# WREATH PRODUCT OF A SEMIGROUP AND A $\Gamma$-SEMIGROUP 

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#### Abstract

Let $S=\{a, b, c, \ldots\}$ and $\Gamma=\{\alpha, \beta, \gamma, \ldots\}$ be two nonempty sets. $S$ is called a $\Gamma$-semigroup if $a \alpha b \in S$, for all $\alpha \in \Gamma$ and $a, b \in S$ and $(a \alpha b) \beta c=a \alpha(b \beta c)$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$. In this paper we study the semidirect product of a semigroup and a $\Gamma$-semigroup. We also introduce the notion of wreath product of a semigroup and a $\Gamma$ semigroup and investigate some interesting properties of this product.

Keywords: semigroup, $\Gamma$-semigroup, orthodox semigroup, right(left) orthodox $\Gamma$-semigroup, right(left) inverse semigroup, right(left) inverse $\Gamma$-semigroup, right(left) $\alpha$ - unity, $\Gamma$-group, semidirect product, wreath product.


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## 1. Introduction

The notion of a $\Gamma$-semigroup has been introduced by Sen and Saha $[7]$ in the year 1986. Many classical notions of semigroup have been extended to $\Gamma$-semigroup. In [1] and [2] we have introduced the notions of right inverse $\Gamma$-semigroup and right orthodox $\Gamma$-semigroup. In [6] we have studied the semidirect product of a monoid and a $\Gamma$-semigroup as a generalization of [4] and [5]. We have obtained necessary and sufficient conditions for a semidirect product of the monoid and a $\Gamma$-semigroup to be right (left) orthodox $\Gamma$-semigroup and right (left) inverse $\Gamma$-semigroup. In [9] Zhang has studied the semidirect product of semigroups and also studied wreath product of semigroups. In this paper we generalize the results of Zhang to the semidirect product of a semigroup and a $\Gamma$-semigroup. We also study the wreath product of a semigroup and a $\Gamma$-semigroup.

## 2. Preliminaries

We now recall some definitions and results relating our discussion.
Definition 2.1. Let $S=\{a, b, c, \ldots\}$ and $\Gamma=\{\alpha, \beta, \gamma, \ldots\}$ be two nonempty sets. $S$ is called a $\Gamma$-semigroup if
(i) $a \alpha b \in S$, for all $\alpha \in \Gamma$ and $a, b \in S$ and
(ii) $(a \alpha b) \beta c=a \alpha(b \beta c)$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

Let $S$ be an arbitrary semigroup. Let 1 be a symbol not representing any element of $S$. We extend the binary operation defined on $S$ to $S \cup\{1\}$ by defining $11=1$ and $1 a=a 1=a$ for all $a \in S$. It can be shown that $S \cup\{1\}$ is a semigroup with identity element 1 . Let $\Gamma=\{1\}$. If we take $a b=a 1 b$, it can be shown that the semigroup $S$ is a $\Gamma$-semigroup where $\Gamma=\{1\}$. Thus a semigroup can be considered to be a $\Gamma$-semigroup.

Let $S$ be a $\Gamma$-semigroup and $x$ be a fixed element of $\Gamma$. We define $a . b=a x b$ for all $a, b \in S$. We can show that $(S,$.$) is a semigroup and we$ denote this semigroup by $S_{x}$.

Definition 2.2. Let $S$ be a $\Gamma$-semigroup. An element $a \in S$ is said to be regular if $a \in a \Gamma S \Gamma a$ where $a \Gamma S \Gamma a=\{a \alpha b \beta a: b \in S, \alpha, \beta \in \Gamma\} . S$ is said to be regular if every element of $S$ is regular.

We now describe some examples of regular $\Gamma$-semigroup.
In $[7]$ we find the following interesting example of a regular $\Gamma$-semigroup.
Example 2.3. Let $S$ be the set of all $2 \times 3$ matrices and $\Gamma$ be the set of all $3 \times 2$ matrices over a field. Then for all $A, B, C \in S$ and $P, Q \in \Gamma$ we have $A P B \in S$ and since the matrix multiplication is associative, we have $(A P B) Q C=A P(B Q C)$. Hence $S$ is a $\Gamma$-semigroup. Moreover it is regular shown in [7].

Here we give another example of a regular $\Gamma$-semigroup.
Example 2.4. Let $S$ be a set of all negative rational numbers. Obviously $S$ is not a semigroup under usual product of rational numbers. Let $\Gamma=\left\{-\frac{1}{p}: p\right.$ is prime $\}$. Let $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. Now if $a \alpha b$ is equal to the usual product of rational numbers $a, \alpha, b$, then $a \alpha b \in S$ and $(a \alpha b) \beta c=a \alpha(b \beta c)$. Hence $S$ is a $\Gamma$-semigroup. Let $a=$ $\frac{m}{n} \in \Gamma$ where $m>0$ and $n<0 . m=p_{1} p_{2} \ldots p_{k}$ where $p_{i}$ 's are prime. $\frac{p_{1} p_{2} \ldots p_{k}}{n}\left(-\frac{1}{p_{1}}\right) \frac{n}{p_{2} \ldots p_{k-1}}\left(-\frac{1}{p_{k}}\right) \frac{m}{n}=\frac{p_{1} p_{2} \ldots p_{k}}{n}$. Thus taking $b=\frac{n}{p_{2} \ldots p_{k-1}}$, $\alpha=\left(-\frac{1}{p_{1}}\right)$ and $\beta=\left(-\frac{1}{p_{k}}\right)$ we can say that $a$ is regular. Hence $S$ is a regular $\Gamma$-semigroup.

Definition 2.5 [7]. Let $S$ be a $\Gamma$-semigroup and $\alpha \in \Gamma$. Then $e \in S$ is said to be an $\alpha$-idempotent if eae $=e$. The set of all $\alpha$-idempotents is denoted by $E_{\alpha}$. We denote $\bigcup_{\alpha \in \Gamma} E_{\alpha}$ by $E(S)$. The elements of $E(S)$ are called idempotent elements of $S$.

Definition 2.6 [7]. Let $a \in M$ and $\alpha, \beta \in \Gamma$. An element $b \in M$ is called an $(\alpha, \beta)$-inverse of $a$ if $a=a \alpha b \beta a$ and $b=b \beta a \alpha b$. In this case we write $b \in V_{\alpha}^{\beta}(a)$.

Definition 2.7 [2]. A regular $\Gamma$-semigroup $M$ is called a right (left) orthodox $\Gamma$-semigroup if for any $\alpha$-idempotent $e$ and $\beta$-idempotent $f$, e $\alpha f$ (resp. $f \alpha e)$ is a $\beta$-idempotent.

Example 2.8 [2]. Let $A=\{1,2,3\}$ and $B=\{4,5\}$. $S$ denotes the set of all mappings from $A$ to $B$. Here members of $S$ are described by the images of the elements $1,2,3$. For example the map $1 \rightarrow 4,2 \rightarrow 5,3 \rightarrow 4$ is written as $(4,5,4)$ and $(5,5,4)$ denotes the map $1 \rightarrow 5,2 \rightarrow 5,3 \rightarrow 4$. A map from $B$ to $A$
is described in the same fashion. For example $(1,2)$ denotes $4 \rightarrow 1,5 \rightarrow 2$. Now $S=\{(4,4,4),(4,4,5),(4,5,4),(4,5,5),(5,5,5),(5,4,5),(5,4,4),(5,5,4)\}$ and let $\Gamma=\{(1,1),(1,2),(2,3),(3,1)\}$. Let $f, g \in S$ and $\alpha \in \Gamma$. We define $f \alpha g$ by $(f \alpha g)(a)=f \alpha(g(a))$ for all $a \in A$. So $f \alpha g$ is a mapping from $A$ to $B$ and hence $f \alpha g \in S$ and we can show that $(f \alpha g) \beta h=f \alpha(g \beta h)$ for all $f, g, h \in S$ and $\alpha, \beta \in \Gamma$. We can show that each element $x$ of $S$ is an $\alpha$-idempotent for some $\alpha \in \Gamma$ and hence each element is regular. Thus $S$ is a regular $\Gamma$-semigroup. It is an idempotent $\Gamma$-semigroup. Moreover we can show that it is a right orthodox $\Gamma$-semigroup.

Theorem 2.9 [2]. A regular $\Gamma$-semigroup $M$ is a right orthodox $\Gamma$-semigroup if and only if for $a, b \in M, V_{\alpha_{1}}^{\beta}(a) \cap V_{\alpha}^{\beta}(b) \neq \phi$ for some $\alpha, \alpha_{1}, \beta \in \Gamma$ implies that $V_{\alpha_{1}}^{\delta}(a)=V_{\alpha}^{\delta}(b)$ for all $\delta \in \Gamma$.
Definition 2.10 [1]. A regular $\Gamma$-semigroup is called a right (left) inverse $\Gamma$ semigroup if for any $\alpha$-idempotent $e$ and for any $\beta$-idempotent $f, e \alpha f \beta e=$ $f \beta e(e \beta f \alpha e=e \beta f)$.
Theorem $2.11[7]$. Let $S$ be a $\Gamma$-semigroup. If $S_{\alpha}$ is a group for some $\alpha \in S$ then $S_{\alpha}$ is a group for all $\alpha \in \Gamma$.

