Discussiones Mathematicae
General Algebra and Applications 28 (2008) 251–259

MINIMAL BOUNDED LATTICES WITH AN ANTITONE INVOLUTION THE COMPLEMENTED ELEMENTS OF WHICH DO NOT FORM A SUBLATTICE

Ivan Chajda

Department of Algebra and Geometry
Palacký University Olomouc
Tomkova 40, 77900 Olomouc, Czech Republic

e-mail: chajda@inf.upol.cz

and

Helmut Länger

Institute of Discrete Mathematics and Geometry
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Vienna, Austria

e-mail: h.laenger@tuwien.ac.at

Abstract

Bounded lattices with an antitone involution the complemented elements of which do not form a sublattice must contain two complemented elements such that not both their join and their meet are complemented. We distinguish (up to symmetry) eight cases and in each of these cases we present such a lattice of minimal cardinality.

Keywords: bounded lattice, antitone involution, complemented element.

2000 Mathematics Subject Classification: 06C15.

*Support of the research of the first author by the Project MSM 6198959214 of the Research and Development Council of the Czech Government is gratefully acknowledged.

*-lattices (these are bounded lattices with an involution, denoted by *, satisfying De Morgan's laws) often serve as models for logics. *-complemented elements of such logics can be considered as sharp assertions corresponding to classical logic. The natural question arises when these elements form a sublogic. The problem of characterizing the structure of bounded lattices with an antitone involution the complemented elements of which form a sublattice seems to be very hard. A partial solution of this problem was obtained in [2]. We consider bounded lattices which do not have this property. The aim of this paper is to present a list of such lattices of minimal cardinality. [1] and [3] are standard references concerning lattice theory.

We start with the definition of a bounded lattice with an antitone involution and of a complemented element.

Definition 1. A *bounded lattice with an antitone involution* is an algebra $\mathcal{L} = (L, \vee, \wedge, ^*, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and

$$(x \vee y)^* = x^* \wedge y^*,$$

$$(x \wedge y)^* = x^* \vee y^* \text{ and}$$

$$(x^*)^* = x$$

hold for all $x, y \in L$. An element a of L is called complemented if $a \vee a^* = 1$ and $a \wedge a^* = 0$. Let $\mathrm{CE}(\mathcal{L})$ denote the set of all complemented elements of $\mathcal{L}$.

In the following let $\mathcal{L} = (L, \vee, \wedge, ^*, 0, 1)$ denote an arbitrary but fixed bounded lattice with an antitone involution.

It is evident that if $\mathcal{L}$ is, moreover, distributive, i.e., a De Morgan algebra, then $\mathrm{CE}(\mathcal{L})$ is the set of its Boolean elements and hence a sublattice of $\mathcal{L}$. A more complex case was solved by the authors in [2]. Further, let us mention that $0, 1 \in \mathrm{CE}(\mathcal{L})$ in each case.

Lemma 2. *Let $a, b \in \mathrm{CE}(\mathcal{L})$. If $a \vee b \notin \mathrm{CE}(\mathcal{L})$ or $a \wedge b \notin \mathrm{CE}(\mathcal{L})$, then $a \wedge b \not\geq a^* \vee b^*$ and $a^* \wedge b^* \not\geq a \vee b$.*

Proof. $a \vee b \notin \mathrm{CE}(\mathcal{L})$ and $a \wedge b \geq a^* \vee b^*$ would imply $a^* \leq a$ and hence $a = a \vee a^* = 1$ whence $1 = a \vee b \notin \mathrm{CE}(\mathcal{L})$ which is a contradiction. The other cases follow in a similar way. ∎

Lemma 3. *Let* $a, b \in \mathrm{CE}(\mathcal{L})$.

(i) *If* $a \vee b \notin \mathrm{CE}(\mathcal{L})$ *then* $0, 1, a, a^*, b, b^*, a \vee b, a^* \wedge b^*$ *are pairwise distinct.*

(ii) *If* $a \wedge b \notin \mathrm{CE}(\mathcal{L})$ *then* $0, 1, a, a^*, b, b^*, a \wedge b, a^* \vee b^*$ *are pairwise distinct.*

(iii) *If* $a \vee b, a \wedge b \notin \mathrm{CE}(\mathcal{L})$ *then* $0, 1, a, a^*, b, b^*, a \vee b, a^* \wedge b^*, a \wedge b, a^* \vee b^*$ *are pairwise distinct.*

Proof.

(i): $0 = 1$ would imply $a \vee b = 0 \in \mathrm{CE}(\mathcal{L})$.

