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ON COVARIETY LATTICES

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Abstract

This paper shows basic properties of covariety lattices. Such lattices are shown to be infinitely distributive. The covariety lattice $L_{\mathcal{CV}}(\mathsf{K})$ of subcovarieties of a covariety K of F-coalgebras, where $F : \mathsf{Set} \to \mathsf{Set}$ preserves arbitrary intersections is isomorphic to the lattice of subcoalgebras of a $\mathcal{P}_\kappa$-coalgebra for some cardinal κ. A full description of the covariety lattice of $\mathcal{I}d$-coalgebras is given. For any topology τ there exist a bounded functor $F : \mathsf{Set} \to \mathsf{Set}$ and a covariety K of F-coalgebras, such that $L_{\mathcal{CV}}(\mathsf{K})$ is isomorphic to the lattice $(\tau, \cup, \cap)$ of open sets of τ.

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1. Introduction

Many mathematicians and computer scientists have been recently studying the universal theory of coalgebras - objects dual to algebras. Many interesting properties of coalgebras have been shown. E.g. an analogue of the Birkhoff Variety Theorem was developed, which describes syntactically the classes of coalgebras called *covarieties*.

This paper studies the basic properties of covariety lattices. We show that the covariety lattices are infinitely distributive. Corollary 3.9 shows that given any F-coalgebra $\mathbb{A}$ there is a covariety K of $A \times F$-coalgebras such that the lattice $L_{\mathcal{CV}}(\mathsf{K})$ of subcovarieties of K is isomorphic to the lattice $\mathsf{S}(\mathbb{A})$ of subcoalgebras of $\mathbb{A}$.

Next, Theorem 4.1 shows that, whenever F preserves arbitrary intersections, the covariety lattice is isomorphic to the lattice $\mathcal{D}(\mathfrak{R}_{\mathsf{Set}_F})$ of subsets of all rooted coalgebras closed under taking rooted subcoalgebras of homomorphic images. As an example, the covariety lattice of $\mathcal{I}d$-coalgebras is described.

Finally, the covariety lattice $L_{\mathcal{CV}}(\mathsf{K})$ of subcovarieties of a covariety K of F-coalgebras, where $F : \mathsf{Set} \to \mathsf{Set}$ preserves arbitrary intersections, is characterized in Theorem 4.5 as the lattice of subcoalgebras of some $\mathcal{P}_\kappa$-coalgebra.

2. Basic definitions and properties

Let Set be the category of all sets and mappings between them. Let $F : \mathsf{Set} \to \mathsf{Set}$ be a functor. An *F-coalgebra* $\mathbb{A}$ is a pair (A, α), where A is a set and α is a mapping $\alpha : A \to F(A)$. The set A is called the *carrier* of the coalgebra (A, α) and the mapping α is called the *structure*.

Let $\mathbb{A} = (A, \alpha)$ and $\mathbb{B} = (B, \beta)$ be two F-coalgebras. A *homomorphism* from the coalgebra $\mathbb{A}$ to the coalgebra $\mathbb{B}$ is a mapping $h : A \to B$, such that $F(h) \circ \alpha = \beta \circ h$.

The class of all F-coalgebras together with homomorphisms as morphisms forms a category denoted by Set_F. An F-coalgebra $\mathbb{B}$ is said to be a *homomorphic image* of an F-coalgebra $\mathbb{A}$ if there exists a surjective homomorphism from $\mathbb{A}$ onto $\mathbb{B}$. An F-coalgebra $\mathbb{S}$ is said to be a *subcoalgebra* of an F-coalgebra $\mathbb{A}$ if there exists an injective homomorphism from $\mathbb{S}$ into $\mathbb{A}$. This is denoted by $\mathbb{S} \leq \mathbb{A}$.

Theorem 2.1 [2]**.** *Let $F : \mathsf{Set} \to \mathsf{Set}$ be a functor. Let $\{\mathbb{S}_i\}_{i \in I}$ be a family of subcoalgebras of an F-coalgebra $\mathbb{A}$. Then*

- *there exists a unique structure $\alpha : \bigcup_{i \in I} S_i \to F(\bigcup_{i \in I} S_i)$ such that the coalgebra $\bigcup_{i \in I} \mathbb{S}_i := (\bigcup_{i \in I} S_i, \alpha)$ is a subcoalgebra of $\mathbb{A}$;*
- *if I is a finite set of indices, then there exists a unique structure $\beta : \bigcap_{i \in I} S_i \to F(\bigcap_{i \in I} S_i)$ such that $\bigcap_{i \in I} \mathbb{S}_i := (\bigcap_{i \in I} S_i, \beta)$ is a subcoalgebra of $\mathbb{A}$.*

In other words, subcoalgebras of a given coalgebra form a topology.

