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POSITIVE SPLITTINGS OF MATRICES AND THEIR NONNEGATIVE MOORE-PENROSE INVERSES

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Abstract

In this short note we study necessary and sufficient conditions for the nonnegativity of the Moore-Penrose inverse of a real matrix in terms of certain spectral property shared by all positive splittings of the given matrix.

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1. Introduction

The concept of monotonicity was proposed first by Collatz (see for instance [4]). A square real matrix A is called monotone if $Ax \geq 0 \Rightarrow x \geq 0$. Here $x = (x_i) \geq 0$ means that $x_i \geq 0$ for all i. Collatz has shown that a matrix is monotone iff it is invertible and the inverse is nonnegative. He also gave a sufficient condition on the entries of A in order for A^{-1} to be nonnegative. Mangasarian [10] studied rectangular real matrices while Berman and Plemmons ([2], [3]) obtained a variety of generalizations. Gil gave new sufficient conditions on the entries of a matrix A that guarantee the nonnegativity of A^{-1} ([5], [6]). We refer the reader to the excellent book by Berman and Plemmons [3] for numerous examples of applications of nonnegative matrices that include Numerical Analysis and linear economic models.

Berman and Plemmons [3] and several others (see [11] and the references cited therein) have characterized monotonicity of matrices using splittings. Recently, Peris [11] considered a positive splitting of a matrix A and characterized the nonnegativity of A^{-1} in terms of an eigenvalue property satisfied by all such splittings of A. He also considered special types of positive splittings called B-splittings. Some of these results were extended to ordered normed spaces by Weber [12]. The objective in this article is to show how Peris' results can be extended to characterize non-negativity of the Moore-Penrose inverse of matrices with real entries. In particular, we show how one of his sufficient conditions on nonnegativity of the inverse of a matrix M (Theorem 1, [11]) can be extended to the nonnegativity of the Moore-Penrose inverse of M, with an entirely similar proof. Moreover, we have improved this particular part of Peris' theorem in not assuming that M is invertible. We show that, in the bargain, the converse fails to hold. However, we succeed in obtaining a partial converse, extending one of Weber's results [12] to obtain a necessary condition for the nonnegativity of the Moore-Penrose inverse in the finite dimensional setting. Finally, we show how Peris' results (that motivated this article), can be obtained as a corollary of our results. In this sense, our main results generalize those of Peris. It is pertinent to point out that we have been able to extend the results of Weber (cited above), in [9], after the completion of the present work. The present paper forms part of the contents of the doctoral thesis of the first author [8].

The paper is organized as follows. In Section 2 we discuss the preliminaries. In Section 3, we prove the main results. We conclude with some observations.

2. Notations, definitions and preliminaries

$\mathbb{R}^n$ denotes the n dimensional real Euclidean space with the Euclidean norm and $\mathbb{R}^n_+$ denotes non-negative orthant in $\mathbb{R}^n$. For a matrix A with nonnegative entries we use the notation $A \geq 0$. We denote the spectral radius of a matrix A by $r(A)$.

A splitting of a matrix $M = B - A$, with $A \geq 0$, $B \geq 0$ is called a positive splitting.

A closed subset K of $\mathbb{R}^n$ is called a *wedge* if $x, y \in K$ and $\alpha \geq 0$ imply that $x + y \in K$ and $\alpha x \in K$. A wedge K is called a *cone* if $K \cap -K = \{0\}$.

A cone K is called reproducing (or generating) if every $x \in \mathbb{R}^n$ can be represented in the form $x = u - v$, where $u, v \in K$. Equivalently, K is reproducing if $\mathbb{R}^n = K - K$. Let the interior of K be denoted by K°. Then K is reproducing iff K° is nonempty [3].

A norm in the space $\mathbb{R}^n$ with respect to the cone K is said to be monotonic on K if $x \leq y$ implies that $\|x\| \leq \|y\|$ for arbitrary $x, y \in K$. It is well known that the Euclidean norm is monotonic on $\mathbb{R}^n_+$ [7].

The Moore-Penrose inverse of a matrix $M \in \mathbb{R}^{m\times n}$ is the unique matrix $M^\dagger$ which satisfies the following equations: $MM^\dagger M = M$; $M^\dagger MM^\dagger = M^\dagger$; $(MM^\dagger)^* = MM^\dagger$; $(M^\dagger M)^* = M^\dagger M$. If M is a (square) invertible matrix, then it is easily seen that $M^{-1} = M^\dagger$. The following properties of $M^\dagger$ are well known ([1]): $R(M^*) = R(M^\dagger)$; $N(M^*) = N(M^\dagger)$; $MM^\dagger = P_{R(M)}$; $M^\dagger M = P_{R(M^*)}$. In particular, if $x \in R(M^*)$ then $x = M^\dagger Mx$.