Definition 2.12 [7]. A $\Gamma$-semigroup $S$ is called a $\Gamma$-group if $S_{\alpha}$ is a group for some $\alpha \in \Gamma$.

Definition 2.13 [8]. A regular semigroup $S$ is said to be a right (left) inverse semigroup if for any $e, f \in E(S)$, efe $=f e(e f e=e f)$.
Definition 2.14 [3]. A semigroup $S$ is called orthodox semigroup if it is regular and the set of all idempotents forms a subsemigroup.

Definition 2.15 [7]. A nonempty subset $I$ of a $\Gamma$-semigroup $S$ is called a right (resp. left) ideal if $I \Gamma S \subseteq I$ (resp. $S \Gamma I \subseteq I$ ). If $I$ is both a right ideal and a left ideal then we say that $I$ is an ideal of $S$.
Definition 2.16 [7]. A $\Gamma$-semigroup $S$ is called right (resp. left) simple if it contains no proper right (resp. left) ideal i.e, for every $a \in S, a \Gamma S=$ $S($ resp. $S \Gamma a=S)$. A $\Gamma$ - semigroup is said to be simple if it has no proper ideals.

Theorem 2.17 [7]. Let $S$ be a $\Gamma$ - semigroup. $S$ is a $\Gamma$ - group if and only if it is both left simple and right simple.

## 3. SEmidirect product of a SEmigroup and a $\Gamma$-SEMIGROUP

Let $S$ be a semigroup and $T$ be a $\Gamma$-semigroup. Let $\operatorname{End}(T)$ denote the set of all endomorphisms on $T$ i.e., the set of all mappings $f: T \rightarrow T$ satisfying $f(a \alpha b)=f(a) \alpha f(b)$ for all $a, b \in T, \alpha \in \Gamma$. Clearly $\operatorname{End}(T)$ is a semigroup. Let $\phi: S \nrightarrow \operatorname{End}(T)$ be a given antimorphism i.e, $\phi(s r)=$ $\phi(r) \phi(s)$ for all $r, s \in S$. If $s \in S$ and $t \in T$, we write $t^{s}$ for $(\phi(s))(t)$ and $T^{s}=\left\{t^{s}: t \in T\right\}$. Let $S \times_{\phi} T=\{(s, t): s \in S, t \in T\}$. We define $\left(s_{1}, t_{1}\right) \alpha\left(s_{2}, t_{2}\right)=\left(s_{1} s_{2}, t_{1}^{s_{2}} \alpha t_{2}\right)$ for all $\left(s_{i}, t_{i}\right) \in S \times_{\phi} T$ and $\alpha \in \Gamma$. Then $S \times{ }_{\phi} T$ is a $\Gamma$-semigroup. This $\Gamma$-semigroup $S \times_{\phi} T$ is called the semidirect product of the semigroup $S$ and the $\Gamma$-semigroup $T$. In [6] we have studied the semidirect product $S \times{ }_{\phi} T$ assuming that $S$ is a monoid. In this paper we investigate the properties of the semidirect product $S \times{ }_{\phi} T$ without taking 1 in $S$.

Lemma 3.1. Let $S \times_{\phi} T$ be a semidirect product of a semigroup $S$ and $a$ $\Gamma$-semigroup $T$. Then
(i) $(t \alpha u)^{s}=t^{s} \alpha u^{s}$ for all $s \in S, t, u \in T$ and $\alpha \in \Gamma$.
(ii) $\left(t^{s}\right)^{r}=(t)^{s r}$ for all $s, r \in S$ and $t \in T$.

Proof. Let $s, r \in S, \alpha \in \Gamma$ and $t, u \in T$. Now $(t \alpha u)^{s}=(\phi(s))(t \alpha u)=$ $(\phi(s))(t) \alpha(\phi(s))(u)=t^{s} \alpha u^{s}$ Hence (i) follows. Again $\left(t^{s}\right)^{r}=(\phi(r))\left(t^{s}\right)=$ $(\phi(r))((\phi(s))(t))=(\phi(r) \phi(s))(t)=(\phi(s r))(t)=(t)^{s r}$. Thus (ii) follows.

Theorem 3.2. Let $S \times_{\phi} T$ be a semidirect product of a semigroup $S$ and a $\Gamma$-semigroup $T$. Then $T^{x}$ is a $\Gamma$-semigroup for all $x \in S$ where $T^{x}=\left\{t^{x}\right.$ : $t \in T\}$. If moreover $S \times_{\phi} T$ is a regular $\Gamma$ - semigroup then $S$ is a regular semigroup and $T^{e}$ is a regular $\Gamma$-semigroup for all $e \in E(S)$.

Proof. The first part is clear from the above lemma. Let $S \times_{\phi} T$ be regular. For $(s, t) \in S \times_{\phi} T$, there exist $\left(s^{\prime}, t^{\prime}\right) \in S \times_{\phi} T$ and $\alpha, \beta \in \Gamma$ such that $(s, t)=(s, t) \alpha\left(s^{\prime}, t^{\prime}\right) \beta(s, t)=\left(s s^{\prime} s, t^{s^{\prime} s} \alpha\left(t^{\prime}\right)^{s} \beta t\right)$ and $\left(s^{\prime}, t^{\prime}\right)=$ $\left(s^{\prime}, t^{\prime}\right) \beta(s, t) \alpha\left(s^{\prime}, t^{\prime}\right)=\left(s^{\prime} s s^{\prime},\left(t^{\prime}\right)^{s s^{\prime}} \beta t^{s^{\prime}} \alpha t^{\prime}\right)$. This implies $s^{\prime} \in V(s)$. Let $e \in E(S)$, then for $\left(e, t^{e}\right)$, there exist $\left(s^{\prime}, t^{\prime}\right) \in S \times_{\phi} T$ and $\alpha, \beta \in \Gamma$ such that $\left(e, t^{e}\right)=\left(e, t^{e}\right) \alpha\left(s^{\prime}, t^{\prime}\right) \beta\left(e, t^{e}\right)=\left(e s^{\prime} e, t^{e s^{\prime} e} \alpha t^{\prime e} \beta t^{e}\right)$ and $\left(s^{\prime}, t^{\prime}\right)=$ $\left(s^{\prime}, t^{\prime}\right) \beta\left(e, t^{e}\right) \alpha\left(s^{\prime}, t^{\prime}\right)=\left(s^{\prime} e s^{\prime},\left(t^{\prime}\right)^{e s^{\prime}} \beta t^{e s^{\prime}} \alpha t^{\prime}\right)$. Hence $s^{\prime} \in V(e)$ and we have $t^{e}=t^{e} \alpha t^{\prime e} \beta t^{e}$ and $t^{\prime e}=t^{\prime e} \beta t^{e} \alpha t^{\prime e}$. i.e, $t^{\prime e} \in V_{\alpha}^{\beta}\left(t^{e}\right)$. Hence $T^{e}$ is a regular $\Gamma$-semigroup.

Theorem 3.3. Let $S$ be a semigroup and $T$ be a $\Gamma$-semigroup, $\phi: S \nrightarrow$ $\operatorname{End}(T)$ be a given antimorphism. If the semidirect product $S \times{ }_{\phi} T$ is
(i) a right (left) orthodox $\Gamma$-semigroup then $S$ is an orthodox semigroup and $T^{e}$ is a right (left) orthodox $\Gamma$-semigroup for every idempotent $e \in S$,
(ii) a right (left) inverse $\Gamma$-semigroup then $S$ is a right (left) inverse semigroup and $T^{e}$ is a right (left) inverse $\Gamma$-semigroup.

