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NORMALIZATION OF BASIC ALGEBRAS

Miroslav Kolařík

Department of Computer Science
Palacký University Olomouc
Tomkova 40, 779 00 Olomouc, Czech Republic

e-mail: kolarik@inf.upol.cz

Abstract

We consider algebras determined by all normal identities of basic algebras. For such algebras, we present a representation based on a q-lattice, i.e., the normalization of a lattice.

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1. Preliminaries, normalization, q-lattices

1.1. Normal identities, normally presentable varieties

Let τ be a similarity type and p, q be n-ary terms of type τ. If either none of them is a variable or both p, q are the same variable, we say that the identity $p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n)$ is *normal.*

Let $\mathcal{V}$ be a variety of type τ. Let $\mathrm{Id}(\mathcal{V})$ and $\mathrm{Id}_N(\mathcal{V})$ denote the sets of all identities and of all normal identities, respectively, valid in $\mathcal{V}$. The variety $\mathcal{V}$ is called *normally presentable* if $\mathrm{Id}(\mathcal{V}) = \mathrm{Id}_N(\mathcal{V})$, cf. [2–4].

If $\mathrm{Id}(\mathcal{V}) \neq \mathrm{Id}_N(\mathcal{V})$ then $\mathcal{V}$ is called *non-normally presentable.* If this is the case then there is a unary term v such that the identity $v(x) = x$ belongs to $\mathrm{Id}(\mathcal{V}) \setminus \mathrm{Id}_N(\mathcal{V})$, see e.g. [3] for details. As usual, for any set Σ of identities of type τ, $\mathrm{Mod}(\Sigma)$ stands for the class of all algebras of type τ that satisfy all identities from Σ. The following lemma was proved in [3, 7, 9].

Lemma 1. *If a non-normally presentable variety $\mathcal{V}$ is given by a system Σ of identities, i.e., $\mathcal{V} = \mathrm{Mod}(\Sigma)$, and $v(x) = x$ belongs to Σ, then there exists a system of normal identities valid in $\mathcal{V}$, $\Sigma_N \subset \mathrm{Id}_N(\mathcal{V})$, such that $\Sigma_N \cup \{v(x) = x\}$ is equivalent to Σ, i.e., $\mathcal{V} = \mathrm{Mod}(\Sigma_N \cup \{v(x) = x\})$.*

Consequently, $w(x) = x$ is satisfied in $\mathcal{V}$ for another unary term w if and only if the identity $v(x) = w(x)$ belongs to $\mathrm{Id}_N(\mathcal{V})$. So v is determined uniquely up to a normal identity valid in $\mathcal{V}$, and it will be called the *assigned term* of $\mathcal{V}$, [2].

1.2. Normalization

The *normalization* of $\mathcal{V}$ (called a *nilpotent shift* of the variety in [2, 4, 9]) is the variety $N(\mathcal{V}) = \mathrm{Mod}(\mathrm{Id}_N(\mathcal{V}))$. That is, $N(\mathcal{V})$ consists of all τ-algebras which satisfy all normal identities of $\mathcal{V}$. Hence $\mathcal{V}$ is a subvariety of $N(\mathcal{V})$, and $\mathcal{V} = N(\mathcal{V})$ holds if and only if the variety $\mathcal{V}$ is normally presentable.
The next result is taken from [7].

Proposition 1. *Let $\mathcal{V}$ be a non-normally presentable variety with an assigned term v. Let $\mathcal{N} = \mathrm{Mod}(\Xi_N)$ be a normally presentable variety with the system of defining identities $\Xi_N \subset \mathrm{Id}_N(\mathcal{V})$. Then $\mathcal{N} = N(\mathcal{V})$ if and only if all defining identities of $\mathcal{V}$ can be proved from the system $\Xi_N \cup \{v(x) = x\}$.*

The following proposition was proved by I. Mel'nik in [9].