We need the following version of the Perron-Frobenius theorem for matrices leaving a closed convex reproducing cone invariant.

Theorem 2.1 (Theorem 1.3.2, [3]). *Let K be a closed, convex and reproducing cone in $\mathbb{R}^n$ and let $A \in \mathbb{R}^{n\times n}$ be such that $AK \subseteq K$. Then $r(A)$, the spectral radius of A is an eigenvalue of A and K contains an eigenvector of A corresponding to $r(A)$.*

3. MAIN RESULTS

In this section we present the main results (Theorem 3.1, Theorem 3.5 and Corollary 3.6). The proof of Theorem 3.1 is entirely similar to the ingenious proof provided by Peris ((b) $\Longrightarrow$ (a), Theorem 1, [11]). However, we obtain a more general result.

Theorem 3.1. *Suppose that $M \in \mathbb{R}^{m\times n}$ satisfies the following condition:*

$$\begin{cases} \textit{Whenever } M = B - A, \textit{ with } B \geq 0 \textit{ and } A \geq 0, \textit{ there exists} \\ 0 \neq v \in \mathbb{R}^n_+ \cap R(M^*) \textit{ and } \mu \in [0,1) \textit{ such that } Av = \mu Bv. \end{cases}$$

Then $M^\dagger \geq 0$.

Proof. Let $M^{\dagger} = (t_{ij})$. Suppose $M^{\dagger} \not\geq 0$. Then there exist i_0 and j_0 such that $t_{i_0 j_0} < 0$. Define am $m \times n$ matrix $B = (b_{ij})$, for all $j = 1, 2, 3, ..., m$ with

$$b_{ij} = \begin{cases} b, & \text{if } \ i \neq j_0 \\ \\ b + q, & \text{i}f \ \ i = j_0, \end{cases}$$

where b and q being arbitrary positive numbers chosen in such a way that $A = B - M \geq 0$. Then $M = B - A$ and by the hypothesis, for this splitting, there exist $\mu \in [0, 1)$ and $0 \neq v \in \mathbb{R}^n_+ \cap R(M^*)$ such that $Av = \mu Bv$. Hence, $Mv = (B - A)v = (1 - \mu)Bv$. Setting $w = Bv$, we then have (using $v = M^{\dagger}Mv$, as $v \in R(M^*)$)

$$M^{\dagger}w = M^{\dagger}Bv = M^{\dagger}\left(\frac{1}{1-\mu}Mv\right) = \frac{1}{1-\mu}v \geq 0.$$

On the other hand, using $v = (v_1, v_2, ..., v_n)^t$, the equation $w = Bv$ gives

$$w_j = \begin{cases} b(v_1 + v_2 + \cdots + v_n), & for \ j \neq j_0 \\ \\ (b+q)(v_1 + v_2 + \cdots + v_n), & for \ j = j_0, \end{cases}$$

where j varies from 1 to m. So, if $j \neq j_0$, then

$$\frac{w_j}{w_{j_0}} = \frac{b}{b+q} \ .$$

Let $s = max_{1 \leq j \leq m}\left\{\frac{|t_{i_0 j}|}{|t_{i_0 j_0}|}\right\}$. Choose q large enough such that

$$\frac{b}{b+q} < \frac{1}{ms+1} \ .$$

Then, $(M^{\dagger}w)_{i_0}$ is given by

$$\begin{aligned}
(M^{\dagger}w)_{i_0} &= (t_{i_01}, t_{i_02}, ..., t_{i_0j_0}, ..., t_{i_0m}) \,.\, (w_1, w_2, ..., w_{j_0}, ..., w_m)^t \\
&= w_{j_0}\,(t_{i_01}, t_{i_02}, ..., t_{i_0j_0}, ..., t_{i_0m}) \,.\, \left(\frac{b}{b+q}, \frac{b}{b+q}, ..., 1, ..., \frac{b}{b+q}\right)^t \\
&\leq |t_{i_0j_0}| w_{j_0}\,(s, s..., -1, ..., s) \,.\, \left(\frac{b}{b+q}, \frac{b}{b+q}, ..., 1, ..., \frac{b}{b+q}\right)^t \\
&= |t_{i_0j_0}| w_{j_0} \left[\frac{sb}{b+q} + \frac{sb}{b+q} + \cdots + (-1) + \cdots + \frac{sb}{b+q}\right] \\
&< |t_{i_0j_0}| w_{j_0} \left[s\frac{(m-1)}{ms+1} - 1\right] < 0,
\end{aligned}$$