$0 = a$ resp. $1 = a^*$ would imply $a \vee b = 0 \vee b = b \in \mathrm{CE}(\mathcal{L})$.

$0 = a^*$ resp. $1 = a$ would imply $a \vee b = 1 \vee b = 1 \in \mathrm{CE}(\mathcal{L})$.

$0 = a \vee b$ resp. $1 = a^* \wedge b^*$ would imply $a \vee b = 0 \in \mathrm{CE}(\mathcal{L})$.

$0 = a^* \wedge b^*$ resp. $1 = a \vee b$ would imply $a \vee b = 1 \in \mathrm{CE}(\mathcal{L})$.

$a = a^*$ would imply $0 = a \wedge a^* = a \wedge a = a = a \vee a = a \vee a^* = 1$.

$a = b$ resp. $a^* = b^*$ would imply $a \vee b = a \vee a = a \in \mathrm{CE}(\mathcal{L})$.

$a = b^*$ resp. $a^* = b$ would imply $a \vee b = b^* \vee b = 1 \in \mathrm{CE}(\mathcal{L})$.

$a = a \vee b$ resp. $a^* = a^* \wedge b^*$ would imply $a \vee b = a \in \mathrm{CE}(\mathcal{L})$.

$a = a^* \wedge b^*$ resp. $a^* = a \vee b$ would imply $a \vee b = a^* \in \mathrm{CE}(\mathcal{L})$.

$a \vee b = a^* \wedge b^*$ is impossible because of Lemma 2.

The rest follows by symmetry of a and b.

(ii): Follows by duality.

(iii): $a \vee b = a \wedge b$ resp. $a^* \wedge b^* = a^* \vee b^*$ would imply $a \vee b = a \vee (a \vee b) = a \vee (a \wedge b) = a \in \mathrm{CE}(\mathcal{L})$.

$a \vee b = a^* \vee b^*$ resp. $a^* \wedge b^* = a \wedge b$ would imply $a \vee b = a \vee (a \vee b) = a \vee (a^* \vee b^*) = 1 \in \mathrm{CE}(\mathcal{L})$.

The rest follows from (i) and (ii). ∎

Lemma 4. *If* $a, b \in \mathrm{CE}(\mathcal{L})$, $a \vee b, a \wedge b \notin \mathrm{CE}(\mathcal{L})$, $a \wedge b < a^* \vee b^*$ *and* $a^* \wedge b^* < a \vee b$ *then* (i)–(iii) *hold:*

(i) $0, 1, a, a^*, b, b^*, a \vee b, a^* \wedge b^*, a \wedge b, a^* \vee b^*, (a \wedge b) \vee (a^* \wedge b^*)$ *are pairwise distinct.*

(ii) $0, 1, a, a^*, b, b^*, a \vee b, a^* \wedge b^*, a \wedge b, a^* \vee b^*, (a \vee b) \wedge (a^* \vee b^*)$ *are pairwise distinct.*

(iii) $(a \wedge b) \vee (a^* \wedge b^*) \leq (a \vee b) \wedge (a^* \vee b^*)$

Proof.

(i): $(a \wedge b) \vee (a^* \wedge b^*) = 0$ would imply $a \wedge b = 0$.

$(a \wedge b) \vee (a^* \wedge b^*) = 1$ would imply $a \vee b = (a \wedge b) \vee (a \vee b) \geq (a \wedge b) \vee (a^* \wedge b^*) = 1$.

$(a \wedge b) \vee (a^* \wedge b^*) = a$ would imply $1 = a \vee a^* = (a \wedge b) \vee (a^* \wedge b^*) \vee a^* \leq a^* \vee b^*$.

$(a \wedge b) \vee (a^* \wedge b^*) = a^*$ would imply $1 = a \vee a^* = a \vee (a \wedge b) \vee (a^* \wedge b^*) \leq a \vee b$.

$(a \wedge b) \vee (a^* \wedge b^*) = a \vee b$ would imply $a \vee b = (a \wedge b) \vee (a^* \wedge b^*) \leq (a \vee b) \wedge (a^* \vee b^*) = a^* \wedge b^* < a \vee b$.

$(a \wedge b) \vee (a^* \wedge b^*) = a^* \wedge b^*$ would imply $a \wedge b = a \wedge (a \wedge b) \leq a \wedge (a^* \wedge b^*) = 0$.

$(a \wedge b) \vee (a^* \wedge b^*) = a \wedge b$ would imply $a^* \wedge b^* = a^* \wedge (a^* \wedge b^*) \leq a^* \wedge (a \wedge b) = 0$.