Proposition 2. *If $\mathcal{V} = \mathrm{Mod}(\Sigma_N \cup \{v(x) = x\})$ is a variety of type τ with the set of operation symbols F where $\Sigma_N \subset \mathrm{Id}_N(\mathcal{V})$ then the normalization $N(\mathcal{V})$ is characterized by the identities $\Sigma_N \cup \Sigma_v$ where the set of additional identities is*

$$\Sigma_v = \{f(x_1, \dots, x_n) = v(f(x_1, \dots, x_n)),$$

$$f(x_1, \dots, x_j, \dots, x_n) = f(x_1, \dots, v(x_j), \dots, x_n);\ \ f \in F,\ \ j = 1, \dots, n\}.$$

1.3. Skeleton

Given a non-normally presentable variety $\mathcal{V}$ (of type τ) with the assigned term v, let $A \in N(\mathcal{V})$. By a *skeleton* of A is meant a set $\mathrm{Sk}A = \{a \in A;\, v^A(a) = a\}$, and its elements are called *skeletal*. Skeletal elements are exactly the results of term operations. In particular, $\mathrm{Sk}A = \{v^A(a);\, a \in A\}$. The skeleton $\mathrm{Sk}A$ is clearly a subalgebra of a given algebra $\mathcal{A}$. An algebra A is decomposed into classes $C_a = \{d \in A;\, v(d) = v(a)\}$, $a \in \mathrm{Sk}A$, called *cells* of A in [2]. The decomposition is formed exactly by congruence classes of the congruence relation $\Phi = \{\langle a, b\rangle;\ t^A(a, a_2, \dots, a_n) = t^A(b, a_2, \dots, a_n),$ $t \in T_\tau, a_2, \dots, a_n \in A\}$. Moreover, the map $[a]_\Phi \mapsto v^A(a)$ is an isomorphism $A/\Phi \to \mathrm{Sk}A$.

The following lemma was proved in [2].

Lemma 2. *If $A \in \mathcal{V}$ then* SkA *is the maximal subalgebra of A belonging to $N(\mathcal{V})$.*

1.4. *q*-lattices as a normalization of lattices

A *quasiorder* on a set A is a reflexive and transitive binary relation $\preceq$ on A, and $(A; \preceq)$ is called a *quasiordered set*.

It is well-known, that lattices have two faces, i.e., they can be viewed as algebras and simultaneously as ordered sets. An analogous situation occurs also for algebras resulting from the normalization of lattices, the so-called q-lattices. A q-lattice can be introduced by identities, but can be characterized as well as a lattice-quasiordered set (with suprema and infima for skeletal elements) endowed with a choice function, [1].

By a *q-lattice* (see [1]) we mean an algebra $A = (A; \vee, \wedge)$ with two binary operations satisfying the following identities:

commutativity:

$(\mathrm{C})_\vee:\ x \vee y = y \vee x, \qquad (\mathrm{C})_\wedge:\ x \wedge y = y \wedge x;$

associativity:

$(\mathrm{AS})_\vee:\ (x \vee y) \vee z = x \vee (y \vee z), \qquad (\mathrm{AS})_\wedge:\ (x \wedge y) \wedge z = x \wedge (y \wedge z);$

weak absorption:

$(\mathrm{WAB})_\vee:\ x \vee (x \wedge y) = x \vee x, \qquad (\mathrm{WAB})_\wedge:\ x \wedge (x \vee y) = x \wedge x;$

weak idempotence:

$(\mathrm{WI})_\vee:\ x \vee y = x \vee (y \vee y), \qquad (\mathrm{WI})_\wedge:\ x \wedge y = x \wedge (y \wedge y);$

equalization:

$(\mathrm{EQ}):\ x \wedge x = x \vee x.$

Of course, all these identities are normal identities of lattices.

A q-lattice A is *bounded* if there exist elements 0 and 1 of A such that $a \wedge 0 = 0$ and $a \vee 1 = 1$ for each $a \in A$.

Evidently, a q-lattice is a lattice if and only if it satisfies the idempotency $x \vee x = x$, i.e., if A is equal to its skeleton.

The proof of the following proposition can be found in [1].

Proposition 3. *Let $\mathcal{A} = (A; \vee, \wedge)$ be a q-lattice. Define*

$$x \preceq y \quad \textit{iff} \quad x \vee y = y \vee y \quad (\textit{iff} \quad x \wedge y = x \wedge x).$$

Then $\preceq$ is a quasioreder on A such that

(α) *for all $x, y \in A$ there exists $z \in A$ such that*

(i) *$x, y \preceq z$;*

(ii) *if $w \in A$ such that $x, y \preceq w$ then $z \preceq w$,*

the element z will be called a q-supremum of x, y.

(β) *for all $x, y \in A$ there exists $t \in A$ such that*

(i)$'$ *$t \preceq x, y$;*

(ii)$'$ *if $u \in A$ such that $u \preceq x, y$ then $u \preceq t$,*

the element t will be called a q-infimum of x, y.