## Proof.

(i) Let $S \times_{\phi} T$ be a right orthodox $\Gamma$-semigroup. Let $e, g \in E(S)$ and $t^{e}$ be an $\alpha$-idempotent and $u^{e}$ be a $\beta$-idempotent in $T^{e}$. Then $\left(e, t^{e}\right) \alpha\left(e, t^{e}\right)=$ $\left(e, t^{e} \alpha t^{e}\right)=\left(e, t^{e}\right)$, i.e., $\left(e, t^{e}\right)$ is an $\alpha$-idempotent. Similarly $\left(e, u^{e}\right)$ is a $\beta$-idempotent. Again $\left(g, u^{e g}\right) \beta\left(g, u^{e g}\right)=\left(g, u^{e g} \beta u^{e g}\right)=\left(g,\left(u^{e} \beta u^{e}\right)^{g}\right)=$ $\left(g, u^{e g}\right)$. Thus $\left(g, u^{e g}\right)$ is a $\beta$-idempotent of $S \times_{\phi} T$. Now $\left(e,\left(t^{e} \alpha u^{e}\right)\right.$ $\left.\beta\left(t^{e} \alpha u^{e}\right)\right)=\left(e,\left(t^{e} \alpha u^{e}\right)\right) \beta\left(e,\left(t^{e} \alpha u^{e}\right)\right)=\left(\left(e, t^{e}\right) \alpha\left(e, u^{e}\right)\right) \beta\left(\left(e, t^{e}\right) \alpha\left(e, u^{e}\right)\right)$ $=\left(e, t^{e}\right) \alpha\left(e, u^{e}\right)=\left(e, t^{e} \alpha u^{e}\right)$ which shows that $t^{e} \alpha u^{e}$ is a $\beta$-idempotent and hence $T^{e}$ is a right orthodox $\Gamma$-semigroup. Again since $S \times_{\phi} T$ is a right orthodox $\Gamma$-semigroup we have $\left((e g)^{2},\left(t^{e g} \alpha u^{e g}\right)^{e g} \beta t^{e g} \alpha u^{e g}\right)=$ $\left(e g, t^{e g} \alpha u^{e g}\right) \beta\left(e g, t^{e g} \alpha u^{e g}\right)=\left(\left(e, t^{e}\right) \alpha\left(g, u^{e g}\right)\right) \beta\left(\left(e, t^{e}\right) \alpha\left(g, u^{e g}\right)\right)=\left(e, t^{e}\right)$ $\alpha\left(g, u^{e g}\right)=\left(e g, t^{e g} \alpha u^{e g}\right)$. Thus $(e g)^{2}=e g$ which shows that $S$ is orthodox.
(ii) Suppose that $S \times_{\phi} T$ is a right inverse $\Gamma$-semigroup. Let $e, g \in E(S)$ and $t^{e}$ be an $\alpha$-idempotent and $u^{e}$ be a $\beta$-idempotent in $T^{e}$. Then $\left(e, t^{e}\right)$ is an $\alpha$-idempotent, $\left(e, u^{e}\right),\left(g, u^{e g}\right)$ are $\beta$-idempotents of $S \times_{\phi} T$. Now $\left(e, t^{e} \alpha u^{e} \beta t^{e}\right)=\left(e, t^{e}\right) \alpha\left(e, u^{e}\right) \beta\left(e, t^{e}\right)=\left(e, u^{e}\right) \beta\left(e, t^{e}\right)=\left(e, u^{e} \beta t^{e}\right)$ and $\left(e g e, t^{e g e} \alpha u^{e g e} \beta t^{e}\right)=\left(e, t^{e}\right) \alpha\left(g, u^{e g}\right) \beta\left(e, t^{e}\right)=\left(g, u^{e g}\right) \beta\left(e, t^{e}\right)$ $=\left(g e, u^{e g e} \beta t^{e}\right)$. So we have $t^{e} \alpha u^{e} \beta t^{e}=u^{e} \beta t^{e}$ and ege $=g e$. Consequently we have $S$ is a right inverse semigroup and $T^{e}$ is a right inverse $\Gamma$-semigroup.

The proofs of the following two theorems are almost similar to our Lemma 3.3 and Lemma 3.4 proved in [6]. For completeness we give the proof here.

Theorem 3.4. Let $S \times_{\phi} T$ be the semidirect product of a semigroup $S$ and $a \Gamma$-semigroup $T$ corresponding to a given antimorphism $\phi: S \nrightarrow \operatorname{End}(T)$ and let $(s, t) \in S \times_{\phi} T$, then
(i) if $\left(s^{\prime}, t^{\prime}\right) \in V_{\alpha}^{\beta}((s, t))$ then $\left(s^{\prime}, t^{\prime}\right) \in V_{\alpha}^{\beta}\left(\left(s, t^{s^{\prime} s}\right)\right)$. In particular if $s \in$ $E(S)$, then $\left(s,\left(t^{\prime}\right)^{s} \beta t^{s^{\prime s}} \alpha t^{\prime}\right) \in V_{\alpha}^{\beta}\left(\left(s, t^{s^{\prime} s}\right)\right)$ and
(ii) if $t^{s}$ is an $\alpha$-idempotent and $s^{\prime} \in V(s)$, then $\left(s^{\prime}, t^{s s^{\prime}}\right) \in V_{\alpha}^{\alpha}\left(\left(s, t^{s}\right)\right)$.

## Proof.

(i) Since $\left(s^{\prime}, t^{\prime}\right) \in V_{\alpha}^{\beta}((s, t))$ we have,

$$
\left(s^{\prime}, t^{\prime}\right)=\left(s^{\prime}, t^{\prime}\right) \beta(s, t) \alpha\left(s^{\prime}, t^{\prime}\right)=\left(s^{\prime} s s^{\prime},\left(t^{\prime}\right)^{s s^{\prime}} \beta t^{s^{\prime}} \alpha t^{\prime}\right)
$$

and

$$
(s, t)=(s, t) \alpha\left(s^{\prime}, t^{\prime}\right) \beta(s, t)=\left(s s^{\prime} s, t^{s^{\prime} s} \alpha\left(t^{\prime}\right)^{s} \beta t\right)
$$

This shows that

$$
\begin{equation*}
s^{\prime} \in V(s) \text { and } t^{s^{\prime} s} \alpha\left(t^{\prime}\right)^{s} \beta t=t \tag{1}
\end{equation*}
$$

$$
\begin{equation*}
\left(t^{\prime}\right)^{s s^{\prime}} \beta t^{s^{\prime}} \alpha t^{\prime}=t^{\prime} \tag{2}
\end{equation*}
$$

From (1) we have, $\left(t^{s^{\prime} s} \alpha\left(t^{\prime}\right)^{s} \beta t\right)^{s^{\prime} s}=(t)^{s^{\prime} s}$ i.e., $t^{s^{\prime s}} \alpha\left(t^{\prime}\right)^{s} \beta t^{s^{\prime} s}=t^{s^{\prime} s}$ and from (2), $\left(\left(t^{\prime}\right)^{s s^{\prime}} \beta t^{s^{\prime}} \alpha t^{\prime}\right)^{s}=\left(t^{\prime}\right)^{s}$ i.e., $\left(t^{\prime}\right)^{s} \beta t^{s^{\prime} s} \alpha\left(t^{\prime}\right)^{s}=\left(t^{\prime}\right)^{s}$. Now $\left(s^{\prime}, t^{\prime}\right) \beta\left(s, t^{s^{\prime} s}\right) \alpha\left(s^{\prime}, t^{\prime}\right)=\left(s^{\prime} s s^{\prime},\left(t^{\prime}\right)^{s s^{\prime}} \beta t^{s^{\prime} s s^{\prime}} \alpha t^{\prime}\right)=\left(s^{\prime}, t^{\prime}\right)$ by (2) and $\left(s, t^{s^{\prime} s}\right) \alpha\left(s^{\prime}, t^{\prime}\right) \beta\left(s, t^{s^{\prime} s}\right)=\left(s s^{\prime} s, t^{s^{\prime} s s^{\prime} s} \alpha\left(t^{\prime}\right)^{s} \beta t^{s^{\prime} s}\right)=\left(s, t^{s^{\prime} s} \alpha\left(t^{\prime}\right)^{s} \beta t^{s^{\prime} s}\right)=$ $\left(s, t^{s^{\prime} s}\right)$. Thus we have $\left(s^{\prime}, t^{\prime}\right) \in V_{\alpha}^{\beta}\left(\left(s, t^{s^{\prime} s}\right)\right)$. Again if $s \in E(S)$, $\left(\left(t^{\prime}\right)^{s} \beta t^{s^{\prime} s} \alpha t^{\prime}\right)^{s}=\left(t^{\prime}\right)^{s} \beta t^{s^{\prime} s} \alpha\left(t^{\prime}\right)^{s}=\left(t^{\prime}\right)^{s}$ and $\left(s, t^{s^{\prime} s}\right) \alpha\left(s,\left(t^{\prime}\right)^{s} \beta t^{s^{\prime} s} \alpha t^{\prime}\right)$ $\beta\left(s, t^{s^{\prime} s}\right)=\left(s s s, t^{s^{\prime} s} \alpha\left(\left(t^{\prime}\right)^{s} \beta t^{s^{\prime} s} \alpha t^{\prime}\right)^{s} \beta t^{s^{\prime} s}\right)=\left(s, t^{s^{\prime} s} \alpha\left(t^{\prime}\right)^{s} \beta t^{s^{\prime} s}\right)=\left(s, t^{s^{\prime s} s}\right)$ and $\left(s,\left(t^{\prime}\right)^{s} \beta t^{s^{\prime} s} \alpha t^{\prime}\right) \beta \quad\left(s, t^{s^{\prime} s}\right) \alpha\left(s,\left(t^{\prime}\right)^{s} \beta t^{s^{\prime} s} \alpha t^{\prime}\right)=\left(s,\left(\left(t^{\prime}\right)^{s} \beta t^{s^{\prime} s} \alpha t^{\prime}\right)^{s}\right.$ $\left.\beta t^{s^{\prime} s s} \alpha\left(t^{\prime}\right)^{s} \beta t^{s^{\prime} s} \alpha t^{\prime}\right)=\left(s,\left(t^{\prime}\right)^{s} \beta t^{s^{\prime} s} \alpha\left(t^{\prime}\right)^{s} \beta t^{s^{\prime} s} \alpha t^{\prime}\right)=\left(s,\left(t^{\prime}\right)^{s} \beta t^{s^{\prime} s} \alpha t^{\prime}\right)$. Hence $\left(s,\left(t^{\prime}\right)^{s} \beta t^{s^{\prime} s} \alpha t^{\prime}\right) \in V_{\alpha}^{\beta}\left(s, t^{s^{\prime} s}\right)$.
(ii) $\left(s, t^{s}\right) \alpha\left(s^{\prime}, t^{s s^{\prime}}\right) \alpha\left(s, t^{s}\right)=\left(s s^{\prime} s, t^{s s^{\prime} s} \alpha t^{s s^{\prime} s} \alpha t^{s}\right)=\left(s, t^{s}\right)$ since $t^{s}$ is an $\alpha$-idempotent and $\left(s^{\prime}, t^{s s^{\prime}}\right) \alpha\left(s, t^{s}\right) \alpha\left(s^{\prime}, t^{s s^{\prime}}\right)=\left(s^{\prime} s s^{\prime}, t^{s s^{\prime} s s^{\prime}} \alpha t^{s s^{\prime}} \alpha t^{s s^{\prime}}\right)$ $=\left(s^{\prime}, t^{s s^{\prime}} \alpha t^{s s^{\prime}} \alpha t^{s s^{\prime}}\right)=\left(s^{\prime},\left(t^{s} \alpha t^{s} \alpha t^{s}\right)^{s^{\prime}}\right)=\left(s^{\prime}, t^{s s^{\prime}}\right)$ i.e., $\left(s^{\prime}, t^{s s^{\prime}}\right) \in V_{\alpha}^{\alpha}\left(s, t^{s}\right)$.

