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A COMMON APPROACH TO DIRECTOIDS WITH AN ANTITONE INVOLUTION AND D-QUASIRINGS*

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Abstract

We introduce the so-called DN-algebra whose axiomatic system is a common axiomatization of directoids with an antitone involution and the so-called D-quasiring. It generalizes the concept of Newman algebras (introduced by H. Dobbertin) for a common axiomatization of Boolean algebras and Boolean rings.

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By a *Newman algebra* (see [1]) is meant a (generally non-associative) semiring $\mathcal{A} = (A; +, \cdot, ', 0, 1)$ with neutral elements 0 and 1 and complementation operation $'$ (i.e. $x \cdot x' = 0$ and $x + x' = 1$ for all $x \in A$). These algebras were introduced by M.H.A. Newman in 1941 when studying the relationship between a non-associative modification of Boolean rings with unit and Boolean algebras. For the associative modification of a Newman algebra, so-called $\mathbf{N}$-*algebra*, the simple axiomatic system is given in [4].

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One of the axioms is as follows

(N) $x + y = ((1 + 1)' \cdot (x \cdot y))' \cdot (x' \cdot y')'$.

H. Dobbertin shows in [4] that if $1 + 1 = 1$ in (N) then the corresponding N-algebra is a Boolean algebra and if $1 + 1 = 0$ then it is a Boolean ring.

Since Boolean algebras and Boolean rings with unit are in a one-to-one correspondence, there was a question if an algebra similar to a Boolean ring can be constructed to obtain a one-to-one correspondence between ortholattices and these algebras. It was solved in [5] where the concept of *Boolean quasiring* is introduced. Hence ortholattices and Boolean quasirings are in the same correspondence as Boolean algebras and Boolean rings with unit. The natural question arised if also the concept of N-algebra can be generalized to serve as a common axiomatization of ortholattices and Boolean quasirings. This was answered in [3] where the concept of a **QN**-*algebra* is introduced by several simple axioms including the axiom (N). The main result of [3] is that if $1 + 1 = 1$ then the QN-algebra is an ortholattice and if $1 + 1 = 0$ then it is a Boolean quasiring. Hence, the analogy is complete.

The concept of an ortholattice can be generalized when the underlying semilattice is substituted by a so-called directoid (see e.g. [6]).

For the readers convenience, we repeat the definition: A *directoid* is an algebra $\mathcal{D} = (D; \sqcap)$ of type (2) satisfying the identities

(D1) $x \sqcap x = x$;

(D2) $(x \sqcap y) \sqcap x = x \sqcap y$;

(D3) $y \sqcap (x \sqcap y) = x \sqcap y$;

(D4) $x \sqcap ((x \sqcap y) \sqcap z) = (x \sqcap y) \sqcap z$.

The *induced order* of a directoid $\mathcal{D} = (D; \sqcap)$ is defined by $x \leq y$ if and only if $x \sqcap y = x$. With respect to $\leq$, the couple $(D; \leq)$ is a downward directed set where for every $x, y \in D$ the element $x \sqcap y$ is a common lower bound of x, y.

Also conversely, if $(D; \leq)$ is a downward directed ordered set and for each $x, y \in D$ we choose a common lower bound d of x, y arbitrarily with the constraint that $x \leq y$ implies $d = x$, then, putting $x \sqcap y = d$, the resulting algebra $(D; \sqcap)$ is a directoid.

Let $(D; \sqcap)$ be a directoid with a least element 0 and a greatest element 1. A mapping from D to D assigning x' to x is called an *antitone involution* if $x'' = x$ and if $x \leq y$ implies $y' \leq x'$ with respect to the induced order. A bounded directoid with an antitone involution will be denoted by $\mathcal{D} = (D; \sqcap, ', 0, 1)$. The term operation $\sqcup$ on D defined by $x \sqcup y = (x' \sqcap y')'$ will be called an *assigned operation*, see e.g. [2] for its properties and further results.

Hence, bounded directoids with an antitone involution are in fact a generalization of ortholattices since the underlying semilattice $(D; \wedge)$ of an ortholattice $(D; \wedge, \vee, ', 0, 1)$ is a directoid and, due to De Morgan laws, $x \vee y = (x' \wedge y')'$ is an assigned operation. The question how the induced ring-like structure looks like was completely solved in [2]:

By a **D**-*quasiring* is meant an algebra $\mathcal{R} = (R; +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ satisfying the identities

(Q1) $(x \cdot y) \cdot x = x \cdot y;$

(Q2) $y \cdot (x \cdot y) = x \cdot y;$

(Q3) $x \cdot ((x \cdot y) \cdot z) = (x \cdot y) \cdot z;$

(Q4) $x \cdot 0 = 0;$

(Q5) $x \cdot 1 = x;$

(Q6) $x + 0 = x;$

(Q7) $1 + ((1 + (x \cdot y)) \cdot (1 + y)) = y.$

The following correspondence is shown in [2] (Theorems 4 and 5):