Conversely, let $(A; \preceq)$ be a quasiordered set satisfying the conditions (α) and (β). Define $x \vee y = z$ where z is a q-supremum of x, y and $x \wedge y = t$ where t is a q-infimum of x, y. Then $(A; \vee, \wedge)$ is a q-lattice.

A quasiordered set $(A; \preceq)$ satisfying (α) where $x \vee y$ denote q-supremum of x, y is called a *join-q-semilattice.*

2. Normalization of basic algebras

A *basic algebra* (see [6]) is an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the identities

(BA1) $x \oplus 0 = x$;

(BA2) $\neg\neg x = x$ (double negation);

(BA3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ (Łukasiewicz axiom);

(BA4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0$;

(BA5) $\neg 0 \oplus x = \neg 0 = x \oplus \neg 0$.

Clearly, also the (normal) identities $\neg\neg x = x \oplus 0$ and $\neg\neg\neg x = \neg x$ hold in every basic algebra.

Remark 1. The axiom (BA5) can be derived from the remaining axioms (BA1)–(BA4), see [8]. On the other hand, for our purposes, it will be more convenient to compute with the axiom (BA5) also.

Let us note that basic algebras serve as a tool for some investigations of nonclassical logics (including MV-algebras, orthomodular lattices and their generalizations).

The basic algebras form a variety **BA** which is not normally presentable, with $v(x) = x \oplus 0$ as the assigned term (or equivalently, $v(x) = \neg\neg x$). According to Proposition 1, the normalization $N(\mathbf{BA})$ has a basis consisting of the following normal identities:

(N1) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$;

(N2) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0$;

(N3) $0 \oplus 0 = 0$;

(N4) $\neg\neg x = x \oplus 0$;

(N5) $x \oplus y = (x \oplus 0) \oplus y$;

(N6) $x \oplus y = x \oplus (y \oplus 0)$;

(N7) $x \oplus \neg 0 = \neg 0$;

(N8) $\neg 0 \oplus x = \neg 0$;

(N9) $(x \oplus y) \oplus 0 = x \oplus y$;

(N10) $\neg\neg\neg x = \neg x$;

(N11) $\neg(x \oplus 0) = \neg x$;

(N12) $\neg x \oplus 0 = \neg x$;

Thus $N(\mathbf{BA}) = \mathrm{Mod}(\mathrm{Id}_N(\mathbf{BA})) = \mathrm{Mod}(\{(\mathrm{N1}) - (\mathrm{N12})\})$. We are going to show that this axiom system can be reduced.

Lemma 3. *The following identities holds in $N(\mathbf{BA})$:*

(1) $\neg\neg x = 0 \oplus x$;

(2) $\neg x \oplus x = \neg 0$.

Proof. (1):

$$\neg\neg x = \neg(\neg\neg\neg x) = \neg(\neg x \oplus 0) = \neg\neg\neg(\neg x \oplus 0)$$
$$= \neg(\neg x \oplus 0) \oplus 0 = \neg(\neg 0 \oplus x) \oplus x = \neg\neg 0 \oplus x = 0 \oplus x.$$

(2):

$$\neg x \oplus x = \neg\neg\neg x \oplus x = \neg(0 \oplus x) \oplus x$$
$$= \neg(\neg\neg 0 \oplus x) \oplus x = \neg(\neg x \oplus \neg 0) \oplus \neg 0 = \neg 0.$$

■

Lemma 4. *The following implications hold:*

(i) (N4) *and* (N3) *imply* $\neg\neg 0 = 0$;

(ii) (N11) *and* (N12) *imply* $\neg x \oplus 0 = \neg(x \oplus 0)$;

(iii) (N4) *and* (N11) *imply* (N10);

(iv) (N10) *and* (N4) *imply* (N11), (N12).

Proof. The first two cases are obvious. Prove (iii): (N4) and (N11) yield $\neg\neg\neg x = \neg(x \oplus 0) = \neg x$. To prove (iv), suppose (N10) and (N4) then $\neg x = \neg(\neg\neg x) = \neg(x \oplus 0)$, and similarly, $\neg x = \neg\neg(\neg x) = \neg x \oplus 0$. ■

So $N(\mathbf{BA}) = \mathrm{Mod}(\{(\mathrm{N1}) - (\mathrm{N10})\})$. Since $v(x) = x \oplus 0$, the skeleton of a basic algebra $\mathcal{M} = (M; \oplus, \neg, 0)$ is $\mathrm{Sk}M = \{a \oplus 0;\, a \in M\}$.