Theorem 3.5. Let $S$ be a semigroup and $T$ be a $\Gamma$-semigroup and $S \times{ }_{\phi} T$ be the semidirect product corresponding to a given antimorphism $\phi: S \nrightarrow$ $\operatorname{End}(T)$. Moreover, if $t \in t^{e} \Gamma T$ for every $e \in E(S)$ and every $t \in T$, then
(i) $(e, t)$ is an $\alpha$-idempotent if and only if $e \in E(S)$ and $t^{e}$ is an $\alpha$ idempotent and
(ii) if $(e, t)$ is an $\alpha$-idempotent, then $\left(e, t^{e}\right) \in V_{\alpha}^{\alpha}((e, t))$.

## Proof.

(i) If $(e, t)$ is an $\alpha$-idempotent then

$$
\begin{equation*}
(e, t)=(e, t) \alpha(e, t)=\left(e^{2}, t^{e} \alpha t\right) \text { i.e., } e=e^{2} \text { and } t^{e} \alpha t=t . \tag{3}
\end{equation*}
$$

So, $t^{e}=\left(t^{e} \alpha t\right)^{e}=t^{e} \alpha t^{e}$ which implies that $t^{e}$ is an $\alpha$-idempotent. Conversely, let $e \in E(S)$ and $t^{e}$ be an $\alpha$-idempotent. Since $t \in t^{e} \Gamma T, t=$ $t^{e} \beta t_{1}$ for some $\beta \in \Gamma, t_{1} \in T$ and hence $t^{e} \alpha t=t^{e} \alpha t^{e} \beta t_{1}=t$. Thus $(e, t) \alpha(e, t)=\left(e, t^{e} \alpha t\right)=(e, t)$ i.e., $(e, t)$ is an $\alpha$-idempotent.
(ii) If $(e, t)$ is an $\alpha$-idempotent, from (i) $e \in E(S)$ and $t^{e}$ is an $\alpha$-idempotent. Now $(e, t) \alpha\left(e, t^{e}\right) \alpha(e, t)=\left(e, t^{e} \alpha t^{e} \alpha t\right)=\left(e, t^{e} \alpha t\right)=(e, t)$ from (3) and $\left(e, t^{e}\right) \alpha(e, t) \alpha\left(e, t^{e}\right)=\left(e, t^{e} \alpha t^{e} \alpha t^{e}\right)=\left(e, t^{e}\right)$. Thus $\left(e, t^{e}\right) \in V_{\alpha}^{\alpha}((e, t))$.

Theorem 3.6. Let $S$ be a semigroup and $T$ be a $\Gamma$-semigroup. Let $\phi: S \nrightarrow$ $\operatorname{End}(T)$ be a given antimorphism. Then the semidirect product $S \times_{\phi} T$ is a right (left) orthodox $\Gamma$-semigroup if and only if
(i) $S$ is an orthodox semigroup and $T^{e}$ is a right (left) orthodox $\Gamma$-semigroup for every $e \in E(S)$,
(ii) for every $e \in E(S)$ and every $t \in T, t \in t^{e} \Gamma T$ and
(iii) for every $\alpha$-idempotent $t^{e}$, $t^{g e}$ is an $\alpha$-idempotent, where $e, g \in E(S)$, $t \in T$.

Proof. Suppose $S \times{ }_{\phi} T$ is a right orthodox $\Gamma$-semigroup. Then by Theorem 3.3 $S$ is an orthodox semigroup and $T^{e}$ is a right orthodox $\Gamma$-semigroup for every $e \in E(S)$. For (ii), let $(e, t) \in S \times{ }_{\phi} T$ with $e \in E(S)$ and let $\left(e^{\prime}, t^{\prime}\right) \in V_{\alpha}^{\beta}((e, t))$ for some $\alpha, \beta \in \Gamma$. Then by Theorem 3.4 $\left(e^{\prime}, t^{\prime}\right),\left(e^{\prime},\left(t^{\prime}\right)^{e} \beta t^{e^{\prime} e} \alpha t^{\prime}\right) \in V_{\alpha}^{\beta}\left(\left(e, t^{e^{\prime} e}\right)\right)$. Thus $V_{\alpha}^{\beta}((e, t)) \cap V_{\alpha}^{\beta}\left(\left(e, t^{t^{\prime} e}\right)\right) \neq \phi$ and hence by Theorem 2.9, $V_{\alpha}^{\beta}((e, t))=V_{\alpha}^{\beta}\left(\left(e, t^{e^{\prime} e}\right)\right)$. So $\left(e,\left(t^{\prime}\right)^{e} \beta t^{e^{\prime} e} \alpha t^{\prime}\right) \in$ $V_{\alpha}^{\beta}((e, t))$. Thus $(e, t)=(e, t) \alpha\left(e,\left(t^{\prime}\right)^{e} \beta t^{e^{\prime} e} \alpha t^{\prime}\right) \beta(e, t)=\left(e, t^{e} \alpha\left(t^{\prime}\right)^{e} \beta t^{e^{\prime} e}\right.$ $\left.\alpha\left(t^{\prime}\right)^{e} \beta t\right)$ and hence $t=t^{e} \alpha\left(t^{\prime}\right)^{e} \beta t^{e^{\prime} e} \alpha\left(t^{\prime}\right)^{e} \beta t \in t^{e} \Gamma T$.

For (iii) we shall first show that for an $\alpha$-idempotent $t^{e}$ of $T$ if $e \in$ $E(S), t^{e^{\prime}}$ is an $\alpha$-idempotent for any $e^{\prime} \in V(e)$. If $e \in E(S)$ and $t^{e}$ is an $\alpha$-idempotent, then by Theorem 3.5, $(e, t)$ is an $\alpha$-idempotent in $S \times{ }_{\phi} T$ and $\left(e, t^{e}\right) \in V_{\alpha}^{\alpha}((e, t))$. Again since $t^{e}$ is an $\alpha$-idempotent $\left(e, t^{e}\right)$ is also an $\alpha$ idempotent and thus $\left(e, t^{e}\right) \in V_{\alpha}^{\alpha}\left(\left(e, t^{e}\right)\right)$ i.e., $V_{\alpha}^{\alpha}\left(\left(e, t^{e}\right)\right) \cap V_{\alpha}^{\alpha}((e, t)) \neq \phi$ and so $V_{\alpha}^{\alpha}\left(\left(e, t^{e}\right)\right)=V_{\alpha}^{\alpha}((e, t))$ and by Theorem $3.5\left(e^{\prime}, t^{e e^{\prime}}\right) \in V_{\alpha}^{\alpha}\left(\left(e, t^{e}\right)\right)$ i.e.,
$\left(e^{\prime}, t^{e e^{\prime}}\right) \in V_{\alpha}^{\alpha}((e, t))$. Thus $(e, t)=(e, t) \alpha\left(e^{\prime}, t^{e e^{\prime}}\right) \alpha(e, t)=\left(e e^{\prime} e, t^{e^{\prime} e} \alpha t^{e e^{\prime} e} \alpha t\right)$ $=\left(e, t^{e^{\prime} e} \alpha t^{e} \alpha t\right)=\left(e, t^{e^{\prime} e} \alpha t\right)$ [since $t=t^{e} \beta u$ for some $\left.\beta \in \Gamma, u \in T, t^{e} \alpha t=t\right]$. So $t=t^{e^{\prime} e} \alpha t$ and hence $t^{e^{\prime}}=\left(t^{e^{\prime} e} \alpha t\right)^{e^{\prime}}=t^{e^{\prime}} \alpha t^{e^{\prime}}$. Thus $t^{e^{\prime}}$ is an $\alpha$-idempotent. Let $e, g \in E(S)$ and suppose that $t^{e}$ is an $\alpha$-idempotent for $t \in T$, then $t^{e g} \alpha t^{e g}=\left(t^{e} \alpha t^{e}\right)^{g}=t^{e g}$ i.e, $t^{e g}$ is an $\alpha$-idempotent and we have $e g \in E(S)$ and $g e \in V(e g)$ since $S$ is orthodox. Then by the above fact $t^{g e}$ is an $\alpha$-idempotent.