a contradiction. Hence $M^{\dagger} \geq 0$. ∎

Remarks 3.2. Verifying that $M^{\dagger} \geq 0$ is arguably simpler than testing whether all possible nonnegative splittings of M satisfy the generalized eigenvalue property stated in Theorem 3.1. However, our intention is to merely demonstrate that with the same proof, Peris' result is true even for Moore-Penrose inverses. And so, a stronger result is obtained.

We next show that the converse of Theorem 3.1 is not true.

Example 3.3. Let $M = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ be the matrix representation of a linear operator on $\mathbb{R}^2$ with the nonnegative orthant as the cone. Then $M^{\dagger} = \frac{1}{2}M^t \geq 0$. Consider $M = B - A$, where $B = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} \geq 0$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \geq 0$. If $Av = \mu Bv$ holds for some μ and for $v \in \mathbb{R}^2_+ \cap R(M^*) \neq \{0\}$ then, $v = 0$.

We next show that, if we impose an additional (natural) condition on B then we can recover the converse. We need a preliminary result that we state next.

Lemma 3.4. *Let S be a subspace of $\mathbb{R}^n$. If $\mathbb{R}^n_+ \cap S \neq \{0\}$ then $\mathbb{R}^n_+ \cap S$ is a closed and generating cone in its linear span. Further, the Euclidean norm on $\mathbb{R}^n$ is monotonic with respect to the cone $\mathbb{R}^n_+ \cap S$.*

Proof. Since $\mathbb{R}^n_+$ is closed in $\mathbb{R}^n$ and S is a subspace of $\mathbb{R}^n$, $\mathbb{R}^n_+ \cap S$ is closed in $\mathbb{R}^n$. Set $P = \mathbb{R}^n_+ \cap S$ and let Q denote the linear span of P. Then P is a cone. The proof follows from the representation of $x \in Q$ as

$$x = \sum_{\{i : b_i \geq 0\}} b_i x^i + \sum_{\{i : b_i < 0\}} b_i x^i$$

for some $b_i \in \mathbb{R}$ and $x^i \in P$. ■

The next result is a partial converse of Theorem 3.1. This generalizes a result of Weber (Theorem 2, [12] for positively invertible operators) to matrices with nonnegative Moore-Penrose inverse.

Theorem 3.5. *Let $0 \neq M \in \mathbb{R}^{m \times n}$. If $M^\dagger \geq 0$, then for any decomposition $M = B - A$, with $A, B \geq 0$ and $R(B) \subseteq R(M)$, there exist $0 \neq v \in \mathbb{R}^n_+ \cap R(M^*)$ and a number $\mu \in [0, 1)$ such that $Av = \mu Bv$.*

Proof. Set $L = \mathbb{R}^n_+ \cap R(M^*)$. Let H denote the linear span of L. (Note that since $M^\dagger \geq 0$, and $M^\dagger \neq 0$, it follows that $L \neq \{0\}$). By Lemma 3.4, L is a closed and generating cone in H. Define the operator $C : H \to H$ by $C = (M^\dagger B)|_H$. Since $M^\dagger \geq 0$ and $B \geq 0$, we have $C \overset{L}{\geq} 0$ (i.e., $C(L) \subseteq L$). So, by Theorem 2.1, there exist $0 \neq v \in \mathbb{R}^n_+ \cap R(M^*)$ such that $(M^\dagger B)v = Cv = rv$, where r is the spectral radius of C. Next, we show $r \geq 1$. Let I_H denote the identity operator on H. Then

$$\begin{aligned} I_H &= (M^\dagger M)|_H \\ &= (M^\dagger (B - A))|_H \\ &= C - (M^\dagger A)|_H. \end{aligned}$$