Theorem 2.2 [2]. *Let $F : \mathsf{Set} \to \mathsf{Set}$ be a functor and $\mathbb{A}$ be an F-coalgebra. If $S \subseteq A$, then there exists at most one structure $\sigma : S \to F(S)$ such that $(S, \sigma) \leq \mathbb{A}$.*

The disjoint union of a family $\{X_j\}_{j\in J}$ of sets is denoted by $\Sigma_{j\in J} X_j$. Now let $\{\mathbb{A}_i\}_{i\in I}$ be a family of F-coalgebras. The *disjoint sum* $\Sigma_{i\in I}\mathbb{A}_i$ of the family $\{\mathbb{A}_i\}_{i\in I}$ of F-coalgebras is an F-coalgebra defined as follows The carrier set of the disjoint sum $\mathbb{A} = \Sigma_{i\in I}\mathbb{A}_i$ is the disjoint union of the carriers of $\mathbb{A}_i$, i.e.

$$A := \Sigma_{i\in I} A_i.$$

The structure $\alpha : A \to F(A)$ of the disjoint sum $\mathbb{A} = \Sigma_{i\in I}\mathbb{A}_i$ is defined as follows

$$\alpha : A \to F(A); A_i \ni a \mapsto F(e_i) \circ \alpha_i(a),$$

where the mapping α_i denotes the structure of the coalgebra $\mathbb{A}_i$ and

$$e_i : A_i \to A; a \mapsto (a, i),$$

for every $i \in I$. We say that an F-coalgebra $\mathbb{A}$ is a *conjunct sum of the family* $\{\mathbb{G}_i\}_{i\in I}$ of F-coalgebras if there exists a family $\{e_i : \mathbb{G}_i \to \mathbb{A}\}_{i\in I}$ of injective homomorphisms such that $A = \bigcup_{i\in I} e_i(G_i)$. We denote it by $\mathbb{A} \in \Sigma^C(\{\mathbb{G}_i\}_{i\in I})$.

A functor $F : \mathsf{Set} \to \mathsf{Set}$ is said to *preserve arbitrary intersections* if for any family of subcoalgebras $\{\mathbb{A}_i\}_{i\in I}$ of an F-coalgebra $\mathbb{A}$, there exists a structure $\alpha : \bigcap A_i \to F(\bigcap A_i)$ such that the F-coalgebra $\bigcap \mathbb{A}_i := (\bigcap A_i, \alpha)$ is a subcoalgebra of $\mathbb{A}$.

A functor $F : \mathsf{Set} \to \mathsf{Set}$ is said to be *bounded by* κ, if κ is the cardinal number such that for every F-coalgebra $\mathbb{A}$ and for every $a \in A$ there exists an F-coalgebra $\mathbb{U}_a$, such that $|U_a| \leq \kappa$, $a \in U_a$ and $\mathbb{U}_a \leq \mathbb{A}$. We say that F *is bounded* if it is bounded by κ for some cardinal κ.

Example 2.3. Let κ be a cardinal number. Let $\mathcal{P}_\kappa : \mathsf{Set} \to \mathsf{Set}$ be the functor given by $\mathcal{P}_\kappa(X) = \{S \subseteq X \mid |S| \leq \kappa\}$ for a set X and

$$\mathcal{P}_\kappa(f) : \mathcal{P}_\kappa(X) \to \mathcal{P}_\kappa(Y); S \to f(S)$$

for a mapping $f : X \to Y$. The functor $\mathcal{P}_\kappa$ is an example of a bounded functor which preserves arbitrary intersections (see [5]).

Example 2.4. The filter functor $\mathcal{F} : \mathsf{Set} \to \mathsf{Set}$ assigns to every set X the set of filters $\mathcal{F}(X)$ on X and to every mapping $f : X \to Y$ the mapping

$$\mathcal{F}(f) : \mathcal{F}(X) \to \mathcal{F}(Y); F \mapsto \uparrow \{f(W) \mid W \in F\},$$

where $\uparrow \{f(W) \mid W \in F\}$ denotes the filter generated by the set $\{f(W) \mid W \in F\}$. This functor is an example of a functor which does not preserve arbitrary intersections.

It is important to mention that any topological space can be turned into an $\mathcal{F}$-coalgebra. Let (X, τ) be a topological space. Define the mapping

$$\sigma : X \to \mathcal{F}(X); x \mapsto \{W \subseteq X \mid \exists O \in \tau \text{ such that } x \in O \subseteq W\}.$$

The subcoalgebras of the $\mathcal{F}$-coalgebra (X, σ) are precisely the open subsets of (X, τ) (see [3]). Since the intersection of an arbitrary family of open sets in a given topological space may not exist, it is clear that $\mathcal{F}$ does not preserve arbitrary intersections. The filter functor $\mathcal{F}$ is not bounded.

Let K be a class of F-coalgebras. We define the following classes of F-coalgebras:

$$\begin{aligned} \mathcal{S}(\mathsf{K}) &:= \{\mathbb{S} \mid \exists \mathbb{A} \in \mathsf{K} \text{ such that } \mathbb{S} \leq \mathbb{A}\}, \\ \mathcal{H}(\mathsf{K}) &:= \{\mathbb{B} \mid \exists \mathbb{A} \in \mathsf{K} \text{ such that } \mathbb{A} \twoheadrightarrow \mathbb{B}\}, \\ \Sigma(\mathsf{K}) &:= \{\Sigma_{i \in I} \mathbb{A}_i \mid \{\mathbb{A}_i\}_{i \in I} \subseteq \mathsf{K}\}. \end{aligned}$$

A class K of F -coalgebras is called a *covariety* if it is closed under $\mathcal{S}, \mathcal{H}$ and Σ, i.e., $\mathcal{S}(\mathsf{K}) \subseteq \mathsf{K}$, $\mathcal{H}(\mathsf{K}) \subseteq \mathsf{K}$ and $\Sigma(\mathsf{K}) \subseteq \mathsf{K}$.

Theorem 2.5 [2]**.** *Let* K *be a class of* F*-coalgebras. The class* $\mathcal{SH}\Sigma(\mathsf{K})$ *is the smallest covariety containing* K.