We now prove the converse part. Suppose $S$ and $T$ satisfy (i), (ii) and (iii). Let $(s, t) \in S \times_{\phi} T$. Since $S$ is regular, there exists $s^{\prime} \in S$ such that $s=s s^{\prime} s$ and $s^{\prime}=s^{\prime} s s^{\prime}$. We take $e=s^{\prime} s$, then $e \in E(S)$. By (ii) $t \in t^{e} \Gamma T$ which implies $t=t^{e} \beta u$ for some $\beta \in \Gamma, u \in T$. Let $t^{\prime}=v^{s^{\prime}}$ where $v^{e} \in$ $V_{\gamma}^{\delta}\left(t^{e}\right)$ where $\gamma, \delta \in \Gamma$. Now $t^{s^{\prime} s} \gamma\left(t^{\prime}\right)^{s} \delta t=t^{s^{\prime} s} \gamma v^{s^{\prime} s} \delta t^{e} \beta u=(t \gamma v \delta t)^{e} \beta u=$ $\left(t^{e} \gamma v^{e} \delta t^{e}\right) \beta u=t^{e} \beta u=t$ i.e, $(s, t)=\left(s s^{\prime} s, t^{s^{\prime} s} \gamma\left(t^{\prime}\right)^{s} \delta t\right)=(s, t) \gamma\left(s^{\prime}, t^{\prime}\right) \delta(s, t)$. Again $\left(t^{\prime}\right)^{s s^{\prime}} \delta t^{s^{\prime}} \gamma t^{\prime}=\left(v^{s^{\prime}}\right)^{s s^{\prime}} \delta t^{s^{\prime}} \gamma v^{s^{\prime}}=v^{s^{\prime}} \delta t^{s^{\prime}} \gamma v^{s^{\prime}}=v^{s^{\prime} s s^{\prime}} \delta t^{s^{\prime} s s^{\prime}} \gamma v^{s^{\prime} s s^{\prime}}=$ $\left(v^{e} \delta t^{e} \gamma v^{e}\right)^{s^{\prime}}=v^{e s^{\prime}}=v^{s^{\prime} s s^{\prime}}=v^{s^{\prime}}=t^{\prime}$ i.e., $\left(s^{\prime}, t^{\prime}\right)=\left(s^{\prime} s s^{\prime},\left(t^{\prime}\right)^{s s^{\prime}} \delta t^{s^{\prime}} \gamma t^{\prime}\right)=$ $\left(s^{\prime}, t^{\prime}\right) \delta(s, t) \gamma\left(s^{\prime}, t^{\prime}\right)$. Thus we have $\left(s^{\prime}, t^{\prime}\right) \in V_{\gamma}^{\delta}(s, t)$ which yields $S \times_{\phi} T$ is a regular $\Gamma$-semigroup.

Now let $(e, t)$ be an $\alpha$-idempotent and $(g, u)$ be a $\beta$-idempotent. Then by Theorem $3.5 e, g \in E(S), t^{e}$ is an $\alpha$-idempotent and $u^{g}$ is a $\beta$-idempotent. By (iii) $t^{g e}$ is an $\alpha$-idempotent, $u^{e g}$ is a $\beta$-idempotent and $t^{g e g} \alpha t^{g e g}=$ $\left(t^{g e} \alpha t^{g e}\right)^{g}=t^{g e g}$ i.e., $t^{g e g}$ is an $\alpha$-idempotent. By our assumption $e, g \in$ $E(S)$ and $\left(t^{g} \alpha u\right)^{e g}=t^{g e g} \alpha u^{e g}$ is a $\beta$-idempotent. Thus by Theorem 3.5 $(e, t) \alpha(g, u)=\left(e g, t^{g} \alpha u\right)$ is a $\beta$-idempotent which shows that $S \times_{\phi} T$ is a right orthodox $\Gamma$-semigroup.

Theorem 3.7. Let $S$ be a semigroup, $T$ be a $\Gamma$-semigroup and $\phi: S \nrightarrow$ $\operatorname{End}(T)$ be a given antimorphism. Then the semidirect product $S \times_{\phi} T$ is a right inverse $\Gamma$-semigroup if and only if
(i) Sis a right inverse semigroup and $T^{e}$ is a right inverse $\Gamma$-semigroup for every $e \in E(S)$ and
(ii) for every $e \in E(S)$ and every $t \in T, t \in t^{e} \Gamma T$.

Proof. Let $S \times{ }_{\phi} T$ be a right inverse $\Gamma$-semigroup. Then by Theorem $3.3 S$ is a right inverse semigroup and $T^{e}$ is a right inverse $\Gamma$-semigroup for every $e \in E(S)$. Again since every right inverse $\Gamma$-semigroup is a right orthodox $\Gamma$-semigroup from the above theorem, condition (ii) holds.

Conversely, suppose that $S$ and $T$ satisfy (i) and (ii). Then by Theorem $3.2 S \times{ }_{\phi} T$ is a regular $\Gamma$-semigroup . Let $(e, t)$ be an $\alpha$-idempotent and $(g, u)$
be a $\beta$-idempotent in $S \times{ }_{\phi} T$. Then by Theorem $3.5 e, g \in E(S)$, $t^{e}$ is an $\alpha$ idempotent, $u^{g}$ is a $\beta$-idempotent. From (ii) $t=t^{e} \gamma v$ for some $\gamma \in \Gamma, v \in T$ and thus $t^{e} \alpha t=t$ and similarly $u^{g} \beta u=u$. So $u^{g e}=\left(u^{g} \beta u\right)^{g e}=u^{g e} \beta u^{g e}$ and $t^{g e}=\left(t^{e} \alpha t\right)^{g e}=t^{e g e} \alpha t^{g e}=t^{g e} \alpha t^{g e}$ since $S$ is a right inverse semigroup. Now by (ii) we have $u^{e} \beta t=\left(u^{e} \beta t\right)^{g e} \delta v_{1}$ for some $\delta \in \Gamma, v_{1} \in T$ and hence $u^{e} \beta t=$ $u^{e g e} \beta t^{g e} \delta v_{1}=u^{g e} \beta t^{g e} \delta v_{1}$. Thus we have (e,t) $\alpha(g, u) \beta(e, t)=\left(e g e, t^{g e} \alpha u^{e} \beta t\right)$ $=\left(g e, t^{g e} \alpha u^{g e} \beta t^{g e} \delta v_{1}\right)=\left(g e, u^{g e} \beta t^{g e} \delta v_{1}\right)=\left(g e, u^{e} \beta t\right)=(g, u) \beta(e, t)$ which implies $S \times_{\phi} T$ is a right inverse $\Gamma$-semigroup.

Theorem 3.8. Let $S$ be a semigroup, $T$ be a $\Gamma$-semigroup and $\phi: S \nrightarrow$ $\operatorname{End}(T)$ be a given antimorphism. Then the semidirect product $S \times_{\phi} T$ is a left inverse $\Gamma$-semigroup if and only if
(i) $S$ is a left inverse semigroup and $T^{e}$ is a left inverse $\Gamma$-semigroup for every $e \in E(S)$ and
(ii) for every $e \in E(S)$ and every $t \in T, t=t^{e}$.

Proof. Let $S \times{ }_{\phi} T$ be a left inverse $\Gamma$-semigroup. Then by Theorem $3.3 S$ is a left inverse semigroup and $T^{e}$ is a left inverse $\Gamma$-semigroup. For (ii) let ( $e, u$ ) be an $\alpha$-idempotent in $S \times_{\phi} T$. Then $(e, u)=(e, u) \alpha(e, u)=\left(e, u^{e} \alpha u\right)$ i.e., $u^{e} \alpha u=u$. Again $\left(e, u^{e}\right) \alpha\left(e, u^{e}\right)=\left(e, u^{e e} \alpha u^{e}\right)=\left(e, u^{e}\right)$ which yields $\left(e, u^{e}\right)$ is an $\alpha$-idempotent and we have $\left(e, u^{e}\right) \alpha(e, u)=\left(e, u^{e} \alpha u\right)=(e, u)$. Since $S \times_{\phi}$ $T$ is a left inverse $\Gamma$-semigroup, $(e, u)=\left(e, u^{e}\right) \alpha(e, u)=\left(e, u^{e}\right) \alpha(e, u) \alpha\left(e, u^{e}\right)$ $=\left(e, u^{e e e} \alpha u^{e e} \alpha u^{e}\right)=\left(e,\left(u^{e} \alpha u\right)^{e e} \alpha u^{e}\right)=\left(e, u^{e e} \alpha u^{e}\right)=\left(e, u^{e}\right)$ i.e., $u=u^{e}$. Thus if $(e, u)$ is an $\alpha$-idempotent then $u=u^{e}$. Now $(e, t) \in S \times_{\phi} T$ with $e \in$ $E(S)$ and let $\left(e^{\prime}, t^{\prime}\right) \in V_{\gamma}^{\delta}((e, t))$ for some $\gamma, \delta \in \Gamma$. Then we get $e^{\prime} \in V(e)$, $t^{e^{\prime} e} \gamma\left(t^{\prime}\right)^{e} \delta t=t$ i.e., $\left.t^{e^{\prime} e} \gamma\left(t^{\prime}\right)\right)^{e e^{\prime} e} \delta t^{e^{\prime} e}=t^{e^{\prime} e}$ which implies $t^{e^{e} e} \gamma\left(t^{\prime}\right)^{e} \delta t^{e^{e} e}=t^{t^{\prime} e}$. Since $\left(e^{\prime} e,\left(t^{\prime}\right)^{e} \delta t\right)=\left(e^{\prime}, t^{\prime}\right) \delta(e, t)$ and $S \times_{\phi} T$ is left orthodox (since it is left inverse), $\left(e^{\prime} e,\left(t^{\prime}\right)^{e} \delta t\right)$ is a $\gamma$-idempotent and hence $\left(t^{\prime}\right)^{e} \delta t=\left(\left(t^{\prime}\right)^{e} \delta t\right)^{e^{\prime} e}$ $=\left(t^{\prime}\right)^{e} \delta t^{e^{\prime} e}$. Thus $t^{e^{\prime} e}=t^{e^{e} e} \gamma\left(t^{\prime}\right)^{e} \delta t^{e^{\prime} e}=t^{e^{\prime} e} \gamma\left(t^{\prime}\right)^{e} \delta t=t$ and hence $t^{e}=$ $\left(t^{e^{\prime} e}\right)^{e}=t^{e^{\prime} e}=t$.

