. A representation of Boolean algebras by binary functions, the so-called "guard functions" was presented in [1]. Also MV-algebras were represented by binary functions which are an extended version of action functions (see [3]). The aim of this section is to give a representation of pseudo MV-algebras by certain binary functions which can be considered as action functions.

At first we define an action function on pseudo MV-algebra.

By an *action* we mean any process which starts whenever certain conditions are satisfied. Formally, if a is a condition and x a process, by an action we mean the proposition

$$\text{IF} \ \ a = 1 \ \ \text{DO} \ \ x, \ \ \text{IF} \ \ a = 0 \ \ \text{DO NOTHING}.$$

Consider a binary function

$$h_a(x, y) = x \wedge (a \oplus y)$$

defined on a pseudo MV-algebra $\mathcal{A} = (A; \oplus, \neg, \sim, 0, 1)$. One can easily see that

$$h_1(x, 0) = x \wedge (1 \oplus 0) = x$$

and

$$h_0(x, 0) = x \wedge (0 \oplus 0) = 0.$$

Hence, $h_a(x, 0)$ describes the aforementioned concept of action and hence $h_a(x, y)$ will be called an *action function* (cf. [3]).

For two action functions on $\mathcal{A}$ we define the following composition

$$(h_a \circ h_b)(x, y) = h_a(x, h_b(x, y)).$$

Of course, $h_a(x, x) = x$ for each $a \in A$, thus this composition can be read as $(h_a \circ h_b)(x, y) = h_a(h_b(x, x), h_b(x, y))$.

The following result can be found in [6]. Note that it actually holds for all involutive residuated lattices (for details see [7]).

Lemma 1. *Let $\mathcal{A} = (A; \oplus, \neg, \sim, 0, 1)$ be a pseudo MV-algebra, let $a, b, c \in A$. Then*

$$a \oplus (b \wedge c) = (a \oplus b) \wedge (a \oplus c).$$

Lemma 2. *Let $\mathcal{A} = (A; \oplus, \neg, \sim, 0, 1)$ be a pseudo MV-algebra and h_a, h_b be action functions on $\mathcal{A}$. Then $h_a \circ h_b = h_{a \oplus b}$.*

Proof. By Theorem 3 in [12], $0 \leq a$ implies $x = 0 \oplus x \leq a \oplus x$. According to this fact and the previous definitions and Lemma 1, we have:

$$(h_a \circ h_b)(x, y) = h_a(x, h_b(x, y)) = x \wedge (a \oplus (x \wedge (b \oplus y)))$$

$$= x \wedge (a \oplus x) \wedge (a \oplus b \oplus y) = x \wedge (a \oplus b \oplus y) = h_{a \oplus b}(x, y).$$

■

Theorem 1. *Let $\mathcal{A} = (A; \oplus, \neg, \sim, 0, 1)$ be a pseudo MV-algebra and $H(A) = \{h_a(x, y);\ a \in A\}$ the set of all action functions on $\mathcal{A}$. Define*

$$\neg h_a(x, y) = h_{\neg a}(x, y), \qquad \sim h_a(x, y) = h_{\sim a}(x, y).$$

Then $\mathcal{H} = (H(A); \circ, \neg, \sim, x \wedge y, x)$ is a pseudo MV-algebra which is isomorphic to $\mathcal{A}$ via $h_a(x, y) \mapsto a$.

Proof. We compute $h_0(x, y) = x \wedge (0 \oplus y) = x \wedge y$ and $h_1(x, y) = x \wedge (1 \oplus y) = x$. By the previous results the map $a \mapsto h_a$ is clearly a homomorphism of a pseudo MV-algebra $\mathcal{A}$ onto $H(A)$ and, therefore, $\mathcal{H}$ is also a psedo MV-algebra. Moreover, if $h_a = h_b$ then $x \wedge (a \oplus y) = x \wedge (b \oplus y)$ for all x, y of A. In particular, for $x = 1$ and $y = 0$ we obtain $a = b$, i.e. this mapping is one-to-one and hence $\mathcal{H}$ is isomorphic to $\mathcal{A}$. ∎