We say that a class K' of F-coalgebras is a *subcovariety* of a covariety K whenever K' is a covariety and $\mathsf{K}' \subseteq \mathsf{K}$.

The assumption of boundedness of a functor F guarantees that the collection of all subcovarieties of the covariety Set_F is a set (see [2]). Since we do not want to focus only on coalgebras for bounded functors we need to allow *class based lattices*, i.e., partially ordered classes in which each pair of elements has a supremum and an infimum. Obviously, any lattice is a class based lattice. We may easily generalize the notion of completeness to the class based lattices. Namely, a partially ordered class $(C, \leq)$ is a complete class based lattice if all its subclasses have a supremum and infimum. We see that whenever $(C, \leq)$ is a complete class based lattice and C is a proper set then $(C, \leq)$ is simply a complete lattice. The following holds.

Theorem 2.6. *The collection of all subcovarieties of a given covariety* K *of* F*-coalgebras ordered by inclusion is a complete class based lattice.*

We denote the class based lattice of all subcovarieties of K by $L_{\mathcal{CV}}(\mathsf{K})$. Let $\{\mathsf{K}_i\}_{i\in I}$ be a collection of subcovarieties of the covariety K of F-coalgebras. Note that the collection $\{\mathsf{K}_i\}_{i\in I}$ and hence I may be a proper class. The infimum and supremum of $\{\mathsf{K}_i\}_{i\in I}$ in $L_{\mathcal{CV}}(\mathsf{K})$ are of the following form.

$$\begin{aligned}\prod_{i\in I}\mathsf{K}_i &:= \bigcap_{i\in I}\mathsf{K}_i,\\ \sum_{i\in I}\mathsf{K}_i &:= \mathcal{SH}\Sigma\left(\bigcup_{i\in I}\mathsf{K}_i\right).\end{aligned}$$

We will clearly distinguish between the class based lattices whose carrier is a proper class and lattices with a set carrier. We will use the term *proper lattice* to emphasize the fact that the latter holds, i.e. a class based lattice is simply a lattice.

3. COVARIETY LATTICES

In this section we discuss the distributivity of covariety class based lattices. Then we describe the lattices $L_{\mathcal{CV}}(\mathcal{SH}\Sigma(\mathbb{A}))$ for certain coalgebras $\mathbb{A}$ and show that the lattice of open sets of any topological space is isomorphic to some covariety lattice $L_{\mathcal{CV}}(\mathsf{K})$.

Suppose F is bounded by $|X|$ for some set X. Then the cofree F-coalgebra $\mathbb{C}_X$ over the set X exists. In this case there is a one-to-one correspondence between the so-called invariant subcoalgebras of $\mathbb{C}_X$ and covarieties of F-coalgebras. This correspondence is given by the following formula:

$$\mathsf{K} = \mathcal{Q}(\mathbb{C}_X, \mathbb{U}) := \{\mathbb{A} \mid \forall\phi : \mathbb{A} \to \mathbb{C}_X, \phi(A) \subseteq U\},$$

where $\mathbb{U} := \bigcup\{\phi(\mathbb{A}) \mid \phi : \mathbb{A} \to \mathbb{C}_X \text{ and } \mathbb{A} \in \mathsf{K}\}$ (see [2]). Therefore, the lattice $L_{\mathcal{CV}}(\mathsf{Set}_F)$ of all covarieties of F-coalgebras is isomorphic to the lattice of invariant subcoalgebras of $\mathbb{C}_X$ ordered by inclusion. Because it is clear that the invariant subcoalgebras are closed under infinite unions and finite intersections the lattice $L_{\mathcal{CV}}(\mathsf{Set}_F)$ is infinitely distributive. If we do not assume boundedness of F then we cannot speak of the above correspondence. Yet, we are able to derive the following result directly.

Theorem 3.1. *The class based lattice $L_{\mathcal{CV}}(\mathsf{Set}_F)$ of covarieties of F-coalgebras is distributive.*

Proof. Let $\{\mathsf{K}_i\}_{i\in I}$ be a collection of covarieties of F-coalgebras and let K be a covariety. Note that I may be a proper class. To show that the covariety class based lattice $L_{\mathcal{CV}}(\mathsf{Set}_F)$ is distributive it is enough to verify that the following inequality is true:

$$\mathsf{K}\cdot\left(\sum_{i\in I}\mathsf{K}_i\right)\leq\sum_{i\in I}\mathsf{K}\cdot\mathsf{K}_i.$$

Let $\mathbb{A}\in\mathsf{K}\cdot(\sum_{i\in I}\mathsf{K}_i)$. This means that $\mathbb{A}\in\mathsf{K}$ and $\mathbb{A}\in\sum_{i\in I}\mathsf{K}_i$. Since $\sum_{i\in I}\mathsf{K}_i=\mathcal{SH}\Sigma(\bigcup_{i\in I}\mathsf{K}_i)$, it follows that $\mathbb{A}\leq h(\Sigma_{j\in J}\mathbb{B}_j)$, where $\mathbb{B}_j\in\bigcup_{i\in I}\mathsf{K}_i$ for any j coming from the set of indices J and h is a homomorphism. Let $e_k:\mathbb{B}_k\to\Sigma_{j\in J}\mathbb{B}_j$ for $k\in J$ denote the canonical embeddings. Then $\mathbb{A}\leq\bigcup_{j\in J}h(e_j(\mathbb{B}_j))$. By Theorem 2.1 we have