Using that $M^\dagger \geq 0$ and $A \geq 0$, the operator $(M^\dagger A)|_H$ is also positive with respect to L, and thus $C = I_H + (M^\dagger A)|_H \overset{L}{\geq} I_H$. Then for all $x \in L$ and for any $n \in \mathbb{N}$, $x \overset{L}{\leq} C^n x$. Since the norm in $\mathbb{R}^n$ is monotonic with respect to L, for any $0 \neq x \in L$,

$$\|x\| \leq \|C^n x\| \leq \|C^n\| \|x\|,$$

so that

$$1 \leq \|C^n\|, \quad n \in \mathbb{N}.$$

Thus
$$1 \le \lim_{n\to\infty} (\|C^n\|)^{\frac{1}{n}} = r(C) = r.$$
Since $Cv = rv$, we have $MCv = rMv$, i.e., $Bv = r(Bv - Av)$, as $R(B) \subseteq R(M)$. This implies $Av = (1 - \frac{1}{r})Bv$. This completes the proof when $\mu = 1 - \frac{1}{r}$. ■

The next result will also be useful in obtaining the result of Peris, a particular case.

Corollary 3.6. *Let $0 \neq M \in \mathbb{R}^{m\times n}$. Suppose that $M^\dagger \ge 0$ and that M has a decomposition $M = B - A$, with $A, B \ge 0$ where $R(B) \subseteq R(M)$. If λ is a real number such that $Au = \lambda Bu$ for some $0 \neq u \in L$, then $\lambda \le \mu = 1 - \frac{1}{r(C)}$, where $C = (M^\dagger B)|_H$ and H is the linear span of L.*

Proof. Let $Au = \lambda Bu$ with $0 \neq u \in L$. Then $Mu = Bu - Au = (1-\lambda)Bu$. If $Mu = 0$, then $u \in R(M^*) \cap N(M)$ and so $u = 0$, a contradiction. Thus $Mu \neq 0$. So $\lambda \neq 1$ and $Cu = \frac{1}{1-\lambda}u$. Then $\frac{1}{1-\lambda} > 0$ as $C \overset{L}{\ge} 0$ and $u \in L$. So $\frac{1}{1-\lambda}$ is a positive eigenvalue with the eigenvector u for C. By Theorem 2.1, $\frac{1}{1-\lambda} \le r(C)$, i.e., $\lambda \le 1 - \frac{1}{r(C)}$. ■

We now show how Peris' theorem follows from our results. This is given mainly for completeness.

Corollary 3.7 (Theorem 1, [11])**.** *For a square non-singular matrix M, the following conditions are equivalent:*

(a) *M is inverse positive ($M^{-1} \ge 0$).*

(b) *For all positive splittings of M i.e., $M = B - A$, $B \ge 0$, $A \ge 0$, there exists $0 \neq v \ge 0$, $\mu \in [0, 1)$ such that $Av = \mu Bv$.*

Furthermore, if there exists $0 \neq u \ge 0$ such that $Au = \lambda Bu$, then $\lambda \le \mu$.

Proof. Indeed, if M^{-1} exists, then $M^\dagger = M^{-1}$ and $R(M) = R(M^*) = \mathbb{R}^n$. So, statement (b) follows by Theorem 3.5. Condition (b) implies statement (a) by Theorem 3.1. The last statement follows from Corollary 3.6. ■

Remarks 3.8.

(i) For the decomposition considered in Example 3.3, $R(B) \nsubseteq R(M)$. Thus the condition $R(B) \subseteq R(M)$ is indispensable in Theorem 3.5, in general.

(ii) Let be $M = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. Let be $M = B - A$ with $A, B \geq 0$. Then $M^\dagger = \frac{1}{2}M^t \geq 0$. If $B = (b_{ij})$ and $A = (a_{ij})$ then $b_{11} = 1 + a_{11}$, $b_{12} = a_{12}$, $b_{21} = 1 + a_{21}$ and $b_{22} = a_{22}$. Let $R(B) \subseteq R(M)$. Consider the equation $Av = \mu Bv$ with $0 \neq v \in \mathbb{R}^2_+ \cap R(M^*)$ and $\mu \in [0,1)$. This reduces to the single equation $a_{11}v_1 = \mu(1 + a_{11})v_1$ for $v_1 > 0$ and $a_{11} \geq 0$, which always has a solution: $\mu = 0$ if $a_{11} = 0$ and $\mu = \frac{a_{11}}{1+a_{11}} < 1$ if $a_{11} > 0$.

(iii) We make the following interesting observation: Suppose that $M^\dagger \geq 0$. Then for all nonnegative matrices A and B that satisfy the conditions of Theorem 3.5, a Perron-Frobenius property holds for $A - \lambda B$, with $\lambda \in [0,1)$. We are presently not aware of any application of this result.

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