It is known (see e.g. [6]) that basic algebras form bounded lattices with respect to the natural order defined by $x \leq y$ if and only if $\neg x \oplus y = \neg 0$ where $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$. An analogous statement can be proved for their normalizations:

Theorem 1. *Let* $\mathcal{A} = (A; \oplus, \neg, 0) \in N(\mathbf{BA})$. *Define* $x \preceq y$ *if and only if* $\neg x \oplus y = \neg 0$. *Then* $(A; \preceq)$ *is a bounded q-lattice with* $0 \preceq x \preceq \neg 0$ *for each* $x \in A$ *and* $x \vee y = \neg(\neg x \oplus y) \oplus y$ *and* $x \wedge y = \neg(\neg x \vee \neg y)$.

Proof. Obviously, $\preceq$ is reflexive by Lemma 3(2). Let $x \preceq y$, i.e., $\neg x \oplus y = \neg 0$. Then, by (N2), we have

$$\begin{aligned} \neg 0 &= \neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus (\neg x \oplus z) \\ &= \neg(\neg(0 \oplus y) \oplus z) \oplus (\neg x \oplus z) \\ &= \neg(\neg\neg\neg y \oplus z) \oplus (\neg x \oplus z) \\ &= \neg(\neg y \oplus z) \oplus (\neg x \oplus z). \end{aligned} \tag{2.1}$$

Assume $x \preceq y$ and $y \preceq z$. Then $\neg y \oplus z = \neg 0$, and, by (2.1)

$$\begin{aligned} \neg(\neg y \oplus z) \oplus (\neg x \oplus z) &= \neg\neg 0 \oplus (\neg x \oplus z) \\ &= 0 \oplus (\neg x \oplus z) = \neg\neg(\neg x \oplus z) \\ &= (\neg x \oplus z) \oplus 0 = \neg x \oplus z, \end{aligned}$$

so that $x \preceq z$. Hence $\preceq$ is really a quasiorder on A. We have $0 \preceq x$ since $\neg 0 \oplus x = \neg 0$, and also $x \preceq \neg 0$ since $\neg x \oplus \neg 0 = \neg 0$.

Let $x \preceq y$, i.e., $\neg x \oplus y = \neg 0$. Then, by (2.1),

$$\neg 0 = \neg(\neg y \oplus z) \oplus (\neg x \oplus z),$$

hence $\neg y \oplus z \preceq \neg x \oplus z$. This also entails

$$x \preceq y \quad \Rightarrow \quad \neg y \preceq \neg x.$$

Further, $\neg\neg y = 0 \oplus y \preceq x \oplus y$, whence $\neg y \oplus (x \oplus y) = \neg 0$, i.e., $y \preceq x \oplus y$. Due to this fact, $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ is a common upper bound of x, y. Assume that $x, y \preceq z$. Then $\neg z \oplus y \preceq \neg x \oplus y$, whence

$$\neg(\neg x \oplus y) \oplus y \preceq \neg(\neg z \oplus y) \oplus y = \neg(\neg y \oplus z) \oplus z = \neg\neg 0 \oplus z = \neg\neg z.$$

Since $\neg 0 = \neg z \oplus z = \neg\neg\neg z \oplus z$, we have $\neg\neg z \preceq z$. Using transitivity, we conclude $\neg(\neg x \oplus y) \oplus y \preceq z$. We put $x \vee y = \neg(\neg x \oplus y) \oplus y$.

Analogously we can show that $x \wedge y = \neg(\neg x \vee \neg y)$ is a lower bound of $\{x, y\}$ such that $z \preceq \neg(\neg x \vee \neg y)$ for any other lower bound of $\{x, y\}$. Hence, $(A; \vee, \wedge)$ is a q-lattice (see e.g. [1] for details). ∎

Example 1. Let us consider the algebra $\mathcal{A} = (A; \oplus, \neg, 0) \in N(\mathbf{BA})$, where $A = \{0, 0', a, a', b, 1\}$, and whose operations $\oplus$ and $\neg$ are given by the following tables