$$\mathbb{A}=\bigcup_{j\in J}h(e_j(\mathbb{B}_j))\cap\mathbb{A}.$$

Since all K_i's are covarieties and $h(e_j(\mathbb{B}_j))\cap\mathbb{A}\leq h(e_j(\mathbb{B}_j))$, it follows that $h(e_j(\mathbb{B}_j))\cap\mathbb{A}\in\mathsf{K}_{i_j}$ for some $i_j\in I$. Moreover, because $h(e_j(\mathbb{B}_j))\cap\mathbb{A}\leq\mathbb{A}$ and $\mathbb{A}\in\mathsf{K}$, we have $h(e_j(\mathbb{B}_j))\cap\mathbb{A}\in\mathsf{K}$. Hence $h(e_j(\mathbb{B}_j))\cap\mathbb{A}\in\mathsf{K}\cdot\mathsf{K}_{i_j}$ and therefore

$$\mathbb{A}\in\sum_{i\in I}\mathsf{K}\cdot\mathsf{K}_i.$$

■

Definition 3.2 ([2])**.** An F-coalgebra $\mathbb{A}$ is called *strongly simple* whenever it does not possess any nontrivial homomorphic images.

We will now show some properties of strongly simple coalgebras, neccessary for characterisation of $L_{\mathcal{CV}}(\mathcal{SH}\Sigma(\mathbb{A}))$.

Lemma 3.3 ([2])**.** *Let $\mathbb{A}$ be a strongly simple F-coalgebra. If $\mathbb{B}$ is an F-coalgebra, then there exists at most one homomorphism $h : \mathbb{B} \to \mathbb{A}$.*

Lemma 3.4. *Let $\mathbb{A} = (A, \alpha)$ be a strongly simple F-coalgebra. Let $\mathbb{S} \leq \mathbb{A}$ and $\mathbb{T} \leq \mathbb{A}$ be such that $\mathbb{S} \cong \mathbb{T}$. Then $\mathbb{S} = \mathbb{T}$.*

Lemma 3.5. *Let $\mathbb{A}$ be a strongly simple F-coalgebra. If $\mathbb{B} \in \mathcal{SH}\Sigma(\mathbb{A})$, then $\mathbb{B} \in \Sigma^C\mathcal{S}(\mathbb{A})$.*

Proof. If $\mathbb{B} \in \mathcal{SH}\Sigma(\mathbb{A})$ then $\mathbb{B} \leq h(\Sigma_{i\in I}\mathbb{A})$, where h is a homomorphism. Let $e_i : \mathbb{A} \to \Sigma_{i\in I}\mathbb{A}$ denote the canonical embeddings. Since $\mathbb{A}$ is strongly simple, it follows that the image coalgebra $h(e_i(\mathbb{A}))$ is isomorphic to $\mathbb{A}$ for each $i \in I$. Since $h(\Sigma_{i\in I}\mathbb{A}) = \bigcup_{i\in I} h(e_i(\mathbb{A}))$, it follows that $\mathbb{B} = \bigcup_{i\in I} h(e_i(\mathbb{A})) \cap \mathbb{B}$. Because $h(e_i(\mathbb{A})) \cap \mathbb{B} \leq h(e_i(\mathbb{A})) \cong \mathbb{A}$, we have $\mathbb{B} \in \Sigma^C\mathcal{S}(\mathbb{A})$. ■

Let $\mathbb{A}$ be an F-coalgebra. Let $\mathtt{S}(\mathbb{A})$ denote the set of carriers of subcoalgebras of $\mathbb{A}$, i.e.

$$\mathtt{S}(\mathbb{A}) := \{B \mid \mathbb{B} \leq \mathbb{A}\}.$$

By Theorem 2.1, the set $\mathtt{S}(\mathbb{A})$ together with the operations of union and intersection forms a lattice.

What we now want to do is to show without any additional assumptions that $L_{\mathcal{CV}}(\mathcal{SH}\Sigma(\mathbb{A}))$ is isomorphic to the proper lattice $(\mathtt{S}(\mathbb{A}), \cup, \cap)$ for any strongly simple coalgebra $\mathbb{A}$. If we assume that F is bounded then the cofree F-coalgebra $\mathbb{C}_1$ over the one-element set 1 exists. The coalgebra $\mathbb{C}_1$ is the terminal object in the category Set_F. Therefore, it is strongly simple. Moreover, strongly simple F-coalgebras are precisely subcoalgebras of $\mathbb{C}_1$ and all subcoalgebras of $\mathbb{C}_1$ are invariant. Hence, $L_{\mathcal{CV}}(\mathcal{SH}\Sigma(\mathbb{C}_1))$ is isomorphic to $(\mathtt{S}(\mathbb{C}_1), \cup, \cap)$. The same thing is clearly true for any subcoalgebra of $\mathbb{C}_1$. If we do not assume that F is bounded the terminal object in Set_F may not exist. Yet, we can expand our category Set_F to class based coalgebras, where the terminal object always exists (see [1]). Using a similar argument and working with class based coalgebras and we get a general result. At the same time if one does not prefer to work with classes then the direct proof of the following theorem is an alternative.