Conversely suppose that $S$ and $T$ satisfy (i) and (ii). Let $(s, t) \in S \times{ }_{\phi} T$. Let $e \in E(S)$. Since $S$ is regular there exists $s^{\prime} \in S$ such that $s^{\prime} \in V(s)$. From (ii) we have $t=t^{e}$. Since $T^{e}$ is regular there exists $v \in T$ such that $v^{e} \in$ $V_{\gamma}^{\delta}\left(t^{e}\right)$. We now take $t^{\prime}=v^{s^{\prime}}$. Now $t^{s s^{\prime}} \gamma\left(t^{\prime}\right)^{s} \delta t=t^{s^{\prime} s} \gamma v^{s^{\prime} s} \delta t^{e}=t^{e} \gamma v^{e} \delta t^{e}=$ $t^{e}=t$ and $\left(t^{\prime}\right)^{s^{\prime}} \delta \delta t^{s^{\prime}} \gamma t^{\prime}=\left(v^{s^{\prime}}\right) s^{s s^{\prime}} \delta t^{s^{\prime}} \gamma v^{s^{\prime}}=v^{s^{\prime}} \delta t^{s^{\prime}} \gamma v^{s^{\prime}}=v^{s^{\prime} s s^{\prime}} \delta t^{s^{\prime} s s^{\prime}} \gamma v^{s^{\prime} s s^{\prime}}=$ $\left(v^{e} \delta t^{e} \gamma v^{e}\right)^{s^{\prime}}=v^{e s^{\prime}}=v^{s^{\prime} s s^{\prime}}=v^{s^{\prime}}=t^{\prime}$. Thus we have $\left(s^{\prime}, t^{\prime}\right) \in V_{\gamma}^{\delta}(s, t)$. Hence $S \times_{\phi} T$ is regular. Now let $(e, t)$ be an $\alpha$-idempotent and $(g, u)$ be a
$\beta$-idempotent. Then $e^{2}=e$ and $t=t^{e} \alpha t=t \alpha t$ [by (ii)] and similarly $g^{2}=g$ and $u \beta u=u$ i.e., $e, g \in E(S)$ and $t$ is an $\alpha$-idempotent, $u$ is a $\beta$-idempotent. Thus we have $(e, t) \beta(g, u) \alpha(e, t)=\left(e g e, t^{g e} \beta u^{e} \alpha t\right)=(e g e, t \beta u \alpha t)$ [by (ii)] $=(e g, t \beta u)=\left(e g, t^{g} \beta u\right)=(e, t) \beta(g, u)$. Thus $S \times_{\phi} T$ is a left inverse $\Gamma$ semigroup.

## 4. Wreath product of a Semigroup and a $\Gamma$-semigroup

In this section we introduce the notion of wreath product of a semigroup $S$ and a $\Gamma$ semigroup $T$. Let $X$ be a nonempty set. Consider the set $T^{X}$ of all mappings from $X$ to $T$. For $f, g \in T^{X}$ and $\alpha \in \Gamma$, define $f \alpha g$ such that $T^{X} \times \Gamma \times T^{X} \rightarrow T^{X}$ by $(f \alpha g)(x)=f(x) \alpha g(x)$.

Before going to establish the relation between $T$ and $T^{X}$ we assume $\Gamma=\{\alpha\}$, a set consisting of single element. Then $(T, \cdot)$ becomes a semigroup where $a \cdot b=a \alpha b$ and $T^{X}$ also becomes a semigroup where $f \cdot g=f \alpha g$. Suppose $T$ is a regular $\Gamma$-semigroup. Then $(T, \cdot)$ is a semigroup. Let $f \in T^{X}$ and let $x \in X$. Now $f(x) \in T$ and $V(f(x)) \neq \phi$. We define $g: X \rightarrow T$ so that $g(x) \in V(f(x))$. Hence for each $x \in X$ we can choose a $g(x)$ such that $f(x) g(x) f(x)=f(x)$. Hence $f g f=f$ which implies that $\left(T^{X}, \cdot\right)$ is a regular semigroup and consequently $T^{X}$ is a regular $\Gamma$-semigroup. In general we cannot extend the process when $\Gamma$ contains more than one element. To explain this we consider the following example.

Example 4.1. Let $T=\{(a, 0): a \in Q\} \cup\{(0, b): b \in Q\}, Q$ denote the set of all rational numbers. Let $\Gamma=\{(0,5),(0,1),(3,0),(1,0)\}$. Defining $T \times \Gamma \times T \rightarrow T$ by $(a, b)(\alpha, \beta)(c, d)=(a \alpha c, b \beta d)$ for all $(a, b),(c, d) \in T$ and $(\alpha, \beta) \in \Gamma$, we can show that $T$ is a $\Gamma$-semigroup. Now let $(a, 0) \in T$. If $a=0$ then $(a, 0)$ is regular. Suppose $a \neq 0$, then $(a, 0)(3,0)\left(\frac{1}{3 a}, 0\right)(1,0)(a, 0)=$ $(a, 0)$. Similarly we can show that $(0, b)$ is also regular. Hence $T$ is a regular $\Gamma$-semigroup. Let us now take a set $X=\{x, y\}$, the set consisting of two elements and let us define a mapping $f: X \rightarrow T$ by $f(x)=(2,0)$ and $f(y)=(0,3)$. We now show that $f$ is not regular in $T^{X}$. If possible let $f$ be regular. Then there exists a mapping $g: X \rightarrow T$ and two elements $\alpha, \beta \in \Gamma$ such that $f \alpha g \beta f=f$. i.e., $f(p) \alpha g(p) \beta f(p)=f(p)$ for all $p \in X$. Now if $p=x$, then $\alpha, \beta \notin\{(0,5),(0,1)\}$, since the first component of $f(x)$ is nonzero but if $p=y$, then $\alpha, \beta \in\{(0,5),(0,1)\}$, since the second component of $f(y)$ is nonzero. Thus a contradiction arises. Hence $T^{X}$ is not a regular $\Gamma$-semigroup.

Before further discussion about the relation between $T$ and $T^{X}$ we now give the following definition.

Definition 4.2. Let $S$ be a $\Gamma$-semigroup. An element $e \in S$ is said to be a left(resp. right) $\gamma$-unity for some $\gamma \in \Gamma$ if $e \gamma a=a($ resp. $a \gamma e=a$ ) for all $a \in S$.

We now consider the following examples.
Example 4.3. Consider the $\Gamma$-semigroup $S$ of Example 2.3. In this $\Gamma$ semigroup $\left(\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0\end{array}\right)$ is a left $\alpha$-unity but not a right $\alpha$-unity of $S$ for $\alpha=\left(\begin{array}{ll}1 & 0 \\ 0 & 1 \\ 0 & 0\end{array}\right)$.

Example 4.4. Let $S$ be the set of all integers of the form $4 \mathrm{n}+1$ and $\Gamma$ be the set of all integers of the form $4 \mathrm{n}+3$ where n is an integer. If $a \alpha b$ is $a+\alpha+b$ for all $a, b \in S$ and $\alpha \in \Gamma$ then $S$ is a $\Gamma$-semigroup. Here 1 is a left $(-1)$ - unity and also right ( -1 )- unity.

Example 4.5. Let us consider $N$, the set of all natural numbers. Let $S$ be the set of all mappings from $N$ to $N \times N$ and $\Gamma$ be the set of all mappings from $N \times N$ to $N$. Then the usual mapping product of two elements of $S$ cannot be defined. But if we take $f, g$ from $S$ and $\alpha$ from $\Gamma$ the usual mapping product $f \alpha g$ can be defined. Also, we find that $f \alpha g \in S$ and $(f \alpha g) \beta h=f \alpha(g \beta h)$. Hence $S$ is a $\Gamma$-semigroup. Now we know that the set $N \times N$ is countable. Hence there exists a bijective mapping $f \in S$. Since $f$ is bijective, there exists $\alpha: N \times N \longrightarrow N$ such that $f \alpha$ is the identity mapping on $N \times N$ and $\alpha f$ is the identity mapping on $N$. Then $f \alpha g=g \alpha f=g$ for all $g \in S$. Hence $f$ is both left $\alpha$-unity and right $\alpha$-unity of $S$.