$\oplus$	0	$0'$	a	a'	b	1
0	0	0	a	a	b	1
$0'$	0	0	a	a	b	1
a	a	a	1	1	b	1
a'	a	a	1	1	b	1
b	b	b	a	a	1	1
1	1	1	1	1	1	1

x	0	$0'$	a	a'	b	1
$\neg x$	1	1	a	a	b	0

Note that e.g. $a' \oplus 0 \neq a'$, $\neg\neg a' \neq a'$. By Theorem 1, we can assign to $\mathcal{A}$ a bounded q-lattice $\mathcal{Q} = (A; \vee, \wedge)$, where $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$ for all $x, y \in A$. The tables for operations $\vee$ and $\wedge$ in $\mathcal{Q}$ are as follows

$\vee$	0	$0'$	a	a'	b	1
0	0	0	a	a	b	1
$0'$	0	0	a	a	b	1
a	a	a	a	a	1	1
a'	a	a	a	a	1	1
b	b	b	1	1	b	1
1	1	1	1	1	1	1

$\wedge$	0	$0'$	a	a'	b	1
0	0	0	0	0	0	0
$0'$	0	0	0	0	0	0
a	0	0	a	a	0	a
a'	0	0	a	a	0	a
b	0	0	0	0	b	b
1	0	0	a	a	b	1

One can easily draw the diagram of $\mathcal{Q}$ in Figure 1.

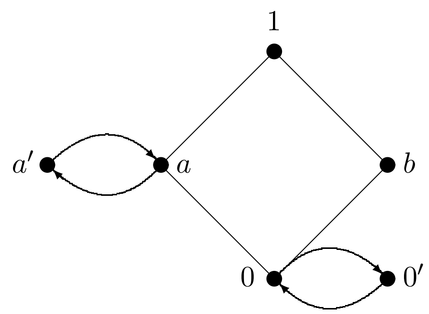

Figure 1

Remark that $1 = \neg 0$ is the greatest element of $\mathcal{Q}$, but 0 is not the least element of $\mathcal{Q}$, since $0 \preceq 0'$ and also $0' \preceq 0$.

The Hasse diagram of the skeleton of $\mathcal{Q}$ is depicted in Figure 2; of course it is a basic algebra.

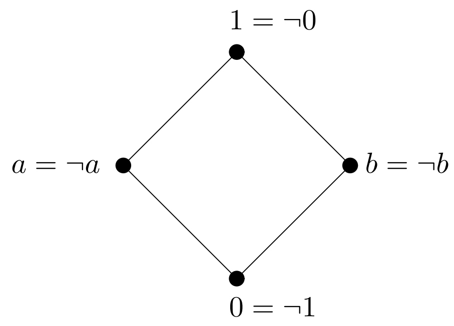

Figure 2

3. q-lattices with sectionally antitone mappings

As usual, under an *involution* on a set A we mean a map ${}^p : A \to A$ such that $a^{pp} = a$ for all $a \in A$.

Given a quasiordered set $(A; \preceq)$, a map ${}^p : A \to A$ is called *antitone* if the implication $x \preceq y \Rightarrow y^p \preceq x^p$ holds.

Let $\mathcal{L} = (L; \vee, \wedge, 1)$ be a q-lattice with the greatest idempotent 1 (i.e., $1 = 1 \vee 1$), and let $\preceq$ denote the induced quasiorder on L. Remember that the skeleton $\mathrm{Sk}\mathcal{L} = \{x \in L;\, x \vee x = x\}$ is a lattice. By an *interval* in $\mathcal{L}$ we understand here the set $[a, b] = \{x \in L;\, a \preceq x \preceq b\}$, and under an *interval in the skeleton* the intersection $\mathrm{Sk}[a, b] = \mathrm{Sk}L \cap [a, b]$ provided $a, b \in \mathrm{Sk}L$.

For example, $[0, a] = \{0, 0', a, a'\}$ and $\mathrm{Sk}[0, a] = \{0, a\}$ for the q-lattice of Example 1 (see Figure 1).

Remark 2. For any $p \in L$, let an antitone involution ${}^p : x \mapsto x^p$, $x \in \mathrm{Sk}L$, be given on the interval $\mathrm{Sk}[p \vee p, 1]$. The mapping ^{p} with $p \in L$ can be extended to a mapping on the whole interval $[p, 1]$ in a natural way as follows. For $x \in [p, 1]$ we define $x^p := (x \vee x)^{p \vee p}$. Note that in general, $x \mapsto x^p$ is not an involution on $[p, 1]$ but only on $\mathrm{Sk}[p \vee p, 1]$. Indeed, $x^{pp} = ((x \vee x)^{p \vee p} \vee (x \vee x)^{p \vee p})^{p \vee p} = ((x \vee x)^{p \vee p})^{p \vee p} = x \vee x \in \mathrm{Sk}L$, however $x^{pp} \neq x$ for $x \notin \mathrm{Sk}L$.