Theorem 3.6. *Let $\mathbb{A}$ be a strongly simple F-coalgebra. Then $L_{\mathcal{CV}}(\mathcal{SH}\Sigma(\mathbb{A}))$ is a proper lattice and*

$$L_{\mathcal{CV}}(\mathcal{SH}\Sigma(\mathbb{A})) \cong (\mathtt{S}(\mathbb{A}), \cup, \cap).$$

Proof. Let K be a subcovariety of the covariety $\mathcal{SH}\Sigma(\mathbb{A})$. Define

$$\mathbb{S}_{\mathsf{K}} := \bigcup\{\mathbb{S}|\ \mathbb{S} \leq \mathbb{A} \text{ and } \mathbb{S} \in \mathsf{K}\}.$$

In other words, the F-coalgebra $\mathbb{S}_{\mathsf{K}}$ is the union of subcoalgebras of $\mathbb{A}$ which are elements of the covariety K.

It is clear that $\mathbb{S}_{\mathsf{K}}$ is the greatest subcoalgebra of $\mathbb{A}$ contained in K.

Let $\mathbb{B} \in \mathsf{K}$. We have $\mathbb{B} \leq f(\Sigma_{i\in I}\mathbb{A})$ for a homomorphism f. By Lemma 3.5, $\mathbb{B} \in \Sigma^C(\{\mathbb{C}_i\}_{i\in I})$, where $\mathbb{C}_i \leq \mathbb{A}$ for $i \in I$. Since $\mathbb{C}_i \leq \mathbb{B}$, it follows that $\mathbb{C}_i \in \mathsf{K}$. Hence $\mathbb{C}_i \leq \mathbb{S}_{\mathsf{K}}$ for $i \in I$ and $\mathbb{B} \in \Sigma^C\mathcal{S}(\mathbb{S}_{\mathsf{K}})$. Therefore any coalgebra $\mathbb{B} \in \mathsf{K}$ is a conjunct sum of subcoalgebras of $\mathbb{S}_{\mathsf{K}}$, i.e. $\mathsf{K} = \Sigma^C\mathcal{S}(\mathbb{S}_{\mathsf{K}})$.

We will now prove that the mapping

$$\mathbb{S}_{(-)} : L_{\mathcal{CV}}(\mathcal{SH}\Sigma(\mathbb{A})) \to \mathsf{S}(\mathbb{A}); \mathsf{K} \mapsto \mathbb{S}_{\mathsf{K}}$$

is a lattice isomorphism. To show that it is injective, let K_1 and K_2 be subcovarieties of the covariety $\mathcal{SH}\Sigma(\mathbb{A})$ such that $\mathbb{S}_{\mathsf{K}_1} = \mathbb{S}_{\mathsf{K}_2}$. Then

$$\mathsf{K}_1 = \Sigma^C\mathcal{S}(\mathbb{S}_{\mathsf{K}_1}) = \Sigma^C\mathcal{S}(\mathbb{S}_{\mathsf{K}_2}) = \mathsf{K}_2.$$

We will now show that $\mathbb{S}_{(-)}$ is a surjection. Let $\mathbb{C} \leq \mathbb{A}$. Then

$$\mathbb{C} \leq \mathbb{S}_{\mathcal{SH}\Sigma(\mathbb{C})}.$$

Since $\mathbb{S}_{\mathcal{SH}\Sigma(\mathbb{C})} \leq \mathbb{A}$ and since $\mathbb{S}_{\mathcal{SH}\Sigma(\mathbb{C})} \in \mathcal{SH}\Sigma(\mathbb{C}) = \Sigma^C\mathcal{S}(\mathbb{C})$, it follows that

$$\mathbb{S}_{\mathcal{SH}\Sigma(\mathbb{C})} \in \Sigma^C(\{\mathbb{D}_j\}_{j\in J}),$$

where $\mathbb{D}_j \leq \mathbb{C}$. This means that for any $j \in J$, the coalgebra $\mathbb{S}_{\mathcal{SH}\Sigma(\mathbb{C})}$ contains a coalgebra $\widetilde{\mathbb{D}}_j$ isomorphic to $\mathbb{D}_j$ as its subcoalgebra. Hence

$$\widetilde{\mathbb{D}}_j \leq \mathbb{S}_{\mathcal{SH}\Sigma(\mathbb{C})} \leq \mathbb{A}$$

for all $j \in J$, and $\mathbb{D}_j \leq \mathbb{C} \leq \mathbb{A}$. By Lemma 3.4, we have $\widetilde{\mathbb{D}}_j = \mathbb{D}_j$. Therefore,

$$\mathbb{S}_{\mathcal{SH}\Sigma(\mathbb{C})} = \bigcup_{j\in J}\widetilde{\mathbb{D}}_j = \bigcup_{j\in J}\mathbb{D}_j \leq \mathbb{C}$$

and $\mathbb{S}_{\mathcal{SH}\Sigma(\mathbb{C})} = \mathbb{C}$. Consequently the mapping $\mathbb{S}_{(-)}$ is a bijection. Since it is clear that $\mathbb{S}_{(-)}$ is order preserving we immediately get that $\mathbb{S}_{(-)}$ is the isomorphism from the lattice $L_{\mathcal{CV}}(\mathcal{SH}\Sigma(\mathbb{A}))$ onto $(\mathsf{S}(\mathbb{A}), \cup, \cap)$. ∎

For an F-coalgebra $\mathbb{A} = (A, \alpha)$ and a set B such that $A \subseteq B$, we define the following $B \times F$- coalgebra:

$$\mathbb{A}_B := \left(A, \left(\subseteq_A^B, \alpha\right)\right).$$

The structure map of $\mathbb{A}_B$ is the following:

$$\left(\subseteq_A^B, \alpha\right) : A \to B \times F(A); a \mapsto (a, \alpha(a)).$$

This easy trick allows us to force the $B \times F$-coalgebra $\mathbb{A}_B$ to be strongly simple and at the same time to leave the subcoalgebras of $\mathbb{A}$ untouched. This property is formally described by the following lemmata.