Let $S$ be a $\Gamma$-semigroup and $e$ be a left $\alpha$-unity. Then $S \Gamma e$ is a left ideal such that $e=e \alpha e \in S \Gamma e$. Also we note that the element $e$ is both left and right $\alpha$-unity of $S \Gamma e$ in $S \Gamma e$.

Suppose $S$ is a regular $\Gamma$-semigroup with a left $\alpha$-unity $e$. Then we show that $S \Gamma e$ is a regular $\Gamma$-semigroup with a unity. We only show that $S \Gamma e$ is regular. Let a $e \in S \Gamma e$. Since $S$ is regular there exist $\beta, \delta \in \Gamma$ and $b \in S$ such that $a \gamma e=a \gamma e \beta b \delta a \gamma e$ i.e., $a \gamma e=a \gamma e \beta b \delta e \alpha a \gamma e=(a \gamma e) \beta(b \delta e) \alpha(a \gamma e)$. Since $b \delta e \in S \Gamma e$, aүe is regular. Hence $S \Gamma e$ is a regular $\Gamma$-semigroup.

Let us now consider $T$ with a left $\gamma$-unity $e$ and a right $\delta$-unity $g$. Then the constant mapping $C_{e}: X \rightarrow T$ which is defined by $C_{e}(x)=e$ for all $x \in X$ is a left $\gamma$-unity of $T^{X}$. Similarly the constant mapping $C_{g}$ is a right $\delta$-unity of $T^{X}$.

Theorem 4.6. Let $T$ be a $\Gamma$-semigroup with a left $\gamma$-unity and a right $\delta$ unity for some $\gamma, \delta \in \Gamma$. Then
(i) $T^{X}$ is a regular $\Gamma$-semigroup if and only if $T$ is a regular $\Gamma$-semigroup,
(ii) $T^{X}$ is a right (resp. left) orthodox $\Gamma$-semigroup if and only if $T$ is so and
(iii) $T^{X}$ is a right (resp. left) inverse $\Gamma$-semigroup if and only if $T$ is a right (resp. left) inverse $\Gamma$-semigroup.

Proof. By $C_{t}, t \in T$ denotes the mapping in $T^{X}$ such that $C_{t}(x)=t$ for all $x \in X$. Then it is clear that $\left(C_{t}\right) \alpha\left(C_{u}\right)=C_{(t \alpha u)}$ which shows that $C_{t}$ is an $\alpha$-idempotent if and only if $t$ is an $\alpha$-idempotent. Again we have that if $f$ is an $\alpha$-idempotent in $T^{X}$ then $f(x)$ is an $\alpha$-idempotent in $T$ for all $x \in X$.
(i) Assume that $T^{X}$ is a regular $\Gamma$-semigroup. Then for each $t \in T$ there exist $f \in T^{X}$ and $\alpha, \beta \in \Gamma$ such that $C_{t} \alpha f \beta C_{t}=C_{t}$ so that $t \alpha f(x) \beta t=t$ for all $x \in X$ which shows that $t$ is regular in $T$. Consequently $T$ is a regular $\Gamma$-semigroup. Conversely let $T$ be regular and let $e$ be a left $\gamma$-unity and $g$ be a right $\delta$-unity of $T$. Then for each $f \in T^{X}$ and for each $x \in X, f(x) \in T$ is a regular element and hence there exists a triplet $\left(\alpha_{x}, t_{x}, \beta_{x}\right) \in \Gamma \times T \times \Gamma$ such that $f(x) \alpha_{x} t_{x} \beta_{x} f(x)=f(x)$. i.e., $f(x)=(f(x) \delta g) \alpha_{x} t_{x} \beta_{x}(e \gamma f(x))=f(x) \delta\left(g \alpha_{x} t_{x} \beta_{x} e\right) \gamma f(x)$. Define $h: X \rightarrow T$ by $h(x)=g \alpha_{x} t_{x} \beta_{x} e$. Then for all $y \in X$, we have

$$
\begin{aligned}
(f \delta h \gamma f)(y) & =f(y) \delta h(y) \gamma f(y) \\
& =f(y) \delta g \alpha_{y} t_{y} \beta_{y} e \gamma f(y) \\
& =f(y) \alpha_{y} t_{y} \beta_{y} f(y) \\
& =f(y) .
\end{aligned}
$$

Hence $f$ is regular in $T^{X}$. Consequently $T^{X}$ is a regular $\Gamma$-semigroup.
(ii) Let $t, u \in T$ such that $t$ be an $\alpha$-idempotent and $u$ be a $\beta$-idempotent. Then $C_{t}$ is an $\alpha$-idempotent and $C_{u}$ is a $\beta$-idempotent in $T^{X}$. Now if $T^{X}$ is a right orthodox $\Gamma$-semigroup then $\left(C_{t} \alpha C_{u}\right) \beta\left(C_{t} \alpha C_{u}\right)=C_{t} \alpha C_{u}$ i.e., $t \alpha u$ is a $\beta$-idempotent in $T$ which implies $T$ is also a right orthodox $\Gamma$-semigroup. Similarly we can show that if $T^{X}$ is a left orthodox $\Gamma$-semigroup then $T$ is so. Let $f$ be an $\alpha$-idempotent and $h$ be a $\beta$-idempotent in $T^{X}$. Let us now suppose that $T$ is a right(resp. left) orthodox $\Gamma$-semigroup. Then $f(x) \alpha h(x)($ resp. $f(x) \beta h(x))$ is a $\beta$-idempotent ( resp. $\alpha$-idempotent ). Hence $T^{X}$ is a right (resp. left) orthodox $\Gamma$-semigroup.
(iii) Let $T^{X}$ be a right (resp. left) inverse $\Gamma$-semigroup and let $t, u \in T$ such that $t$ is an $\alpha$-idempotent and $u$ be a $\beta$-idempotent. Then $C_{t}$ is an $\alpha$-idempotent and $C_{u}$ is a $\beta$-idempotent in $T^{X}$ and $C_{t} \alpha C_{u} \beta C_{t}=$ $C_{u} \beta C_{t}\left(\right.$ resp. $\left.C_{t} \beta C_{u} \alpha C_{t}=C_{t} \beta C_{u}\right)$. Thus we have $t \alpha u \beta t=u \beta t($ resp. $t \beta u \alpha t=t \beta u)$ which implies that $T$ is a right(resp. left) inverse $\Gamma$ semigroup. Again let $T$ be a right (resp. left) inverse $\Gamma$-semigroup. Let $f$ be an $\alpha$-idempotent and $h$ be a $\beta$-idempotent in $T^{X} . f(x) \alpha h(x) \beta f(x)=$ $h(x) \beta f(x)$ (resp. $f(x) \beta h(x) \alpha f(x)=f(x) \beta h(x))$ for all $x \in X$ i.e, $f \alpha h \beta f=h \beta f$ (resp. $f \beta h \alpha f=f \beta h$ ). Thus $T^{X}$ is a right (resp. left)inverse $\Gamma$-semigroup.

Let us now suppose that the semigroup $S$ acts on $X$ from the left i.e., $s x \in X, s(r x)=(s r) x$ and $1 x=x$ if $S$ is a monoid, for every $r, s \in S$ and every $x \in X$. If $S$ acts on $X$ from left we call it left $S$ set $X$.

For every $\Gamma$-semigroup $T$, it is known that $\operatorname{End}(T)$ is a semigroup. Hence $\operatorname{End}\left(T^{X}\right)$ is also a semigroup.

Let $S$ be a semigroup, $T$ a $\Gamma$-semigroup and $X$ a nonempty set. Suppose $S$ acts on $X$ from left. Define $\phi: S \rightarrow \operatorname{End}\left(T^{X}\right)$ by $((\phi(s))(f))(x)=f(s x)$ for all $s \in S, f \in T^{X}$ and $x \in X$. We now verify that $\phi(s) \in \operatorname{End}\left(T^{X}\right)$. For this, let $f, g \in T^{X}, \alpha \in \Gamma$ and $x \in X$. Then $((\phi(s))(f \alpha g))(x)=(f \alpha g)(s x)=$ $f(s x) \alpha g(s x)=((\phi(s))(f))(x) \alpha((\phi(s))(g))(x)=((\phi(s))(f)) \alpha((\phi(s))(g))(x)$. Hence $(\phi(s))(f \alpha g)=((\phi(s))(f)) \alpha((\phi(s))(g))$, which implies that $\phi(s) \in$ $\operatorname{End}\left(T^{X}\right)$.