Lemma 5. *Let $\mathcal{L} = (L; \vee, \wedge, 1)$ be a q-lattice with $1 = 1 \vee 1$. For any $p \in L$, let an antitone mapping ${}^p : x \mapsto x^p$, be given on the interval $[p \vee p, 1]$ such that its restriction to $\mathrm{Sk}[p \vee p, 1]$ is an involution. For $x, y \in L$, let us introduce a binary operation $x \circ y := (x \vee y)^{y \vee y}$. Then the following identities hold:*

(1) $x \circ x = 1$, $x \circ 1 = 1$;

(2) $1 \circ (x \circ y) = x \circ y$;

(3) $(x \circ y) \circ y = (y \circ x) \circ x$ *(quasi-commutativity);*

(4) $(((x \circ y) \circ y) \circ z) \circ (x \circ z) = 1$;

(5) $x \circ ((x \circ y) \circ y) = 1$.

Moreover,

(6) *if* $x \vee y \vee z = z \vee z$ *then* $((x \circ y) \circ y) \circ z = 1$;

(7) *if* $x \vee y = y \vee y$ *then* $(y \circ z) \circ (x \circ z) = 1$.

Proof. Indeed, $x \circ x = (x \vee x)^{x \vee x} = 1$, $x \circ 1 = (x \vee 1)^{1 \vee 1} = 1^1 = 1$, $1 \circ (x \circ y) = 1 \circ (x \vee y)^{y \vee y} = (1 \vee (x \vee y)^{y \vee y})^{(x \vee y)^{y \vee y}} = 1^{(x \vee y)^{y \vee y}} = x \circ y$. Further, $(x \circ y) \circ y = ((x \vee y)^{y \vee y} \vee y)^{y \vee y}$. Here $(x \vee y)^{y \vee y} \vee y = (x \vee y)^{y \vee y}$ since $(x \vee y)^{y \vee y} \succeq y \vee y \succeq y$, therefore $((x \vee y)^{y \vee y} \vee y)^{y \vee y} = x \vee y$, and (3) follows. To prove (4), we compute $(((x \circ y) \circ y) \circ z) \circ (x \circ z) = ((x \vee y) \circ z) \circ (x \circ z) = (((x \vee y) \vee z)^{z \vee z} \vee (x \vee z)^{z \vee z})^{(x \vee z)^{z \vee z} \vee (x \vee z)^{z \vee z}} = ((x \vee z)^{z \vee z})^{(x \vee z)^{z \vee z}} = 1$.

(5): $x \circ ((x \circ y) \circ y) = x \circ (x \vee y) = (x \vee (x \vee y))^{(x \vee y) \vee (x \vee y)} = (x \vee y)^{x \vee y} = 1$

(6): $((x \circ y) \circ y) \circ z = (x \vee y) \circ z = ((x \vee y) \vee z)^{z \vee z} = (z \vee z)^{z \vee z} = 1$

(7): $(y \circ z) \circ (x \circ z) = ((y \vee z)^{z \vee z} \vee (x \vee z)^{z \vee z})^{(x \vee z)^{z \vee z} \vee (x \vee z)^{z \vee z}} =$
$((x \vee z)^{z \vee z})^{(x \vee z)^{z \vee z}} = 1.$

∎

Lemma 6. *Let $(A; \circ, 1)$ be an algebra of type $(2, 0)$ satisfying the identities (1), (2) and (4). Then the relation $\preceq$ introduced by*

$$x \preceq y \qquad \text{if and only if} \qquad x \circ y = 1$$

is a quasiorder on A and for all $x \in A$, we have $x \preceq 1$. Moreover, $x \circ y = 1$ if and only if $x \vee y = y \vee y$.

Proof. By (1), $\preceq$ is reflexive. For transitivity, let $x \preceq y$, $y \preceq z$, that is, $x \circ y = y \circ z = 1$. Then by (4), (1) and (2), $1 = (((x \circ y) \circ y) \circ z) \circ (x \circ z) = ((1 \circ y) \circ z) \circ (x \circ z) = (y \circ z) \circ (x \circ z) = 1 \circ (x \circ z) = x \circ z$, so that $x \preceq z$. Clearly, $x \circ 1 = 1$ gets $x \preceq 1$ for all $x \in A$.