$(a \wedge b) \vee (a^* \wedge b^*) = a^* \vee b^*$ would imply $a^* \vee b^* = (a \wedge b) \vee (a^* \wedge b^*) \leq (a \vee b) \wedge (a^* \vee b^*) = a \wedge b < a^* \vee b^*$.

The rest follows by symmetry of a and b.

(ii): Follows by duality.

(iii): Follows from the assumptions. ∎

In the following, if two elements a, b of L are incomparable, we write $a \parallel b$.

Theorem 5. *Let $\mathcal{L} = (L, \vee, \wedge, ^*, 0, 1)$ be a bounded lattice with an antitone involution the set $\mathrm{CE}(\mathcal{L})$ of all complemented elements of which does not form a sublattice. Then there exist $a, b \in \mathrm{CE}(\mathcal{L})$ such that either $a \vee b \notin \mathrm{CE}(\mathcal{L})$ or $a \wedge b \notin \mathrm{CE}(\mathcal{L})$ or both and, up to symmetry, the following cases are possible:*

(i) $a \vee b, a \wedge b \notin \mathrm{CE}(\mathcal{L})$, $a \wedge b < a^* \vee b^*$ *and* $a^* \wedge b^* < a \vee b$

(ii) $a \vee b, a \wedge b \notin \mathrm{CE}(\mathcal{L})$, $a \wedge b < a^* \vee b^*$ *and* $a^* \wedge b^* \parallel a \vee b$

(iii) $a \vee b, a \wedge b \notin \mathrm{CE}(\mathcal{L})$, $a \wedge b \parallel a^* \vee b^*$ *and* $a^* \wedge b^* < a \vee b$

(iv) $a \vee b, a \wedge b \notin \mathrm{CE}(\mathcal{L})$, $a \wedge b \parallel a^* \vee b^*$ *and* $a^* \wedge b^* \parallel a \vee b$

(v) $a \vee b \in \mathrm{CE}(\mathcal{L})$, $a \wedge b \notin \mathrm{CE}(\mathcal{L})$, $a \vee b = 1$ *and* $a \wedge b < a^* \vee b^*$

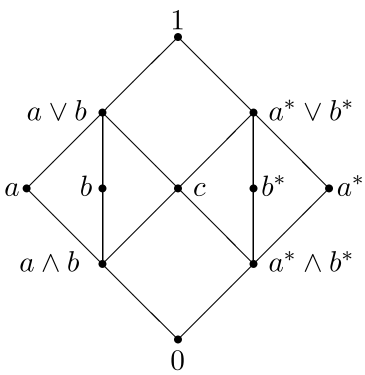
(vi) $a \vee b \in \mathrm{CE}(\mathcal{L})$, $a \wedge b \notin \mathrm{CE}(\mathcal{L})$, $a \vee b = 1$ *and* $a \wedge b \parallel a^* \vee b^*$

(vii) $a \vee b \in \mathrm{CE}(\mathcal{L})$, $a \wedge b \notin \mathrm{CE}(\mathcal{L})$, $a \vee b \neq 1$ *and* $a \wedge b < a^* \vee b^*$

(viii) $a \vee b \in \mathrm{CE}(\mathcal{L})$, $a \wedge b \notin \mathrm{CE}(\mathcal{L})$, $a \vee b \neq 1$ *and* $a \wedge b \parallel a^* \vee b^*$

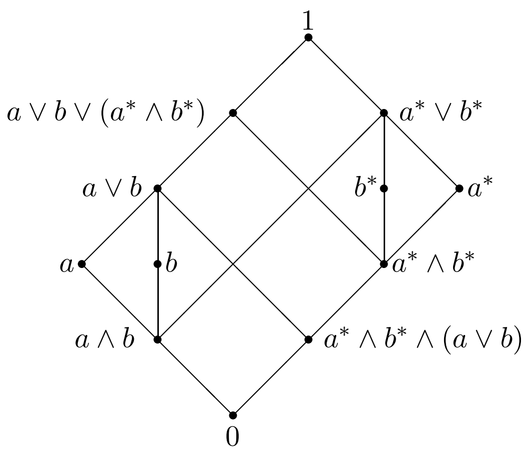
In the listed cases the following minimal (*with respect to the cardinality*) *lattices exist:*

(i):

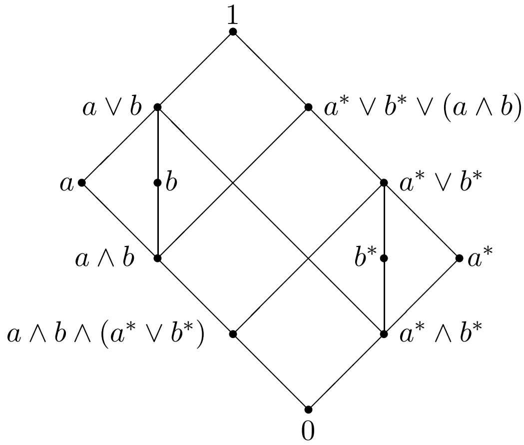


Here $c := (a \wedge b) \vee (a^* \wedge b^*) = (a \vee b) \wedge (a^* \vee b^*)$.