Let us now verify that $\phi: S \rightarrow \operatorname{End}\left(T^{X}\right)$ is a semigroup antimorphism. For this let $s_{1}, s_{2} \in S, f \in T^{X}$ and $x \in X$. Then $\left(\left(\phi\left(s_{1}\right) \phi\left(s_{2}\right)\right)(f)\right)(x)=$ $\left(\phi\left(s_{1}\right)\left(\phi\left(s_{2}\right)(f)\right)\right)(x)=\left(\phi\left(s_{2}\right)(f)\right)\left(s_{1} x\right)=f\left(\left(s_{2}\left(s_{1}(x)\right)\right)=f\left(\left(s_{2} s_{1}\right) x\right)=\right.$ $\left(\phi\left(s_{2} s_{1}\right)(f)\right)(x)$. Hence $\phi\left(s_{2} s_{1}\right)=\phi\left(s_{1}\right) \phi\left(s_{2}\right)$.

For this antimorphism $\phi: S \nrightarrow \operatorname{End}\left(T^{X}\right)$ we can define the semidirect product $S \times_{\phi} T^{X}$ of the semigroup $S$ and the $\Gamma$-semigroup $T^{X}$. We call this semidirect product the wreath product of the semigroup $S$ and the $\Gamma$ semigroup $T$ relative to the left $S$-set $X$. We denote it by $S W_{X} T$. We also denote $\phi(s)(f)(x)$ by $f^{s}(x)$. Hence $f^{s}(x)=f(s x)$.

If $|T|=1$, then $\left|T^{X}\right|=1$ and hence throughout the paper we assume that $|T| \geq 2$. We now give the relation between $T$ and $\left(T^{X}\right)^{e}$ for all $e \in E(S)$.

Similar to the Theorems 3.6 and 3.7 we have following Theorems.
Theorem 4.7. Let $S$ be a semigroup acting on the set $X$ from the left and $T$ be a $\Gamma$-semigroup with a left $\gamma$-unity and a right $\delta$-unity for some $\gamma, \delta \in \Gamma$. Then
(i) $T$ is a regular $\Gamma$-semigroup if and only if $\left(T^{X}\right)^{e}$ is a regular $\Gamma$-semigroup,
(ii) $T$ is a right (resp. left) orthodox $\Gamma$-semigroup if and only if $\left(T^{X}\right)^{e}$ is so and
(iii) $T$ is a right (resp. left) inverse $\Gamma$-semigroup if and only if $\left(T^{X}\right)^{e}$ is a right (resp. left) inverse $\Gamma$-semigroup.

Theorem 4.8. Let $S$ be a semigroup acting on the set $X$ from the left and $T$ be a $\Gamma$-semigroup with a left $\gamma$-unity and a right $\delta$-unity for some $\gamma, \delta \in \Gamma$. Then the wreath product $S W_{X} T$ is a right(left) orthodox $\Gamma$-semigroup if and only if
(i) $S$ is an orthodox semigroup and $\left(T^{X}\right)^{e}$ is a right(left) orthodox $\Gamma$ semigroup for every $e \in E(S)$
(ii) for every $x \in X, f \in T^{X}$ and $e \in E(S), f(x) \in f(e x) \Gamma T$ and
(iii) $f(e x)$ is an $\alpha$-idempotent for every $x \in X$, implies that $f(g e x)$ is an $\alpha$-idempotent for every $g \in E(S)$ where $e \in E(S), f \in T^{X}$.
We now prove the following Theorem.
Theorem 4.9. Let $S$ be an orthodox semigroup acting on the set $X$ from the left and $T$ be a right orthodox $\Gamma$-semigroup with a left $\gamma$-unity and a right $\delta$-unity for some $\gamma, \delta \in \Gamma$. Then the following statements are equivalent.
(a) $S$ and $T^{X}$ satisfy (ii) and (iii) of Theorem 4.8.
(b) $S$ permutes $X$ or $T$ is a $\Gamma$ - group and ge $X \subseteq e X$ for every $e, g \in E(S)$.

Proof. (a) $\Longrightarrow(\mathrm{b})$ : Let us suppose that $T$ is not a $\Gamma$-group. Then there exists $z \in T$ such that $z \Gamma T \neq T$. Let $e_{\delta}$ be a left $\delta$ - unity in $T$. For $x \in X$, define $f_{x}: X \rightarrow T$ by $f_{x}(y)=e_{\delta}$ if $y=x$ and $f_{x}(y)=z$ if $y \neq x$. Then by (ii), $e_{\delta}=f_{x}(x) \in f_{x}(g x) \Gamma T$ for every $g \in E(S)$. If $f_{x}(g x)=z$ then $e_{\delta} \in z \Gamma T$. Thus $e_{\delta}=z \alpha v$ for some $v \in T$ and $\alpha \in \Gamma$. This implies that $u=e_{\delta} \delta u=z \alpha v \delta u$ for all $u \in T$. Hence $T=z \Gamma T$ which is a contradiction. Hence $f_{x}(g x)=e_{\delta}$ Thus we can conclude that $g x=x$ for all $g \in E(S)$. Let $a \in S$ and $x, y \in X$ such that $a x=a y$. For $a^{\prime} \in V(a), a^{\prime} a \in E(S)$ and $x=\left(a^{\prime} a\right) x=\left(a^{\prime} a\right) y=y$. Again $\left(a a^{\prime}\right) x=x$ implies that $a\left(a^{\prime} x\right)=x$. Hence for each $a \in S$, the mapping $f_{a}: X \rightarrow X$ defined by $f_{a}(x)=a x$ is a permutation on $X$. This means that $S$ permutes $X$.

Now $T$ is a $\Gamma$ - group. Note that $e_{\delta}$ is a $\delta$-idempotent and since $T$ is a $\Gamma$ - group, $E_{\delta}(T)=\left\{e_{\delta}\right\}$. Let $t \neq e_{\delta} \in T$ and $e \in E(S)$. Define $h: X \longrightarrow T$ by $h(x)=e_{\delta}$ if $x \in e X$, otherwise $h(x)=t$. Now $h(e x)=e_{\delta}$ for every $x \in X$ and hence by (iii), $h($ gex $)=e_{\delta}$. This implies that gex $\in e X$ and hence $g e X \subseteq e X$ for all $e, g \in E(S)$.
$(\mathrm{b}) \Longrightarrow(\mathrm{a})$ : The proof is almost similar to the proof of $(2) \Rightarrow(1)$ of Lemma 3.2 [5].

From Theorem 4.7 and 4.9 we conclude that
Theorem 4.10. Let $S$ be a semigroup acting on the set $X$ from the left and $T$ be a $\Gamma$-semigroup with a left $\gamma$-unity and a right $\delta$-unity for some $\gamma, \delta \in \Gamma$. Then the wreath product $S W_{X} T$ is a right orthodox $\Gamma$-semigroup if and only if
(1) $S$ is an orthodox semigroup and $T$ is a right orthodox $\Gamma$-semigroup and
(2) $S$ permutes $X$ or $T$ is a $\Gamma$ - group and ge $X \subseteq e X$ for every $e, g \in E(S)$.

Theorem 4.11. Let $S, T$ and $X$ be as in Theorem 4.10. Then the wreath product $S W_{X} T$ is a right inverse $\Gamma$-semigroup if and only if
(i) $S$ is a right inverse semigroup and $T$ is a right inverse $\Gamma$-semigroup and
(ii) $S$ permutes $X$ or $T$ is a $\Gamma$-group.

Proof. Suppose that $S W_{X} T$ is a right inverse $\Gamma$-semigroup. Then by Theorem 3.7 and Theorem 4.7 we have $S$ is a right inverse semigroup and $T$ is a right inverse $\Gamma$-semigroup and by Theorem 4.10 we have $S$ permutes $X$ or $T$ is a $\Gamma$-group.

Conversely suppose that $S, T$ and $X$ satisfy (i) and (ii). Then by Theorem $4.6 T^{X}$ is a right inverse $\Gamma$-semigroup. If $T$ is a $\Gamma$-group, then $f(x) \in$ $f(e x) \Gamma T$ for every $f \in T^{X}, e \in E(S), x \in X$. If $S$ permutes $X$, then $f(x) \in f(x) \Gamma T=f(e x) \Gamma T$ since $e x=x$ for every $e \in E(S)$. Then by Theorem 3.7 $S \times{ }_{\alpha} T^{X}=S W_{X} T$ is a right inverse $\Gamma$-semigroup.

Theorem 4.12. Let $S, T$ and $X$ be as in Theorem 4.10. Then the wreath product $S W_{X} T$ is a left inverse $\Gamma$-semigroup if and only if $S$ is a left inverse semigroup and $T$ is a left inverse $\Gamma$-semigroup and $S$ permutes $X$.

Proof. By Theorem 3.8 and Theorem 4.7, we have $S W_{X} T$ is a left inverse $\Gamma$-semigroup if and only if $S$ is a left inverse semigroup and $T$ is a left inverse $\Gamma$-semigroup and $f(e x)=f(x)$ for every $f \in T^{X}, e \in E(S), x \in X$. The remaining part of the proof is almost similar to the proof of Corollary 3.7 [5].

## Open problem:

(i) Find relation between $T$ and $T^{X}$ without assuming the existence of left $\alpha$-unity and right $\beta$-unity in the $\Gamma$-semigroup $T$ for some $\alpha, \beta \in \Gamma$.
(ii) Study the Wreath product of a semigroup $S$ and a $\Gamma$-semigroup $T$ without assuming the existence of left $\alpha$-unity and right $\beta$-unity in $T$ for some $\alpha, \beta \in \Gamma$.

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