Proposition. *If* $\mathcal{R} = (R; +, \cdot, 0, 1)$ *is a D-quasiring and*

$$x \sqcap y = x \cdot y, \quad x' = 1 + x \quad \text{and} \quad x \sqcup y = 1 + ((1 + x) \cdot (1 + y))$$

then $\mathcal{D}(R) = (R; \sqcap, ', 0, 1)$ *is a bounded directoid with an antitone involution where* $\sqcup$ *is the assigned operation.*

If $\mathcal{D} = (D; \sqcap, ', 0, 1)$ *is a bounded directoid with an antitone involution and* $\sqcup$ *its assigned operation then for*

$$x + y = (x \sqcup y) \sqcap (x \sqcap y)' \quad \text{and} \quad x \cdot y = x \sqcap y$$

the algebra $\mathcal{R}(D) = (D; +, \cdot, 0, 1)$ *is a D-quasiring.*

Due to an analogy of the relationship between ortholattices and Boolean quasirings, we can search for the analogy of a QN-algebra. A suitable candidate can be as follows.

Definition 1. By a **DN**-*algebra* we mean an algebra $\mathcal{A} = (A; +, \cdot, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ satisfying the following identities

(1) $(x \cdot y) \cdot x = x \cdot y$;

(2) $y \cdot (x \cdot y) = x \cdot y$;

(3) $x \cdot ((x \cdot y) \cdot z) = (x \cdot y) \cdot z$;

(4) $x \cdot 0 = 0 = 0 \cdot x$;

(5) $x \cdot 1 = x = 1 \cdot x$;

(6) $x'' = x$;

(7) $(x' \cdot y')' \cdot x = x$;

(N) $x + y = ((1 + 1)' \cdot (x \cdot y))' \cdot (x' \cdot y')'$.

Let us note that not only (N) but also the axioms (4) and (6) are common with a Newman algebra. Moreover, every N-algebra and every QN-algebra is a DN-algebra as well.

We are going to show that the aforementioned algebras can be derived from DN-algebras in the same way as described before.

Theorem 1. *Let $\mathcal{A} = (A; +, \cdot, ', 0, 1)$ be a DN-algebra satisfying $1 + 1 = 1$. Then its reduct $\mathcal{D}(A) = (A; \cdot, ', 0, 1)$ is a bounded directoid with an antitone involution where the assigned operation is $+$.*

Proof. Of course, (1) is (D2), (2) is (D3) and (3) is (D4). Putting $y = z = 1$ in (3) and using (5) we obtain (D1). Hence, $(A; \cdot)$ is a directoid. Let $\leq$ be its induced order. By (4) and (5) we conclude $0 \leq x \leq 1$ for any $x \in A$. By (6) we see that the mapping $x \mapsto x'$ is an involution. Suppose $y' \leq x'$. Then $x' \cdot y' = y'$ and, by (6) and (7), $y \cdot x = (y')' \cdot x = (x' \cdot y')' \cdot x = x$ thus $x \leq y$. Due to (6), also $x \leq y$ implies $y' \leq x'$ thus this involution is antitone. Hence also $0' = 1$ and $1' = 0$. Assume $1 + 1 = 1$. We compute

$$x + y = (1' \cdot (x \cdot y))' \cdot (x' \cdot y')' = 0' \cdot (x' \cdot y')' = (x' \cdot y')'$$

whence $+$ is the assigned operation. ∎

We show that also the second conclusion is analogous.

Theorem 2. *Let $\mathcal{A} = (A; +, \cdot, ', 0, 1)$ be a DN-algebra. If $1 + 1 = 0$ then its reduct $\mathcal{R}(A) = (A; +, \cdot, 0, 1)$ is a D-quasiring.*

Proof. Of course, (1) is (Q1), (2) is (Q2), (3) is (Q3), (4) implies (Q4) and (5) implies (Q5). By (6) and (7), $x \mapsto x'$ is an antitone involution as shown in the proof of Theorem 1 (where the induced order is that of the directoid $(A; \cdot)$). Hence $0' = 1$ and $1' = 0$.

Then by (N) we have

$$x + 0 = ((1 + 1)' \cdot (x \cdot 0))' \cdot (x' \cdot 0')' = 0' \cdot (x' \cdot 0')' = (x' \cdot 1)' = x'' = x$$

proving (Q6).