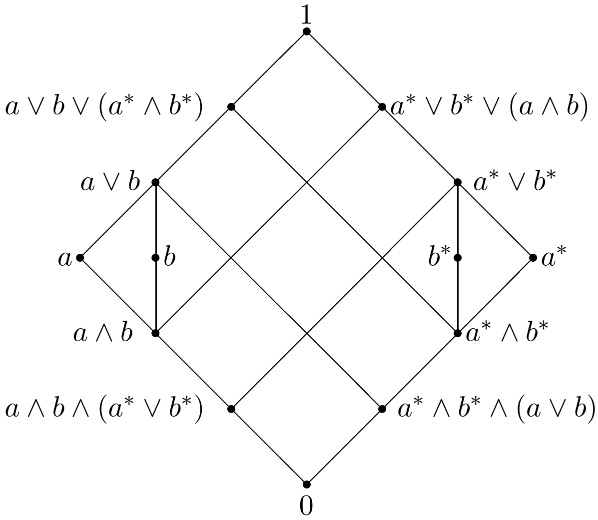
(ii):



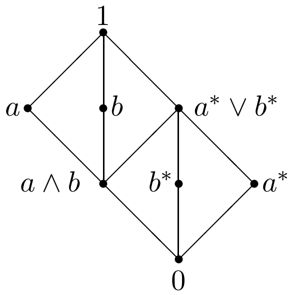
(iii):


1
$a \vee b$
$a^* \vee b^* \vee (a \wedge b)$
a
b
$a^* \vee b^*$
$a \wedge b$
b^*
a^*
$a \wedge b \wedge (a^* \vee b^*)$
$a^* \wedge b^*$
0

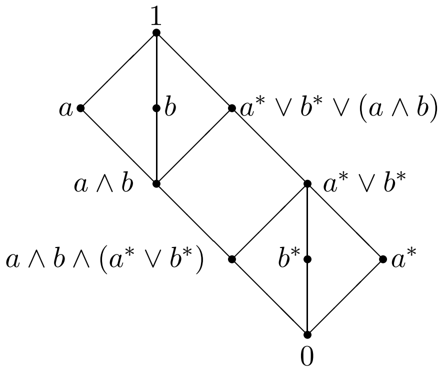
(iv):


1
$a \vee b \vee (a^* \wedge b^*)$
$a^* \vee b^* \vee (a \wedge b)$
$a \vee b$
$a^* \vee b^*$
a
b
b^*
a^*
$a \wedge b$
$a^* \wedge b^*$
$a \wedge b \wedge (a^* \vee b^*)$
$a^* \wedge b^* \wedge (a \vee b)$
0

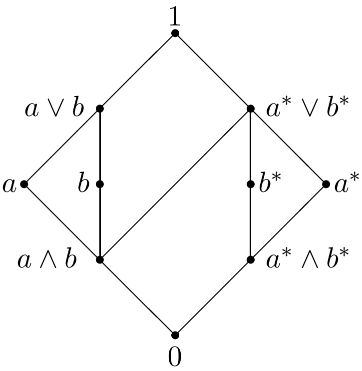
(v):


1
a
b
$a^* \vee b^*$
$a \wedge b$
b^*
a^*
0

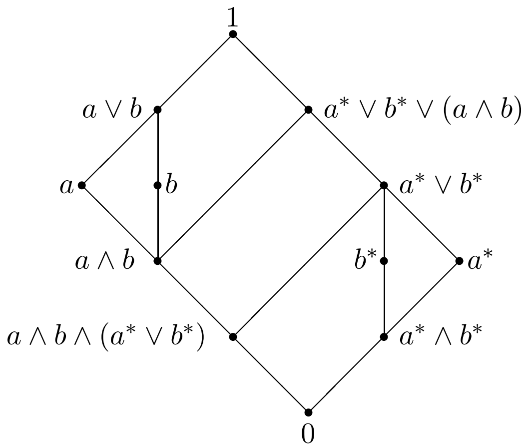
(vi):


1
a
b
$a^* \vee b^* \vee (a \wedge b)$
$a \wedge b$
$a^* \vee b^*$
$a \wedge b \wedge (a^* \vee b^*)$
b^*
a^*
0