Lemma 3.7. *Let $\mathbb{A} = (A, \alpha)$ be an F-coalgebra and let B be a set such that $A \subseteq B$. Then the $B \times F$-coalgebra $\mathbb{A}_B$ is strongly simple.*

Lemma 3.8. *Let $\mathbb{A} = (A, \alpha)$ be an F-coalgebra and let B be a set such that $A \subseteq B$. Then $(\mathsf{S}(\mathbb{A}), \cup, \cap) = (\mathsf{S}(\mathbb{A}_B), \cup, \cap)$.*

Corollary 3.9. *Let (X, τ) be a topological space. There exists a bounded functor $F : \mathsf{Set} \to \mathsf{Set}$ and a covariety K of F-coalgebras such that $L_{\mathcal{CV}}(\mathsf{K})$ is isomorphic to the lattice $(\tau, \cup, \cap)$ of open sets in τ.*

Proof. It follows by Example 2.4, Lemma 3.8, Lemma 3.7 and Theorem 3.6. ■

4. Covariety lattices for functors preserving arbitrary intersections

Throughout this section we will assume that F is a bounded functor. Therefore, the collection of all covarieties of F-coalgebras is a set. It is worth noting that almost all of the results presented here naturally generalize to the case when classes of covarieties are allowed.

Given a strongly simple F-coalgebra $\mathbb{A}$, Theorem 3.6 describes the lattice of subcovarieties of the covariety $\mathcal{SH}\Sigma(\mathbb{A})$ in terms of the lattice of subcoalgebras of $\mathbb{A}$. The following question arises: can we describe the covariety lattice of any covariety K of F-coalgebras in a similar way in terms of subcoalgebras of an F-coalgebra? In general the answer is "no", which is seen in the Example 4.4. But first, we will characterize the lattice $L_{\mathcal{CV}}(\mathsf{Set}_F)$ in the case the functor F preserves arbitrary intersections.

An F-coalgebra $\mathbb{A}$ is called *rooted* (or *one-generated*) if there exists an element $a \in A$, called a *root*, such that the coalgebra $\mathbb{A}$ is the smallest subcoalgebra of $\mathbb{A}$ containing the element a. If $a \in A$ is a root of a rooted coalgebra $\mathbb{A}$, then we say that $\mathbb{A}$ is generated by a.

If $F : \mathsf{Set} \to \mathsf{Set}$ preserves arbitrary intersections, then all rooted F-coalgebras are of the following form

$$\langle a \rangle := \bigcap \{\mathbb{S} \mid a \in S \text{ and } \mathbb{S} \leq \mathbb{A}\},$$

for some F-coalgebra $\mathbb{A}$ and $a \in A$. For any F-coalgebra $\mathbb{A}$, we have $\mathbb{A} = \bigcup_{a \in A} \langle a \rangle$. It follows that $\mathbb{A} \in \Sigma^C(\{\langle a \rangle\}_{a \in A})$.

Let K be a class of F-coalgebras. Let $\mathfrak{R}_{\mathsf{K}}$ denote the collection of rooted F-coalgebras consisting of exactly one representative from each class of isomorphic rooted F-coalgebras from the class K. If $\mathbb{A}, \mathbb{B} \in \mathfrak{R}_{\mathsf{K}}$ and are isomorphic, then $\mathbb{A} = \mathbb{B}$. By the assumption of boundedness of F we know that $\mathfrak{R}_{\mathsf{K}}$ is a proper set. Let $\mathcal{D}(\mathfrak{R}_{\mathsf{K}})$ denote the set of subsets of $\mathfrak{R}_{\mathsf{K}}$ closed under taking subcoalgebras of homomorphic images, i.e.:

$$\mathcal{D}(\mathfrak{R}_{\mathsf{K}}) := \{U \subseteq \mathfrak{R}_{\mathsf{K}} \mid \mathfrak{R}_{\mathsf{K}} \cap \mathcal{SH}(U) = U\}.$$

Theorem 4.1. *If $F : \mathsf{Set} \to \mathsf{Set}$ preserves arbitrary intersections then the lattice $L_{\mathcal{CV}}(\mathsf{Set}_F)$ of subcovarieties of Set_F is isomorphic to the lattice $(\mathcal{D}(\mathfrak{R}_{\mathsf{Set}_F}), \cup, \cap)$.*

Proof. Let K be a covariety of F-coalgebras. Let $\mathbb{A} \in \mathfrak{R}_{\mathsf{K}}$. Then $\mathbb{A}$ is a rooted coalgebra in the covariety K. The rooted subcoalgebras of homomorphic images of $\mathbb{A}$ are elements of the set $\mathfrak{R}_{\mathsf{K}}$. This means that $\mathfrak{R}_{\mathsf{K}} \in \mathcal{D}(\mathfrak{R}_{\mathsf{Set}_F})$. We define the following mapping.

$$r : L_{\mathcal{CV}}(\mathsf{Set}_F) \to \mathcal{D}(\mathfrak{R}_{\mathsf{Set}_F}); \mathsf{K} \mapsto \mathfrak{R}_{\mathsf{K}}.$$

We will show that r is an isomorphism. Let K_1 and K_2 be two covarieties such that $r(\mathsf{K}_1) = r(\mathsf{K}_2)$. Let $\mathbb{A} \in \mathsf{K}_1$. For any $a \in A$ the rooted coalgebra $\langle a \rangle$ is a subcoalgebra of $\mathbb{A}$. Hence $\langle a \rangle \in \mathsf{K}_1$ and $\langle a \rangle \in \mathsf{K}_2$. Since $\mathbb{A} = \bigcup_{a \in A} \langle a \rangle$, the coalgebra $\mathbb{A}$ belongs to K_2. Therefore, $\mathsf{K}_1 = \mathsf{K}_2$ and the mapping r is injective.