Further, by (N) we conclude $1+x = x'$ and, applying (6), $1+(1+x) = x$. Since $(A; \cdot)$ is a directoid, by (D3) we have $y \cdot (x \cdot y) = x \cdot y$, i.e. $x \cdot y \leq y$. Hence $y' \leq (x \cdot y)'$ and $(x \cdot y)' \cdot y' = (1 + (x \cdot y)) \cdot (1 + y) = 1 + y$. Applying the previous identity, we conclude

$$1 + ((1 + (x \cdot y)) \cdot (1 + y)) = 1 + (1 + y) = y$$

which is (Q7). Hence $\mathcal{R}(A) = (A; +, \cdot, 0, 1)$ is a D-quasiring. ∎

Corollary 1. *QN-algebras are exactly the DN-algebras satisfying $x \cdot (y \cdot z) = (y \cdot x) \cdot z$ and $x \cdot x' = 0$.*

Proof. It is evident that the identity $x \cdot (y \cdot z) = (y \cdot x) \cdot z$ yields commutativity and associativity of the operation " $\cdot$ ". Moreover, taking x' and y' instead of x and y in (7) and using (6) we obtain $(x \cdot y)' \cdot x' = x'$, i.e. $((x \cdot y)' \cdot x')' = x$. Hence, if the given DN-algebra satisfies also $x \cdot x' = 0$ then it ia QN-algebra. ∎

Corollary 2. *Ortholattices are exactly the DN-algebras satisfying $x \cdot x' = 0$, $x \cdot (y \cdot z) = (y \cdot x) \cdot z$ and $1 + 1 = 1$.*

Proof. As shown by Corollary 1, the corresponding DN-algebra is a QN-algebra. Hence, if it is satisfies also $1 + 1 = 1$, it is an ortholattice by Theorem 2.3. in [3]. ∎

Definition 2. *Let $\mathcal{A} = (A; +, \cdot, ', 0, 1)$ be a DN-algebra and $a \in A$. Define $x +_a y = (a' \cdot (x \cdot y))' \cdot (x' \cdot y')'$. Then $\mathcal{A}_a = (A; +_a, \cdot, ', 0, 1)$ will be called the a-mutation of $\mathcal{A}$.*

An analogous concept was defined for N-algebras in [4] and for QN-algebras in [3].

Theorem 3. *Let $\mathcal{A} = (A; +, \cdot, ', 0, 1)$ be a DN-algebra and $a, b \in A$. Then the following hold:*

- (i) $1 +_a 1 = a$;
- (ii) $\mathcal{A}_a$ *is a DN-algebra;*
- (iii) $\mathcal{A}_1$ *is a bounded directoid with an antitone involution;*
- (iv) $\mathcal{A}_0$ *is a D-quasiring;*
- (v) $\mathcal{A}_{1+1} = \mathcal{A}$;
- (vi) $(\mathcal{A}_a)_b = \mathcal{A}_b$;
- (vii) $\{\mathcal{A}_a;\ a \in A\}$ *is the set of all DN-algebras with base set A having the same multiplication and the same unary operation as* $\mathcal{A}$;
- (viii) $\mathcal{A}$ *and* $\mathcal{A}_a$ *admit the same congruences.*

Proof.

(i) $1 +_a 1 = (a' \cdot (1 \cdot 1))' \cdot (1' \cdot 1')' = (a')' \cdot (1')' = a \cdot 1 = a$.

(ii) By (i), $x +_a y = (a' \cdot (x \cdot y))' \cdot (x' \cdot y')' = ((1 +_a 1)' \cdot (x \cdot y))' \cdot (x' \cdot y')'$ for all $x, y \in A$.

(iii) According to (ii), $\mathcal{A}_1$ is a DN-algebra and according to (i), $1 +_1 1 = 1$. Hence $\mathcal{A}_1$ is a bounded directoid with an antitone involution where the assigned operation is $+_1$ (by Theorem 1).

(iv) According to (ii), $\mathcal{A}_0$ is a DN-algebra and according to (i), $1 +_0 1 = 0$ thus $\mathcal{A}_0$ is a D-quasiring (by Theorem 2).

(v) $x +_{1+1} y = ((1 + 1)' \cdot (x \cdot y))'(x' \cdot y')' = x + y$ for all $x, y \in A$.

(vi) Since $\mathcal{A}$ is a DN-algebra, the same is true for $\mathcal{A}_a = (A; +_a, \cdot, ', 0, 1)$ according to (ii), and $x +_a y = (a' \cdot (x \cdot y))' \cdot (x' \cdot y')'$ for all $x, y \in A$.

Since $\mathcal{A}_a$ is a DN-algebra, the same is true for $(\mathcal{A}_a)_b = (A; (+_a)_b, \cdot, ', 0, 1)$ according to (ii) and $x(+_a)_b y = (b' \cdot (x \cdot y))' \cdot (x' \cdot y')' = x +_b y$ for all $x, y \in A$.

(vii) Let $\mathcal{S} = (A; \oplus, \cdot, ', 0, 1)$ be a DN-algebra. Then $x \oplus y = ((1 \oplus 1)' \cdot (x \cdot y))' \cdot (x' \cdot y')' = x +_{1\oplus 1} y$ for all $x, y \in A$ and hence $\mathcal{S} = \mathcal{A}_{1\oplus 1}$.

(viii) $\mathcal{A}$ and $\mathcal{A}_a$ admit the same congruences as $(A; \cdot, ')$ because $+$ and $+_a$ are polynomials of the algebra $(A; \cdot, ')$.

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