Further, if $x \vee y = y \vee y$ then $1 = (y \vee y)^{y \vee y} = (x \vee y)^{y \vee y} = x \circ y$. Conversely, if $x \circ y = 1$ then $1 = (x \vee y)^{y \vee y}$ which immediately yields that $x \vee y = y \vee y$. ∎

The quasiorder $\preceq$ given by $x \preceq y \Leftrightarrow x \circ y = 1$ will be called the *induced quasiorder* of $(A; \circ, 1)$.

Theorem 2. *Let $\mathcal{A} = (A; \circ, 1)$ be an algebra satisfying the identities* (1)–(7). *Then $(A; \preceq)$ is a join-q-semilattice in which $x \vee y = (x \circ y) \circ y$ for all $x, y \in A$. For each $p \in A$, the interval $[p \vee p, 1]$ is a q-lattice with an antitone mapping*

$$a \mapsto a^p = a \circ p, \qquad a \in [p \vee p, 1].$$

Proof. For $x, y \in A$, $x \circ ((x \circ y) \circ y) = 1$, $y \circ ((y \circ x) \circ x) = y \circ ((x \circ y) \circ y) = 1$ holds by (5) and (3), hence $(x \circ y) \circ y$ is an upper bound of x, y. The element $(x \circ y) \circ y$ is an idempotent with respect to $\vee$ since $((x \circ y) \circ y) \vee ((x \circ y) \circ y) = (((x \circ y) \circ y) \circ ((x \circ y) \circ y)) \circ ((x \circ y) \circ y) = 1 \circ ((x \circ y) \circ y) = (x \circ y) \circ y$ (by (1) and (2)).

Let z be an idempotent such that $x \preceq z$, $y \preceq z$. Then, according to (6), $((x \circ y) \circ y) \circ z = 1$ that is $(x \circ y) \circ y \preceq z$, and $(x \circ y) \circ y$ is the least idempotent above the elements x and y and hence $(x \circ y) \circ y$ is a q-supremum of x, y, i.e., $x \vee y$. For any element $a \in [p \vee p, 1]$, the map $a \mapsto a^p = a \circ p$ is antitone, because, by (7), $x \preceq y \Rightarrow y \circ z \preceq x \circ z$. For $a, b \in [p \vee p, 1]$ define $a \wedge b = (a^p \vee b^p)^p$. Obviously, $a \wedge b$ is a q-infimum of a, b thus, by Proposition 3, $([p \vee p, 1]; \vee, \wedge)$ is a q-lattice. ∎

Theorem 3. *Let $\mathcal{A} = (A; \oplus, \neg, 0) \in N(\mathbf{BA})$. Define $x \circ y := \neg x \oplus y$ and $1 = \neg 0$. Further, let $\vee$, $\wedge$ are defined as in Theorem* 1. *Then $\mathcal{L}(A) = (A; \vee, \wedge, \circ, 1, 0)$ is a bounded q-lattice with sectionally antitone mappings such that their restrictions to* $\mathrm{Sk}[p \vee p, 1]$ *are involutions where for each $p \in A$ and $x \in [p \vee p, 1]$ we define $x^p = x \circ p$.*

Proof. Let us prove that the mapping ${}^p : [p \vee p, 1] \to [p \vee p, 1]$ where $x \mapsto x^p = \neg x \oplus p$, $p \in A$, is antitone. Indeed, if $x \preceq y$ then $\neg x \succeq \neg y$, hence $x^p = \neg x \oplus p \succeq \neg y \oplus p = y^p$ for all $x, y \in [p \vee p, 1]$. If $x, y \in [p, 1]$ and

$y \vee y = x \vee x = x$ then $\neg x \oplus p = \neg(y \vee y) \oplus p = \neg y \oplus p$ since $N(\mathbf{BA})$ satisfies all normal identities of $\mathbf{BA}$. ■

Theorem 4. *Let* $\mathcal{L} = (L; \vee, \wedge, \circ, 0, 1)$ *be a bounded q-lattice with sectionally antitone mappings such that their restrictions to* $\mathrm{Sk}[p \vee p, 1]$ *are involutions. Define* $\neg x := x \circ 0$ *and* $x \oplus y := (x \circ 0) \circ y$*. Then* $\mathcal{A}(\mathcal{L}) = (L; \oplus, \neg, 0) \in N(\mathbf{BA})$.

Proof. We shall verify the axioms (N1)–(N10).