Now let $U \in \mathcal{D}(\mathfrak{R}_{\mathsf{Set}_F})$. The smallest covariety containing U is given by the class $\mathcal{SH}\Sigma(U)$. It is clear that $U \subseteq r(\mathcal{SH}\Sigma(U))$. Now let $\mathbb{A} \in r(\mathcal{SH}\Sigma(U))$. This means that $\mathbb{A}$ is a rooted coalgebra, say $\mathbb{A} = \langle a \rangle$, and is a subcoalgebra of $\mathbb{B}$, where $\mathbb{B} = h(\Sigma_{i\in I}\mathbb{C}_i)$ is a homomorphic image of the disjoint sum of a family $\{\mathbb{C}_i\}_{i\in I}$ of rooted coalgebras in U. Let $e_i : \mathbb{C}_i \to \Sigma_{i\in I}\mathbb{C}_i$ denote the canonical embeddings. It is easy to see that $\mathbb{B} = h(\Sigma_{i\in I}\mathbb{C}_i) = \bigcup_{i\in I} h(e_i(\mathbb{C}_i))$. Since $\langle a\rangle \le \mathbb{B}$, it follows that $a \in h(e_j(\mathbb{C}_j))$ for some $j \in I$. Hence $\langle a\rangle \le h(e_j(\mathbb{C}_j))$. Since U is closed under taking rooted subcoalgebras of homomorphic images, it follows that $\mathbb{A} = \langle a\rangle \in U$. Therefore $U = r(\mathcal{SH}\Sigma(U))$ and the mapping r is surjective. Consequently r is bijective. It is clear that the mapping r is an order embedding. Hence r is a lattice isomorphism. ∎

Remark 4.2. It is worth noting that the mapping r in the proof of Theorem 4.1 is in fact a complete lattice isomorphism.

Corollary 4.3. *Let $F : \mathsf{Set} \to \mathsf{Set}$ preserve arbitrary intersections and let K be a covariety of F-coalgebras. Then $L_{\mathcal{CV}}(\mathsf{K}) \cong (\mathcal{D}(\mathfrak{R}_{\mathsf{K}}), \cup, \cap)$.*

Example 4.4. We will describe the covariety lattice $L_{\mathcal{CV}}(\mathsf{Set}_{\mathcal{I}d})$. By Theorem 4.1, the first step is to find all rooted $\mathcal{I}d$-coalgebras. Note that $\mathcal{I}d$-coalgebras are exactly mono-unary algebras. Therefore, we can speak of an *index* and *period* of a rooted $\mathcal{I}d$-coalgebra. Let $\mathsf{N}_0 = \mathsf{N} \cup \{0\}$. It is easy to see that every rooted $\mathcal{I}d$-coalgebra can be represented by a pair $(i,p) \in \mathsf{N}_0 \times \mathsf{N} \cup \{(\infty,0)\}$, where i denotes an index and p a period of a given coalgebra. E.g. $(0,2)$ denotes the coalgebra given by the diagram $\bullet \leftrightarrows \bullet$ and $(1,2)$ by the diagram $\bullet \to \bullet \leftrightarrows \bullet$. Given a finite rooted $\mathcal{I}d$-coalgebra (i,p), it is not hard to notice that any rooted subcoalgebra of (i,p) is of the form (i',p), where $i' \le i$. Any subcoalgebra of $(\infty,0)$ is of the form $(\infty,0)$. Moreover, any rooted homomorphic image of a coalgebra (i,p) is of the form (i',p'), where $i' \le i$ and $p' |\, p$. Therefore,

$$\mathcal{SH}((i,p)) = \{(i',p') \in \mathsf{N}_0 \times \mathsf{N} \cup \{(\infty,0)\} \mid i' \le i \text{ and } p'|p\}.$$

We can introduce a partial order on $\mathsf{N}_0 \times \mathsf{N} \cup \{(\infty,0)\}$ as follows: $(i',p') \preccurlyeq (i,p) :\Longleftrightarrow i' \le i$ and $p' |\, p$. Then

$$\mathcal{SH}((i,p)) = \downarrow (i,p) := \{(i',p') \mid (i',p') \preccurlyeq (i,p)\}.$$

By Theorem 4.1, the lattice $L_{\mathcal{CV}}(\mathsf{Set}_{\mathcal{I}d})$ of subcovarieties of $\mathsf{Set}_{\mathcal{I}d}$ is isomorphic to the lattice of downsets $(\mathcal{O}(\mathsf{N}_0 \times \mathsf{N} \cup \{(\infty,0)\}), \cup, \cap)$ of the poset $\mathsf{N}_0 \times \mathsf{N} \cup \{(\infty,0)\}$.