Similarly as for Boolean algebras [1], for q-algebras [2] and for MV-algebras [3], we can define *guard functions* $g_a(x, y)$ on a pseudo MV-algebra by

$$g_a(x, y) = (a \oplus x) \wedge (\neg a \oplus y).$$

The meaning of this is as follows. The element a is considered as a "condition" similarly as for action functions. Hence, if $a = 0$ then

$$g_0(x, y) = (0 \oplus x) \wedge (1 \oplus y) = x \wedge 1 = x$$

and if $a = 1$ then

$$g_1(x, y) = (1 \oplus x) \wedge (0 \oplus y) = 1 \wedge y = y.$$

Thus $g_a(x, y)$ makes decisions on the action x or y in dependency of the value of a. Of course, in many-valued logic we have also another values of a distinct from 0 and 1 and hence this guard function is much more complex. However, guard functions can be constructed by means of action functions as follows

$$g_a(x, y) = h_0(h_a(1, x), h_{\neg a}(1, y)).$$

On the contrary, action functions are not expressible in guard functions. Nevertheless, we can use a guard function to express an action function and its negations $\neg$, $\sim$ in the function algebra $\mathcal{H}$ as follows

$$h_a(x,y) = h_0(x, g_a(y,1))$$

and

$$\neg h_a(x,y) = h_0(x, g_a(1,y)), \quad \sim h_a(x,y) = h_0(x, g_{\sim\sim a}(1,y)).$$

Of course, we can define another sort of guard functions by

$$k_a(x,y) = (a \oplus x) \wedge (\sim a \oplus y)$$

whose behavior is similar.

3. Difference functions in pseudo MV-algebras

The concept of difference function was already introduced for MV-algebras. Due to the fact that the binary operation of pseudo MV-algebra is not commutative and there are two unary operations $\neg$ and $\sim$, we should define two difference functions as follows.

Definition 2. Let $\mathcal{A} = (A; \oplus, \neg, \sim, 0, 1)$ be a pseudo MV-algebra. Define so-called *difference functions* $+_1$, $+_2$ as follows:

$$x +_1 y = (x \odot \sim y) \oplus (y \odot \sim x),$$

$$x +_2 y = (\neg y \odot x) \oplus (\neg x \odot y).$$

Of course, $+_1$, $+_2$ need not be commutative since $\oplus$ does not have this property either. By (c) of Proposition 1 and (A2), the difference functions satisfy the expected properties:

$$x +_1 x = 0, \quad x +_2 x = 0.$$

Now, we give some basic properties of difference functions.

Lemma 3. *Let* $\mathcal{A} = (A; \oplus, \neg, \sim, 0, 1)$ *be a pseudo MV-algebra and* $+_1$, $+_2$ *difference functions. Then*

(i) $\neg x +_1 \neg y = y +_2 x$;

(ii) $\sim x +_2 \sim y = y +_1 x$;

(iii) $x +_1 y = 0 \Leftrightarrow x = y$;

(iv) $x +_2 y = 0 \Leftrightarrow x = y$.

Proof.

(i): $\neg x +_1 \neg y = (\neg x \odot \sim \neg y) \oplus (\neg y \odot \sim \neg x) = (\neg x \odot y) \oplus (\neg y \odot x) = y +_2 x$.

(ii): $\sim x +_2 \sim y = (\neg \sim y \odot \sim x) \oplus \neg \sim x \odot \sim y) = (y \odot \sim x) \oplus (x \odot \sim y) = y +_1 x$.

(iii): If $x = y$ then $x +_1 x = 0$. Conversely, let $x +_1 y = 0$. Then

$$(x \odot \sim y) \oplus (y \odot \sim x) = 0,$$

thus (according to Theorem 3 in [12])

$$(x \odot \sim y) \vee (y \odot \sim x) \leq (x \odot \sim y) \oplus (y \odot \sim x) = 0,$$

whence $x \odot \sim y = 0$ and $y \odot \sim x = 0$. By Theorem 5 in [12], $x \leq y$ and $y \leq x$, i.e. $x = y$.

(iv): In the same way as (iii).

■

Moreover, we can express the operation $\oplus$ by means of $+_1$, $+_2$ and $\odot$ as follows.