(vii):


1
$a \vee b$
$a^* \vee b^*$
a
b
b^*
a^*
$a \wedge b$
$a^* \wedge b^*$
0

(viii):


1
$a \vee b$
$a^* \vee b^* \vee (a \wedge b)$
a
b
$a^* \vee b^*$
$a \wedge b$
b^*
a^*
$a \wedge b \wedge (a^* \vee b^*)$
$a^* \wedge b^*$
0

Remark 6. The remaining case $a \vee b \notin \mathrm{CE}(\mathcal{L})$, $a \wedge b \in \mathrm{CE}(\mathcal{L})$ need not be considered since in this case a^*, b^* satisfy one of the conditions (v)–(viii).

***Proof of Theorem* 5.**

(i): According to Lemma 4 (i) or (ii) the elements $0, 1, a, a^*, b, b^*, a \vee b, a^* \wedge b^*, a \wedge b, a^* \vee b^*, c$ are pairwise distinct.

(ii): According to Lemma 3 (iii) the elements $0, 1, a, a^*, b, b^*, a \vee b, a^* \wedge b^*, a \wedge b, a^* \vee b^*$ are pairwise distinct.

$a \vee b \vee (a^* \wedge b^*) = 0$ would imply $a = 0$.

$a \vee b \vee (a^* \wedge b^*) = 1$ would imply $a \vee b \in \mathrm{CE}(\mathcal{L})$.

$a \vee b \vee (a^* \wedge b^*) = a$ would imply $a \vee b = a$.

$a \vee b \vee (a^* \wedge b^*) = a^*$ would imply $a^* = a^* \vee a^* = a \vee b \vee (a^* \wedge b^*) \vee a^* = 1$.

$a \vee b \vee (a^* \wedge b^*) = a \vee b$ would imply $a^* \wedge b^* \leq a \vee b$.

$a \vee b \vee (a^* \wedge b^*) = a^* \wedge b^*$ would imply $a \vee b \leq a^* \wedge b^*$.

$a \vee b \vee (a^* \wedge b^*) = a \wedge b$ would imply $b = a \wedge b$.

$a \vee b \vee (a^* \wedge b^*) = a^* \vee b^*$ would imply $a^* \vee b^* = a^* \vee b^* \vee a^* \vee b^* = a \vee b \vee (a^* \wedge b^*) \vee a^* \vee b^* = 1$.

$a \vee b \vee (a^* \wedge b^*) = a^* \wedge b^* \wedge (a \vee b)$ would imply $a \vee b \leq a^* \wedge b^*$.

$a^* \wedge b^* \wedge (a \vee b) = 0$ would imply $a \vee b \in \mathrm{CE}(\mathcal{L})$.

$a^* \wedge b^* \wedge (a \vee b) = 1$ would imply $a^* = 1$.

$a^* \wedge b^* \wedge (a \vee b) = a$ would imply $a = a \wedge a = a \wedge a^* \wedge b^* \wedge (a \vee b) = 0$.

$a^* \wedge b^* \wedge (a \vee b) = a^*$ would imply $a^* \wedge b^* = a^*$.

$a^* \wedge b^* \wedge (a \vee b) = a \vee b$ would imply $a \vee b \leq a^* \wedge b^*$.

$a^* \wedge b^* \wedge (a \vee b) = a^* \wedge b^*$ would imply $a^* \wedge b^* \leq a \vee b$.

$a^* \wedge b^* \wedge (a \vee b) = a \wedge b$ would imply $a \wedge b = a \wedge b \wedge a \wedge b = a \wedge b \wedge a^* \wedge b^* \wedge (a \vee b) = 0$.

$a^* \wedge b^* \wedge (a \vee b) = a^* \vee b^*$ would imply $a^* = a^* \vee b^*$.

Hence, the elements $0, 1, a, a^*, b, b^*, a \vee b, a^* \wedge b^*, a \wedge b, a^* \vee b^*, a \vee b \vee (a^* \wedge b^*), a^* \wedge b^* \wedge (a \vee b)$ are pairwise distinct. (Some cases follow by symmetry of a and b.)

(iii)–(viii): These cases can be proved in an analogous way. ■

References

[1] G. Birkhoff, Lattice Theory, AMS, Providence, R. I., 1979.

[2] I. Chajda and H. Länger, *Bounded lattices with antitone involution the complemented elements of which form a sublattice*, J. Algebra Discrete Structures **6** (2008), 13–22.

[3] G. Grätzer, General Lattice Theory, Birkhäuser, Basel 1998.

Received 13 June 2008
Revised 30 August 2008