Now, consider the $\mathcal{I}d$-coalgebra $(1,2)$. The covariety lattice of $\mathcal{SH}\Sigma(1,2)$ looks as follows:

$$\begin{tikzcd}[column sep=small]
 & \mathcal{SH}\Sigma(1,2) \arrow[d, dash] & \\
 & \mathcal{SH}\Sigma\{(0,2),(1,1)\} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{SH}\Sigma(0,2) \arrow[dr, dash] & & \mathcal{SH}\Sigma(1,1) \arrow[dl, dash] \\
 & \mathcal{SH}\Sigma(0,1) \arrow[d, dash] & \\
 & \emptyset &
\end{tikzcd}$$

At the beginning of this section we stated a question whether it was possible to describe a covariety lattice $L_{\mathcal{CV}}(\mathsf{K})$ of any covariety K of F-coalgebras in terms of subcoalgebras of an F-coalgebra. We will show that it is impossible to construct an $\mathcal{I}d$-coalgebra $\mathbb{A}$, whose subcoalgebra lattice is isomorphic to the covariety lattice $\mathcal{SH}\Sigma(1,2)$. By contradiction, assume that there exists $\mathcal{I}d$-coalgebra $\mathbb{A}$ whose subcoalgebra lattice is the following:

$$\begin{tikzcd}[column sep=small, row sep=small]
 & A \arrow[d, dash] & \\
 & B \cup C \arrow[dl, dash] \arrow[dr, dash] & \\
B \arrow[dr, dash] & & C \arrow[dl, dash] \\
 & B \cap C \arrow[d, dash] & \\
 & \emptyset &
\end{tikzcd}$$

Join irreducible elements, i.e. $\mathbb{B}, \mathbb{C}$ and $\mathbb{B}\cap\mathbb{C}$, must be rooted $\mathcal{I}d$-coalgebras. The rooted coalgebra $\mathbb{B}\cap\mathbb{C}$ does not contain any proper subcoalgebras. This means that $\mathbb{B}\cap\mathbb{C}$ is a cycle. The coalgebras $\mathbb{B} = \langle b\rangle$ and $\mathbb{C} = \langle c\rangle$ cover the coalgebra $\mathbb{B}\cap\mathbb{C}$. Hence the coalgebra $\mathbb{B}\cup\mathbb{C}$ has the following form.

$$\begin{tikzcd}[column sep=small]
b \arrow[d] & & & c \arrow[d] \\
\bullet \arrow[d] & \cdots \arrow[l] & & \bullet \arrow[ll] \\
\bullet \arrow[r] & \cdots \arrow[rr] & & \bullet \arrow[u]
\end{tikzcd}$$

Since $\mathbb{A}$ itself is join irreducible, it follows that it is rooted, i.e. $\mathbb{A} = \langle a\rangle$. On one hand the element a has to be connected directly with the element b and on the other with the element c, which is a contradiction. $\square$

Theorem 4.5. *Let $F : \mathsf{Set} \to \mathsf{Set}$ be a functor preserving arbitrary intersections. Then the lattice $L_{\mathcal{CV}}(\mathsf{K})$ of subcovarieties of a covariety K of F-coalgebras is isomorphic to the lattice of subcoalgebras of some $\mathcal{P}_\kappa$-coalgebra.*

Conversely, for any $\mathcal{P}_\kappa$-coalgebra $\mathbb{A}$, there exists a functor $F : \mathsf{Set} \to \mathsf{Set}$ preserving arbitrary intersections and a covariety K of F-coalgebras such that the lattice $L_{\mathcal{CV}}(\mathsf{K})$ is isomorphic to the lattice of subcoalgebras of $\mathbb{A}$.

Proof. If F preserves arbitrary intersection, then by Theorem 4.1, the lattice $L_{\mathcal{CV}}(\mathsf{Set}_F)$ of subcovarieties of Set_F is isomorphic to the lattice $(\mathcal{D}(\mathfrak{R}_{\mathsf{Set}_F}), \cup, \cap)$. Take $\kappa := |\mathfrak{R}_{\mathsf{Set}_F}|$. Define a $\mathcal{P}_\kappa$-coalgebra $(\mathfrak{R}_{\mathsf{Set}_F}, \eta)$ as follows. For $\langle a \rangle \in \mathfrak{R}_{\mathsf{Set}_F}$ define

$$\eta(\langle a \rangle) := \mathcal{SH}(\langle a \rangle) \cap \mathfrak{R}_{\mathsf{Set}_F}.$$

Then clearly

$$\mathsf{S}((\mathfrak{R}_{\mathsf{Set}_F}, \eta)) \cong \mathcal{D}(\mathfrak{R}_{\mathsf{Set}_F}) \cong L_{\mathcal{CV}}(\mathsf{Set}_F).$$

Conversely let $\mathbb{A} = (A, \alpha)$ be a $\mathcal{P}_\kappa$-coalgebra. Then by Theorem 3.6, the lattice $L_{\mathcal{CV}}(\mathcal{SH}\Sigma(\mathbb{A}_A))$ of subcovarieties of the covariety $\mathcal{SH}\Sigma(\mathbb{A}_A)$ of $A \times \mathcal{P}_\kappa$-coalgebras is isomorphic to $\mathsf{S}(\mathbb{A})$ and the functor $A \times \mathcal{P}_\kappa$ is bounded and preserves arbitrary intersections. ∎

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