Lemma 4. *Let $\mathcal{A} = (A; \oplus, \neg, \sim, 0, 1)$ be a pseudo MV-algebra and $+_1$, $+_2$ difference functions. Then $1 +_1 x = \sim x$, $x +_2 1 = \neg x$ and*

$$x \oplus y = (1 +_1 x) \odot (1 +_1 y) +_2 1.$$

Proof.

$$1 +_1 x = (1 \odot \sim x) \oplus (x \odot \sim 1) = \sim x \oplus 0 = \sim x,$$

$$x +_2 1 = (\neg 1 \odot x) \oplus (\neg x \odot 1) = 0 \oplus \neg x = \neg x.$$

Further,

$$(1 +_1 x) \odot (1 +_1 y) +_2 1 = (\sim x \odot \sim y) +_2 1$$

$$= \neg(\sim x \odot \sim y) = \neg \sim x \oplus \neg \sim y = x \oplus y.$$

■

Our next task is to set up axioms characterizing these difference functions and, further, to show that also conversely, a pseudo MV-algebra can be reconstructed by means of these difference functions and the binary operation $\odot$.

Theorem 2. *Let $\mathcal{A} = (A; \oplus, \neg, \sim, 0, 1)$ be a pseudo MV-algebra and $+_1$, $+_2$ be the difference functions. Then the following identities* (D1)–(D5), (M1)–(M3) *are satisfied:*

(D1) $((1 +_1 x) \odot (1 +_1 y) +_2 1) +_2 1 = (x +_2 1) \odot (y +_2 1)$,

$1 +_1 ((1 +_1 x) \odot (1 +_1 y) +_2 1) = (1 +_1 x) \odot (1 +_1 y)$;

(D2) $(1 +_1 x) +_2 1 = x$, $\quad 1 +_1 (x +_2 1) = x$;

(D3) $1 +_1 1 = 0, \quad 1 +_2 1 = 0;$

(D4) $0 +_1 x = x = x +_1 0, \quad x +_2 0 = x = 0 +_2 x;$

(D5) $(1+_1 x)\odot(1+_1(y\odot(1+_1 x)))+_2 1 = (1+_1 y)\odot(1+_1(x\odot(1+_1 y)))+_2 1$

$= (1 +_1 ((y +_2 1) \odot x) \odot (1 +_1 y)) +_2 1 = (1 +_1 ((x +_2 1) \odot y)$

$\odot (1 +_1 x)) +_2 1;$

(M1) $1 \odot x = x = x \odot 1;$

(M2) $0 \odot x = 0 = x \odot 0;$

(M3) $(x \odot y) \odot z = x \odot (y \odot z).$

Proof.

(D1): $((1 +_1 x) \odot (1 +_1 y) +_2 1) +_2 1 = \neg(\neg(\sim x \odot \sim y))$

$= \neg(\neg(\sim (\neg \sim x \oplus \neg \sim y))) \overset{(a)}{=} \neg(x \oplus y) \overset{(A7)}{=} \neg(\sim \neg x \oplus \sim \neg y)$

$\overset{(A5)}{=} \sim (\neg\neg x \oplus \neg\neg y) = \neg x \odot \neg y = (x +_2 1) \odot (y +_2 1),$

$1 +_1 ((1 +_1 x) \odot (1 +_1 y) +_2 1) = \sim (\neg(\sim x \odot \sim y))$

$\overset{(A7)}{=} \sim x \odot \sim y = (1 +_1 x) \odot (1 +_1 y);$

(D2): $(1 +_1 x) +_2 1 = \neg \sim x \overset{(a)}{=} x,$

$1 +_1 (x +_2 1) = \sim \neg x = x$ by (A7);

(D3): $1 +_1 1 = (1 \odot \sim 1) \oplus (1 \odot \sim 1) \overset{(c)}{=} 0 \oplus 0 \overset{(A2)}{=} 0,$

$1 +_2 1 = (\neg 1 \odot 1) \oplus (\neg 1 \odot 1) \overset{(A4)}{=} (0 \odot 1) \oplus (0 \odot 1) \overset{(d)}{=} 0 \oplus 0 \overset{(A2)}{=} 0;$