(N1): $\neg(\neg x \oplus y) \oplus y = (x \circ y) \circ y = (y \circ x) \circ x = \neg(\neg y \oplus x) \oplus x$;

(N2): $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg(\neg((x \vee 0)^{0 \vee 0} \vee y) \oplus z) \oplus ((x \circ 0) \circ z) = (((x \vee 0)^{0 \vee 0} \vee y) \vee z)^{z \vee z} \circ ((x \vee 0)^{0 \vee 0} \vee z)^{z \vee z} = 1 = 0 \circ 0 = \neg 0$;

(N3): $0 \oplus 0 = (0 \circ 0) \circ 0 = 1 \circ 0 = 0$;

(N4): $\neg\neg x = (x \circ 0) \circ 0 = x \oplus 0$;

(N5): $(x \oplus 0) \oplus y = ((x \circ 0) \circ 0) \oplus y = x \oplus y$;

(N6): $x \oplus (y \oplus 0) = x \oplus ((y \circ 0) \circ 0) = x \oplus y$;

(N7): $\neg 0 \oplus x = (0 \circ 0) \oplus x = 1 \oplus x = (1 \circ 0) \circ x = 0 \circ x = 1$;

(N8): $x \oplus \neg 0 = x \oplus (0 \circ 0) = x \oplus 1 = (x \circ 0) \circ 1 = 1 = 0 \circ 0 = \neg 0$;

(N9): $(x \oplus y) \oplus 0 = ((x \oplus y) \circ 0) \circ 0 = x \oplus y$;

(N10): $\neg\neg\neg x = ((x \circ 0) \circ 0) \circ 0 = x \circ 0 = \neg x$.

■

Theorem 5. *Let* $\mathcal{A} = (A; \circ, 1)$ *be an algebra satisfying* (1)–(7) *where* $x \vee y = (x \circ y) \circ y$*. Let* $p \in A$ *with* $1 \circ p = p$ *and define* $\neg_p x := x \circ p$, $x \oplus_p y := (x \circ p) \circ y$*. Then the algebra* $([p, 1]; \oplus_p, \neg_p, p)$ *belongs to* $N(\mathbf{BA})$.

Proof. Let $x, y, z \in [p, 1]$. Clearly, $\neg_p$ and $\oplus_p$ are well-defined operations on the interval $[p, 1]$, since $p \preceq \neg x \oplus p = x \circ p$ and $p \preceq y \preceq \neg(\neg x \oplus p) \oplus y = (x \circ p) \circ y$. We check the axioms (N1)–(N10).

(N1): $\neg_p(\neg_p x \oplus_p y) \oplus_p y = (((((x \circ p) \circ p) \circ y) \circ p) \circ p) \circ y = (x \circ y) \circ y = (y \circ x) \circ x = (((((y \circ p) \circ p) \circ x) \circ p) \circ p) \circ x = \neg_p(\neg_p y \oplus_p x) \oplus_p x$;

(N2): $\neg_p(\neg_p(\neg_p(x \oplus_p y) \oplus_p y) \oplus_p z) \oplus_p (x \oplus_p z) = ((((((x \circ p) \circ y) \circ y) \circ z) \circ p) \circ p) \circ ((x \circ p) \circ z) = ((((x \circ p) \vee y) \circ z) \vee p) \circ ((x \circ p) \circ z) = 1$;

(N3): $p \oplus_p p = (p \circ p) \circ p = 1 \circ p = p$;

(N4): $\neg_p \neg_p x = (x \circ p) \circ p = x \oplus_p p$;

(N5): $(x \oplus_p p) \oplus_p y = ((x \circ p) \circ p) \oplus_p y = x \oplus_p y$;

(N6): $x \oplus_p (y \oplus_p p) = x \oplus_p ((y \circ p) \circ p) = x \oplus_p y$;

(N7): $\neg_p p \oplus_p x = (p \circ p) \oplus_p x = 1 \oplus_p x = (1 \circ p) \circ x = p \circ x = 1 = p \circ p = \neg_p p$;

(N8): $x \oplus_p \neg_p p = x \oplus_p (p \circ p) = x \oplus_p 1 = (x \circ p) \circ 1 = 1 = p \circ p = \neg_p p$;

(N9): $(x \oplus_p y) \oplus_p p = ((x \oplus_p y) \circ p) \circ p = x \oplus_p y$;

(N10): $\neg_p \neg_p \neg_p x = ((x \circ p) \circ p) \circ p = x \circ p = \neg_p x$.

■

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