(D4): $0 +_1 x = (0 \odot \sim x) \oplus (x \odot \sim 0) \overset{(d),(e)}{=} 0 \oplus (x \odot 1) \overset{(d)}{=} 0 \oplus x \overset{(A2)}{=} x,$

analogously $x +_1 0 = x,$

$x +_2 0 = (\neg 0 \odot x) \oplus (\neg x \odot x) \overset{(d),(e)}{=} x \oplus 0 \overset{(A2)}{=} x,$

analogously $x +_2 0 = x;$

(D5): Using Lemma 4 we have:

$$(1 +_1 x) \odot (1 +_1 (y \odot (1 +_1 x))) +_2 1 = x \oplus (y \odot \sim x),$$

$$(1 +_1 y) \odot (1 +_1 (x \odot (1 +_1 y))) +_2 1 = y \oplus (x \odot \sim y),$$

$$(1 +_1 ((y +_2 1) \odot x) \odot (1 +_1 y)) +_2 1 = (\neg y \odot x) \oplus y,$$

$$(1 +_1 ((x +_2 1) \odot y) \odot (1 +_1 x)) +_2 1 = (\neg x \odot y) \oplus x.$$

The rest of the proof of (D5) follows directly by (A6).
(M1), (M2) and (M3) follows immediately by (b) and (d) of Proposition 1.

■

Now, we can prove the converse.

Theorem 3. *Let A be a non-void set, $1 \in A$ and $+_1$, $+_2$, $\odot$ be binary operations on A satisfying the identities* (D1)–(D5) *and* (M1), (M2), (M3). *Then for*

$$x \oplus y = (1 +_1 x) \odot (1 +_1 y) +_2 1,$$

$$\sim x = 1 +_1 x,$$

$$\neg x = x +_2 1,$$

$$0 = 1 +_1 1,$$

the algebra $\mathcal{A} = (A; \oplus, \neg, \sim, 0, 1)$ is a pseudo MV-algebra.

Proof. We must verify the axioms (A1)–(A7).

(A1): $(x \oplus y) \oplus z = (1 +_1 (x \oplus y)) \odot (1 +_1 z) +_2 1$

$$= (1 +_1 ((1 +_1 x) \odot (1 +_1 y) +_2 1)) \odot (1 +_1 z) +_2 1$$

$$\overset{(D2)}{=} ((1 +_1 x) \odot (1 +_1 y)) \odot (1 +_1 z) +_2 1$$

$$\overset{(M3)}{=} (1 +_1 x) \odot ((1 +_1 y) \odot (1 +_1 z)) +_2 1$$

$$\overset{(D2)}{=} (1 +_1 x) \odot (1 +_1 ((1 +_1 y) \odot (1 +_1 z) +_2 1)) +_2 1$$

$$= (1 +_1 x) \odot (1 +_1 (y \oplus z)) +_2 1 = x \oplus (y \oplus z).$$

(A2): $x \oplus 0 = (1 +_1 x) \odot (1 +_1 0) +_2 1 \overset{(D4)}{=}$

$\overset{(D4)}{=} (1 +_1 x) \odot 1 +_2 1 \overset{(M1)}{=} (1 +_1 x) +_2 1 \overset{(D2)}{=} x,$

analogously $0 \oplus x = x$.

(A3): $x \oplus 1 = (1 +_1 x) \odot (1 +_1 1) +_2 1 \overset{(D3)}{=}$

$\overset{(D3)}{=} (1 +_1 x) \odot 0 +_2 1 \overset{(M2)}{=} 0 +_2 1 \overset{(D4)}{=} 1,$

analogously $1 \oplus x = 1$.

(A4): $\neg 1 = 1 +_2 1 \overset{(D3)}{=} 0, \quad \sim 1 = 1 +_1 1 \overset{(D3)}{=} 0.$

(A5): $\neg(\sim x \oplus \sim y) = ((1 +_1 x) \oplus (1 +_1 y)) +_2 1$

$= ((1 +_1 (1 +_1 x)) \odot (1 +_1 (1 +_1 y)) +_2 1) +_2 1 \overset{(D1)}{=}$

$\overset{(D1)}{=} ((1 +_1 x) +_2 1) \odot ((1 +_1 y) +_2 1) \overset{(D2)}{=} x \odot y \overset{(D2)}{=}$

$\overset{(D2)}{=} (1 +_1 (x +_2 1)) \odot (1 +_1 (y +_2 1)) \overset{(D1)}{=}$

$\overset{(D1)}{=} 1 +_1 ((1 +_1 (x +_2 1)) \odot (1 +_1 (y +_2 1)) +_2 1)$

$= 1 +_1 ((x +_2 1) \oplus (y +_2 1)) = \sim (\neg x \oplus \neg y).$

(A6): Follows directly by (D5).

(A7): $\sim\neg x = 1 +_1 (x +_2 1) \overset{(D2)}{=} x.$ ∎

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